Boundary conditions for the single-factor term structure equation

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We consider the classical case of one-factor models for the short rate. Modelling the short rate $X(t)$ directly under the pricing measure as

$$dX(t) = \beta(X(t), t)\,dt + \sigma(X(t), t)\,dW,$$

the option price u corresponding to a pay-off function g is given, using risk neutral valuation, by

$$u(x, t) = E_{x,t}\left[e^{-\int_t^T X(s)\,ds} g(X(T))\right].$$

Note that if the payoff $g \equiv 1$, then bond prices are obtained. Also note that the set-up covers the case of bond options. The function u satisfies the term structure equation

$$u_t(x, t) + \frac{1}{2}\sigma^2(x, t)u_{xx}(x, t) + \beta(x, t)u_x(x, t) = xu(x, t)$$

with terminal condition $u(x, T) = g(x)$. The term structure equation holds at all interior points of the domain of X. However, to have uniqueness of solutions to the equation one also needs to specify boundary conditions at $x = 0$.

Assumptions

Hypothesis

The drift $\beta \in C([0,\infty) \times [0, T])$ *is continuously differentiable in* x *with bounded derivative, and* $\beta(0,t) \geq 0$ *for all* t*. The volatility* $\sigma \in C([0,\infty) \times [0, T])$ *is such that* $\alpha(x,t) := \frac{1}{2}\sigma^2(x,t)$ *is continuously differentiable in* x *with a Hölder continuous derivative (for some parameter* $\delta > 0$*), and* $\sigma(x,t) = 0$ *if and only if* $x = 0$*. The functions* β*,* σ *and* α_x *are all of at most linear growth:*

$$|\beta(x,t)| + |\sigma(x,t)| + |\alpha_x(x,t)| \leq C(1+x) \tag{1}$$

for all x *and* t*. The pay-off function* $g : [0,\infty) \to [0,\infty)$ *is continuously differentiable with both* g *and* g' *bounded.*

Model specification

It follows that there exists a unique strong solution $X(t)$ to

$$dX(t) = \beta(X(t), t)\, dt + \sigma(X(t), t)\, dW \tag{2}$$

The option price $u : [0, \infty) \times [0, T] \to [0, \infty)$ corresponding to a pay-off function $g : [0, \infty) \to [0, \infty)$ is given by

$$u(x, t) = E_{x,t}\left[e^{-\int_t^T X(s)\, ds} g(X(T))\right], \tag{3}$$

where the indices indicate that $X(t) = x$. The corresponding term structure equation is given by

$$u_t(x, t) + \frac{1}{2}\sigma^2(x, t)u_{xx}(x, t) + \beta(x, t)u_x(x, t) = xu(x, t) \tag{4}$$

for $(x, t) \in (0, \infty) \times [0, T)$, with terminal condition

$$u(x, T) = g(x). \tag{5}$$

Moreover, by formally inserting $x = 0$ in the equation we get the boundary condition

$$u_t(0, t) + \beta(0, t)u_x(0, t) = 0 \tag{6}$$

for all $t \in [0, T)$, since $\sigma(0, t) = 0$ by assumption. One of the main efforts in this paper is to show that the option price u is continuously differentiable up to the boundary $x = 0$, and that it indeed satisfies the boundary condition (6) in the classical sense.

Main Result

Definition

A *classical solution* to the term structure equation is a function $v \in C([0,\infty)\times[0,T]) \cap C^1([0,\infty)\times[0,T)) \cap C^{2,1}((0,\infty)\times[0,T))$ which satisfies (4), (5) and (6).

Our main result in this article is the following.

Theorem

In addition to Hypothesis 1, also assume that Assumption 6 below holds. Then the option price u given by (3) *is the unique bounded classical solution to the term structure equation.*

Examples

Classical short rate models such as the Cox-Ingersoll-Ross model

$$dX(t) = (a - bX(t))\, dt + \sigma \sqrt{X(t)}\, dW, \tag{7}$$

and the Dothan model

$$dX(t) = aX(t)\, dt + \sigma X(t)\, dW, \tag{8}$$

have boundary conditions at $x = 0$ that are immediate to write down. These conditions are

$$u_t + au_x = 0 \qquad \text{and} \qquad u_t = 0,$$

respectively. We note that the boundary condition $u_t = 0$ for the Dothan model means that u is constant along the boundary, i.e. $u(0, t) = g(0)$. This is the same type of boundary condition that appears for options on stocks in [4], which can be explained by the fact that the Dothan model is a geometric Brownian motion.

The theorem also covers the Hull-White model

$$dX(t) = (a(t) - b(t)X(t))\,dt + \sigma(t)\sqrt{X(t)}\,dW \tag{9}$$

(which is a time-dependent generalization of the Cox-Ingersoll-Ross model), and models of, for example, the form

$$dX(t) = (b - aX(t))\,dt + \sigma X^{\gamma}(t)\,dW \qquad \gamma \in (1/2, 1], \tag{10}$$

which also would be natural to consider for bond pricing.

Some remarks

Remark

It seems that many of the classical models for the short rate are proposed for their analytical tractability. In particular, if the drift β *and the diffusion coefficient* σ^2 *are affine, then the model admits an affine term structure. It is easy to check that known explicit formulas for bond prices and bond options satisfy the boundary condition (6). In particular, for models admitting an affine term structure, it is a consequence of the associated Riccati equations, see Equation (22.25) in [3], that these boundary conditions are fulfilled.*

Remark

The assumption that g is continuously differentiable is satisfied for bonds, but not in general for bond options. However, using the Markov property, Theorem 3 readily extends to Lipschitz pay-offs provided one can show that the corresponding option price $x \mapsto u(x, T - \epsilon)$ *is continuously differentiable on* $[0, \infty)$ *for any* $\epsilon > 0$. *The regularizing effect of parabolic equations guarantees continuous differentiability on* $(0, \infty)$, *so the main difficulty is to show that* $x \mapsto u_x(x, T - \epsilon)$ *is continuous also at 0. If the model is convexity preserving, this is easily done in certain cases including for example call and put options written on bond prices. For details on which short rate models are convexity preserving, see [2]. To our knowledge, all models used in practice belong to this class.*

Proof of Main Result

Proof of uniqueness. Standard maximum principle argument.

Proof of continuity

To show that u is continuous, denote by $X^{x,t}$ the solution to (2) with initial condition $X^{x,t}(t) = x$. Let (x,t) and (y,r) be two points in $[0,\infty) \times [0,T]$. Then, if $r \leq t$, we have

$$\begin{aligned}
|u(y,r) - u(x,t)| &\leq E\left[e^{-\int_r^T X^{y,r}(s)\,ds}\left|g(X^{y,r}(T)) - g(X^{x,t}(T))\right|\right] \\
&\quad + E\left[g(X^{x,t}(T))\left|e^{-\int_r^T X^{y,r}(s)\,ds} - e^{-\int_t^T X^{x,t}(s)\,ds}\right|\right] \\
&\leq E\left[\left|g(X^{y,r}(T)) - g(X^{x,t}(T))\right|\right] \\
&\quad + C\int_t^T E\left[\left|X^{y,r}(s) - X^{x,t}(s)\right|\right]\,ds \\
&\quad + C\int_r^t E\left[X^{y,r}(s)\right]\,ds
\end{aligned}$$

for some constant C, where we have used that g is bounded. A similar expression can be derived if $r > t$.

It follows from Remark 1 in §8, Chapter 2 in [3] that $X^{y,r}(t) \to x$ in L^2 as $(y, r) \to (x, t)$. Therefore, from Theorem 2.1 in [1] we have

$$E\left[\sup_{t \le s \le T} (X^{y,r}(s) - X^{x,t}(s))^2\right] \to 0$$

as $(y, r) \to (x, t)$ (Theorem 2.1 in [1] also holds in the case of random starting points). Since g is assumed continuous and bounded, all three terms on the right hand side of (11) tend to 0 as $(y, r) \to (x, t)$. Thus u is continuous on $[0, \infty) \times [0, T]$. $\square$

Interior regularity

Proof that $u \in C^{2,1}((0,\infty) \times [0,T))$ *and satisfies* (4). For a given point $(x,t) \in (0,\infty) \times [0,T)$, let

$$R = (x_1, x_2) \times [t_1, t_2) \subseteq (0,\infty) \times [0,T)$$

be a rectangle which contains (x,t), where $x_1 > 0$. Since u is continuous, it follows from standard parabolic theory, see [2], that there exists a unique solution $U \in C^{2,1}(R)$ to the boundary value problem

$$\begin{cases} U_t + \frac{1}{2}\sigma^2 U_{xx} + \beta U_x - xU = 0 & \text{in } R \\ U = u & \text{on } \partial_p R, \end{cases}$$

where $\partial_p R = ([x_1, x_2] \times \{t_2\}) \cup (\{x_1, x_2\} \times [t_1, t_2])$ is the parabolic boundary of R.

From Ito's formula, the process

$$Z(s) = e^{-\int_t^s X^{x,t}(r)\,dr} U(X^{x,t}(s), s)$$

is a martingale on the time interval $[t, \tau_R]$, where

$$\tau_R = \inf\{s \geq t : X^{x,t}(s) \notin R\}$$

is the first exit time from the rectangle R. Therefore,

$$U(x,t) = E\left[e^{-\int_t^{\tau_R} X^{x,t}(r)\,dr} u(X^{x,t}(\tau_R), \tau_R)\right] = u(x,t),$$

where the second equality follows from the strong Markov property. Consequently, $u \in C^{2,1}((0,\infty) \times [0,T))$. Since $u \equiv U$ on R, we also see that u satisfies (4). $\square$

Continuity of the First Spatial Derivate

Recall that α_x is assumed to be continuous on $[0, \infty) \times [0, T]$, where $\alpha(x,t) = \frac{1}{2}\sigma^2(x,t)$. Let the process Y be modeled by the stochastic differential equation

$$dY(t) = (\alpha_x + \beta)(Y(t), t)\, dt + \sigma(Y(t), t)\, dW. \qquad (11)$$

Rather than specifying precise conditions under which (11) has a unique solution, we simply assume what we need.

Assumption

The coefficients σ and β are such that pathwise uniqueness holds for equation (11).

Remark

Note that Assumption 6 holds for example if α is twice continuously differentiable in space, since then the drift $\alpha_x + \beta$ is locally Lipschitz continuous. Moreover, if σ and β are time-independent, then it follows from [1], [2] and Section IX.3 in [3] that Assumption 6 automatically holds. Thus the Cox-Ingersoll-Ross model (7)*, the Dothan model* (8)*, the Hull-White model* (9)*, and the model* (10) *all satisfy Assumption 6.*

Also note that since $\alpha(0, t) = 0$, we have $\alpha_x(0, t) \geq 0$. Thus Y remains nonnegative since it has the same volatility as X but a larger drift at 0*.*

Next, define the function v by

$$\begin{aligned} v(x,t) &= E\left[g'(Y(T))\exp\left\{\int_t^T \beta_x(Y(s),s) - Y(s)\,ds\right\}\right] \\ &- E\left[\int_t^T \exp\left\{\int_t^s \beta_x(Y(r),r) - Y(r)\,dr\right\} u(Y(s),s)\,ds\right] \end{aligned}$$

where Y is the solution to (11) with initial condition $Y(t) = x$.

Remark

If the term structure equation (4) *is formally differentiated with respect to* x, *then the derivative* u_x *satisfies*

$$(u_x)_t + \alpha(u_x)_{xx} + (\alpha_x + \beta)(u_x)_x + (\beta_x - x)u_x - u = 0$$

with terminal condition $u_x(x, T) = g'(x)$. *The function* v *defined in* (12) *is the corresponding stochastic representation. In Theorem 11 below we show that* v *indeed equals the spatial derivative of* u.

Proposition

The function $v(x, t)$ *is continuous on* $[0, \infty) \times [0, T]$.

Proof.

The result follows along the same lines as for the continuity of u above. □

Continuity in the volatility parameters

Let $\{\sigma^n(x,t)\}_{n=1}^{\infty}$ be a sequence of functions satisfying Hypothesis 1 uniformly in n, i.e. with the same constant C in the bound (1). Moreover, assume that $\sigma^n(x,t)$ converges to $\sigma(x,t)$ and α_x^n converges to α_x uniformly on compacts as $n \to \infty$, where $\alpha^n = \frac{1}{2}(\sigma^n)^2$. Let u^n and v^n be defined as u and v but using the volatility function σ^n instead of σ. More explicitly,

$$u^n(x,t) = E\left[e^{-\int_t^T X^n(s)\,ds} g(X^n(T))\right]$$

and

$$\begin{aligned} v^n(x,t) &= E\left[g'(Y^n(T)) \exp\left\{\int_t^T \beta_x(Y^n(s),s) - Y^n(s)\,ds\right\}\right] \\ &\quad - E\left[\int_t^T \exp\left\{\int_t^s \beta_x(Y^n(r),r) - Y^n(r)\,dr\right\} u^n(Y^n(s),s)\,ds\right], \end{aligned}$$

where X^n and Y^n satisfy

$$\begin{cases} dX^n(s) = \beta(X^n(s), s)\, ds + \sigma^n(X^n(s), s)\, dW(s) \\ X^n(t) = x \end{cases}$$

and

$$\begin{cases} dY^n(s) = (\alpha_x^n + \beta)(Y^n(s), s)\, ds + \sigma^n(Y^n(s), s)\, dW(s) \\ Y^n(t) = x, \end{cases}$$

respectively.

Proposition

The functions u and v are continuous in the volatility parameter. More precisely, $u^n(x,t) \longrightarrow u(x,t)$ and $v^n(x,t) \longrightarrow v(x,t)$ as $n \longrightarrow \infty$ for any fixed point $(x,t) \in [0,\infty) \times [0,T]$.

Proof.

It follows from Theorem 2.5 in [1] that

$$\lim_{n\to\infty} E\left[\sup_{s\in[t,T]} (X(s) - X^n(s))^2\right] = 0.$$

Therefore

$$\begin{aligned} |u^n(x,t) - u(x,t)| &\leq E\left[|e^{-\int_t^T X^n(s)\,ds} - e^{-\int_t^T X(s)\,ds}|g(X^n(T))\right] \\ &\quad + E\left[e^{-\int_t^T X(s)\,ds}|g(X^n(T)) - g(X(T))|\right] \\ &\leq C\int_t^T E\left[|X(s) - X^n(s)|\right] ds \\ &\quad + E\left[|g(X^n(T)) - g(X(T))|\right] \to 0 \end{aligned}$$

as $n \to \infty$. Thus u is continuous in the volatility function. The continuity of v in the volatility function is similar. □

Applying Girsanov's theorem

Theorem

We have $u_x(x,t) = v(x,t)$ *on* $[0,\infty) \times [0,T]$*. Consequently,* u_x *is continuous on* $[0,\infty) \times [0,T]$*.*

Proof

It suffices to prove $u_x(x,0) = v(x,0)$. We first assume that σ is continuously differentiable in x with a bounded derivative. It then follows from Section 5.5 in [1] or §8 in [3], that the derivative

$$\xi(t) := \frac{\partial X(t)}{\partial x}$$

of $X(t) = X^{x,0}(t)$ with respect to the initial point x exists and is continuous, and it satisfies

$$\begin{cases} d\xi(t) = \xi(t)\beta_x(X(t),t)\,dt + \xi(t)\sigma_x(X(t),t)\,dW(t) \\ \xi(0) = 1. \end{cases}$$

Moreover,

$$
\begin{aligned}
u_x(x,0) &= E\left[g'(X(T))\xi(T)\exp\left\{-\int_0^T X(s)\,ds\right\}\right] \qquad (12)\\
&\quad -E\left[g(X(T))\exp\left\{-\int_0^T X(s)\,ds\right\}\int_0^T \xi(s)\,ds\right]\\
&=: I_1 - I_2.
\end{aligned}
$$

We claim that $I_i = J_i$, $i = 1, 2$, where

$$
J_1 = E\left[g'(Y(T))\exp\left\{\int_0^T \beta_x(Y(s),s) - Y(s)\,ds\right\}\right]
$$

and

$$
J_2 = E\left[\int_0^T \exp\left\{\int_0^s \beta_x(Y(r),r) - Y(r)\,dr\right\} u(Y(s),s)\,ds\right],
$$

compare (12) above. Here Y is defined as in (11) with initial condition $Y(0) = x$.

To show that $I_1 = J_1$, define a new measure Q on $\mathcal{F}_T$ by $dQ = M(T)\,dP$, where the process M is defined by

$$M(t) = \xi(t) \exp\left\{ - \int_0^t \beta_x(Y(s))\, ds \right\}. \tag{13}$$

By Ito's formula,

$$dM(t) = M(t)\sigma_x(X(t))\, dW(t),$$

so M is a martingale since σ_x is bounded. In particular, $E[M(T)] = 1$, so Q is a probability measure.

From Girsanov's theorem it follows that

$$\tilde{W}(t) = W(t) - \int_0^t \sigma_x(X(s))\,ds$$

is a Q-Brownian motion, and

$$dX = (\sigma\sigma_x + \beta)(X(t), t)\,dt + \sigma(X(t), t)\,d\tilde{W}.$$

Here $\sigma\sigma_x = \alpha_x$, so by weak uniqueness, the Q-law of X is the same as the law of Y under P. Consequently,

$$\begin{aligned} I_1 &= E\left[g'(X(T))\xi^x(T)\exp\left\{-\int_0^T X(s)\,ds\right\}\right] \\ &= E^Q\left[g'(X(T))\exp\left\{\int_0^T \beta_x(X(s), s) - X(s)\,ds\right\}\right] = J_1. \end{aligned}$$

To prove $I_2 = J_2$, note that

$$\begin{aligned} I_2 &= E\left[g(X(T))\exp\left\{-\int_0^T X(s)\,ds\right\}\int_0^T \xi(s)\,ds\right] \\ &= \int_0^T E\left[\exp\left\{-\int_0^s X(r)\,dr\right\}\xi(s)\right. \\ &\qquad\qquad \left.\times E\left[g(X(T))\exp\left\{-\int_s^T X(r)\,dr\right\}\middle|\,\mathcal{F}_s\right]\right]\,ds \\ &= \int_0^T E\left[\exp\left\{-\int_0^s X(r)\,dr\right\}\xi(s)u(X_s,s)\right]\,ds \end{aligned}$$

by the Markov property.

Define a new measure $Q = Q_s$ on $\mathcal{F}_s$ by

$$dQ = M(s)\, dP,$$

where M is defined as in (13). Girsanov's theorem yields

$$\begin{aligned}
& E\left[\exp\left\{-\int_0^s X(r)\, dr\right\}\xi(s)u(X_s, s)\right] \\
& \quad = \quad E^Q\left[\exp\left\{\int_0^s \beta_x(X(r), r) - X(r)\, dr\right\} u(X_s, s)\right] \\
& \quad = \quad E\left[\exp\left\{\int_0^s \beta_x(Y(r), r) - Y(r)\, dr\right\} u(Y_s, s)\right].
\end{aligned}$$

Consequently, $I_2 = J_2$, which finishes the proof in the case of continuously differentiable σ.

The general case follows by approximation as follows.
Let σ^n, u^n and v^n be as described before Proposition 10, with each σ^n being continuously differentiable in x with bounded derivative.
From above we then know that $v^n(x,t) = u^n_x(x,t)$ at all points.
Moreover, by Proposition 10, $v^n(x,t) \to v(x,t)$ pointwise as $n \to \infty$.
On the other hand, since u^n converges to u pointwise and is uniformly bounded, it follows from standard parabolic theory that also u^n_x converges to u_x pointwise for all points (x,t) with $x > 0$.
Consequently, $v = u_x$ on $(0,\infty) \times [0,T]$. Since v is continuous on $[0,\infty) \times [0,T]$ by Proposition 9, it is easy to check that $u_x(0,t)$ exists and that we have $v = u_x$ everywhere on $[0,\infty) \times [0,T]$. The continuity of u_x thus follows. □

An Estimate of the Second Spatial Derivative

Since the function v defined in (12) is continuous, it follows (by a similar argument as in the proof that u satisfies (4)) that it indeed solves the differentiated equation

$$v_t = \alpha v_{xx} + (\alpha_x + \beta)v_x + (\beta_x - x)v - u$$

on $(0, \infty) \times [0, T)$. In this section we use interior estimates to show that $\alpha v_x \to 0$ as $x \to 0$. Since $v = u_x$ by Theorem 11, this shows that the term αu_{xx} in (4) approaches zero close to the boundary.

Proposition

The function $v = u_x$ satisfies

$$\lim_{(x,t)\to(0,t_0)} \alpha(x,t) v_x(x,t) = 0$$

for any t_0. Consequently, $\lim_{(x,t)\to(0,t_0)} \alpha(x,t) u_{xx}(x,t) = 0$.

Proof

Let $\{(x_n, t_n)\}_{n=1}^{\infty} \subseteq (0, \infty) \times [0, T)$ be a sequence of points converging to $(0, t_0)$, where $t_0 \in [0, T)$. Define new coordinates (y, s) by letting $y = kx$ and $s = k(t - t_0)$, where k is specified more precisely below. Then the function w defined by

$$w(y, s) = v(x, t)$$

satisfies

$$w_s = \tilde{\alpha} w_{yy} + \tilde{\beta} w_y + \gamma w + h, \tag{14}$$

where

$$\tilde{\alpha}(y, s) = \alpha(\frac{y}{k}, t_0 + \frac{s}{k})k,$$

$$\tilde{\beta}(y, s) = (\alpha_x + \beta)(\frac{y}{k}, t_0 + \frac{s}{k}),$$

$$\gamma(y, s) = \frac{1}{k}\beta_x(\frac{y}{k}, t_0 + \frac{s}{k}) - \frac{y}{k^2},$$

and

$$h(y, s) = -\frac{1}{k}u(\frac{y}{k}, t_0 + \frac{s}{k}).$$

Now consider a region $\mathcal{R} = \mathcal{R}^n$ which contains the point (x_n, t_n), and such that

$$1 \leq \alpha(x,t)k \leq 2 \tag{15}$$

in $\mathcal{R}$. Since $\alpha_x(x,t)$ is continuous up to the boundary, the region $\mathcal{R}$ in (y,s)-coordinates does not collapse as $n \to \infty$, but it can rather be chosen to consist of a rectangle of fixed size.In this rectangle, the coefficients of the equation (14) satisfy

$$1 \leq \tilde{\alpha}(y,s) \leq 2,$$

$$|\tilde{\beta}(y,s)| \leq C,$$

$$|\gamma(y,s)| \leq C,$$

and

$$|h(y,s)| \leq C/k$$

for some constant C which is independent of n.

Since $w(y, s) = v(x, t)$ we have that w converges to the constant $v(0, t_0) = u_x(0, t_0)$ uniformly on $\mathcal{R}$ as $n \to \infty$. By interior Schauder estimates, w_y tends to 0 as $n \to \infty$. Since

$$\alpha(x, t) v_x(x, t) = \tilde{\alpha}(y, s) w_y(y, s),$$

and since $\tilde{\alpha}(y, s)$ is bounded on $\mathcal{R}$, the conclusion follows. $\square$

The Time Derivative at the Boundary

It follows from Proposition 12 and (4) that

$$\lim_{(x,t)\to(0,t_0)} u_t(x,t) + \beta(0,t_0)u_x(0,t_0) = 0 \tag{16}$$

for any $t_0 \in [0, T)$. In this section we show that the boundary condition (6) also holds *at* the boundary, i.e. not merely in the limit.

Proposition

The function $u_t(x,t) + \beta(x,t)u_x(x,t)$ defines a continuous function on $[0,\infty) \times [0,T)$. Moreover, it vanishes for $x = 0$.

Proof

In view of (16) above, it suffices to show that u_t exists at the boundary and that it equals $-\beta u_x$. To do this, fix a point on the boundary with coordinates $(0, t_0)$. For notational simplicity we assume that $t_0 = 0$.

The time (left) derivative u_t at the boundary is defined by

$$u_t(0,0) = \lim_{k\to\infty} k\left(u(0,0) - u(0,-\frac{1}{k})\right), \tag{17}$$

provided the limit exists. To determine $u_t(0,0)$, we let X^k be defined by

$$\begin{cases} dX^k = \beta(X^k(t),t)\,dt + \sigma(X^k(t),t)\,dW \\ X^k(-1/k) = 0. \end{cases}$$

However, instead of considering the process X with different starting times, we perform a change of variables so that the starting time is independent of k. We thus introduce the process $Y^k(s)$ by

$$Y^k(s) = kX^k(\frac{s}{k}).$$

With respect to the time variable s, the dynamics of Y^k has the form

$$\begin{cases} dY^k(s) = \beta(\frac{1}{k}Y^k(s), \frac{s}{k})\, ds + \sqrt{k\sigma^2(\frac{1}{k}Y^k(s), \frac{s}{k})}\, dW^k \\ Y^k(-1) = 0, \end{cases} \tag{18}$$

where $W^k(s)$ denotes some Brownian motion.

By the Markov property,

$$\begin{aligned} u(0,-1/k) &= E\left[e^{-\int_{-1/k}^{0} X^k(s)\,ds} u(X^k(0),0)\right] \\ &= E\left[e^{-\int_{-1}^{0} \frac{1}{k^2} Y^k(s)\,ds} u(\frac{1}{k} Y^k(0),0)\right]. \end{aligned}$$

Hence,

$$\begin{aligned} u_t(0,0) &= \lim_{k\to\infty} k E_{0,-1}\left[u(0,0) - e^{-\int_{-1}^{0} \frac{1}{k^2} Y^k(s)\,ds} u(\frac{1}{k} Y^k(0),0)\right] \\ &= \lim_{k\to\infty} E_{0,-1}\left[k\left(u(0,0) - u(\frac{1}{k} Y^k(0),0)\right)\right], \end{aligned}$$

where the second equality follows using the inequality
$e^{-x} - 1 \geq -x$ since

$$\begin{aligned} &E_{0,-1}\left[k u(\frac{1}{k} Y^k(0),0)\left|e^{-\int_{-1}^{0} \frac{1}{k^2} Y^k(s)\,ds} - 1\right|\right] \\ &\leq C\frac{1}{k} E_{0,-1}\left[\int_{-1}^{0} Y^k(s)\,ds\right] \to 0 \end{aligned}$$

as $k \to \infty$.

Now, define the process Y by

$$\begin{cases} dY = \beta(0,0)\,ds + \sqrt{2\alpha_x(0,0)Y}\,dW \\ Y(-1) = 0, \end{cases}$$

and redefine Y^k as in (18) above but using the same Brownian motion W (this does not change the law of Y^k). Since

$$\beta^k(y,s) := \beta(\frac{y}{k}, \frac{s}{k}) \to \beta(0,0)$$

and

$$\sigma^k(y,s) := \sqrt{k\sigma^2(\frac{y}{k}, \frac{s}{k})} \to \sqrt{2\alpha_x(0,0)y}$$

uniformly on compacts as $k \to \infty$ (here we used the assumption that α is continuously differentiable in space), it follows from [1] that $Y^k(0) \to Y(0)$ in L^2 as $k \to \infty$.

From Theorem 11 above we know that u is differentiable in x, so $k(u(0,0) - u(\frac{y}{k},0))$ converges to $-u_x(0,0)y$. By dominated convergence, we have

$$kE\left[u(0,0) - u(\frac{1}{k}Y(0),0)\right] \to -u_x(0,0)E[Y(0)] = -\beta(0,0)u_x(0,0)$$

as $k \to \infty$. Moreover, the Lipschitz property of u yields that

$$kE\left[u(\frac{1}{k}Y(0),0) - u(\frac{1}{k}Y^k(0),0)\right] \le CE\left[|Y(0) - Y^k(0)|\right] \to 0$$

as $k \to \infty$. It follows that

$$u_t(0,0) + u_x(0,0)\beta(0,0) = 0.$$

As $t_0 = 0$ was chosen only for notational convenience, we have that

$$u_t(0,t) + \beta(0,t)u_x(0,t) = 0$$

for any t. To be precise we have shown the result above only for the left t-derivative. However, this left t-derivative is continuous by the equation above, so it follows from a simple calculus lemma that in fact u is differentiable in time, thus finishing our proof. □

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