# ON LIEB-THIRRING INEQUALITIES FOR SCHRÖDINGER OPERATORS WITH VIRTUAL LEVEL

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ABSTRACT. We consider the operator $H = -\Delta - V$ in $L_2(\mathbb{R}^d)$, $d \geq 3$. For the moments of its negative eigenvalues we prove the estimate

$$\operatorname{tr} H_-^\gamma \leq C_{\gamma,d} \int_{\mathbb{R}^d} \left( V(x) - \frac{(d-2)^2}{4|x|^2} \right)_+^{\gamma + \frac{d}{2}} dx, \qquad \gamma > 0.$$

Similar estimates hold for the one-dimensional operator with a Dirichlet condition at the origin and for the two-dimensional Aharonov-Bohm operator.

## INTRODUCTION

The Lieb-Thirring inequalities estimate a quantum mechanical quantity, namely moments of negative eigenvalues of the Schrödinger operator $-\Delta - V$ in $L_2(\mathbb{R}^d)$, by means of the classical phase space volume. They state for suitable values of $\gamma$ and $d$ (see [LiTh] and, for more recent results, the survey [LaWei2]) that

$$\operatorname{tr}(-\Delta - V)_-^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_+(x)^{\gamma + \frac{d}{2}} dx. \tag{0.1}$$

Lately the main topic in connection with these inequalities has been to establish their sharp constants $L_{\gamma,d}$.

We are interested in a different question. As is well-known, in dimension $d \geq 3$ a sufficiently weak potential cannot bind a particle. Put differently, if $V \in C_0^\infty(\mathbb{R}^d)$ then $-\Delta - \beta V$ is non-negative for small $\beta > 0$. This follows, *e.g.*, from the Hardy inequality

$$\frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} dx \leq \int_{\mathbb{R}^d} |\nabla u|^2 dx, \qquad u \in C_0^\infty(\mathbb{R}^d). \tag{0.2}$$

We see that the Lieb-Thirring estimate does not yield a good bound for weak potentials. In the particular case $V(x) = \frac{(d-2)^2}{4|x|^2}$ the l.h.s. of (0.1) is zero whereas the r.h.s. is infinite!

In this paper we show the rather unexpected result that the part of the potential which is stronger than the Hardy weight is sufficient to estimate the moments of the negative eigenvalues of $-\Delta - V$. More precisely, we

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prove the inequality

$$\operatorname{tr}(-\Delta - V)_-^\gamma \leq C_{\gamma,d} \int_{\mathbb{R}^d} \left( V(x) - \frac{(d-2)^2}{4|x|^2} \right)_+^{\gamma+\frac{d}{2}} dx \tag{0.3}$$

for any $d \geq 3$ and $\gamma > 0$. Note that a direct approach based only on (0.1) and (0.2) leads to $-\Delta - V \geq -\varepsilon\,\Delta + (1-\varepsilon)\frac{(d-2)^2}{4|x|^2} - V$ for $\varepsilon \in (0,1)$ and hence

$$\operatorname{tr}(-\Delta - V)_-^\gamma \leq L_{\gamma,d}\,\varepsilon^{-\frac{d}{2}} \int_{\mathbb{R}^d} \left( V(x) - (1-\varepsilon)\frac{(d-2)^2}{4|x|^2} \right)_+^{\gamma+\frac{d}{2}} dx.$$

However, as $\varepsilon$ tends to zero the constant in front of the integral diverges. For a deeper analysis in the case of *positive* $\varepsilon$ we refer to [St].

In order to prove (0.3) we will choose a slightly different (but equivalent) point of view and establish Lieb-Thirring inequalities for the operator

$$-\Delta - \frac{(d-2)^2}{4|x|^2} - \beta V \qquad \text{in } L_2(\mathbb{R}^d),\ d \geq 3, \tag{0.4}$$

see Theorem 1.1. We note that several works have been devoted to the investigation of this operator. In [Bi] sufficient conditions for the finiteness of the negative spectrum were given. In [BiLa] the asymptotic number of negative eigenvalues in the strong coupling regime $\beta \to \infty$ was investigated and, in particular, it was shown that the Weyl-type formula may be violated. Indeed, an additional contribution of a one-dimensional auxiliary problem appears. We emphasize that such a term does *not* appear in our Lieb-Thirring estimates, which demonstrates the 'smoothing effect' of taking $\gamma > 0$. In [Wei] the weak coupling regime $\beta \to 0$ is investigated and a necessary and sufficient condition on $V$ is given for the operator (0.4) to have a negative eigenvalue for any $\beta > 0$. This is in particular the case if $V \geq 0$ and stands in sharp contrast to the operator $-\Delta - \beta V$ if $d \geq 3$. This is what we mean by a *virtual level*.

It will turn out that the operator (0.4) has both two- and $d$-dimensional features and the main difficulty in establishing a Lieb-Thirring inequality is to estimate the former. Here we rely on weighted Lieb-Thirring inequalities by Egorov-Kondrat'ev [EgKo].

The same approach allows us to obtain Lieb-Thirring inequalities for the one-dimensional analogue of (0.4) with a Dirichlet boundary condition at the origin (see Theorem 1.6) and for the two-dimensional magnetic Schrödinger operator corresponding to the Aharonov-Bohm field when the critical Hardy weight is subtracted (see Theorem 1.10). Our estimates allow also the inclusion of a weight in the spirit of [GlGrMaTh], [BlReSt] and [EgKo].

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## 1. Statement of the results

1.1. **Schrödinger operators in $d \geq 3$.** The main result of this paper is

**Theorem 1.1.** *Let $d \geq 3$, $\gamma > 0$ and $\alpha \geq 0$. Then*

$$\operatorname{tr}\left(-\Delta - \frac{(d-2)^2}{4|x|^2} - V\right)_-^\gamma \leq C_{\gamma,d,\alpha} \int_{\mathbb{R}^d} V(x)_+^{\gamma+\frac{d+\alpha}{2}} |x|^\alpha \, dx \tag{1.1}$$

*with a constant $C_{\gamma,d,\alpha}$ independent of $V$.*

To be more precise, we prove that if $V \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ and if the r.h.s. of (1.1) is finite then the quadratic form

$$\int_{\mathbb{R}^d} \left(|\nabla u|^2 - \frac{(d-2)^2}{4|x|^2}|u|^2 - V|u|^2\right) dx \tag{1.2}$$

is lower semi-bounded and closable on $C_0^\infty(\mathbb{R}^d)$ and the estimate (1.1) holds for the operator associated with the closure of this form.

Specializing to the case $\alpha = 0$ we obtain for the standard Schrödinger operator

**Corollary 1.2.** *Let $d \geq 3$ and $\gamma > 0$. Then*

$$\operatorname{tr}(-\Delta - V)_-^\gamma \leq \; C_{\gamma,d,0} \int_{\mathbb{R}^d} \left(V(x) - \frac{(d-2)^2}{4|x|^2}\right)_+^{\gamma+\frac{d}{2}} dx \tag{1.3}$$

*with the constant $C_{\gamma,d,0}$ from* (1.1).

*Remark* 1.3. In particular, if we replace $V$ by $\beta V$, where $V$ is bounded and compactly supported, then the r.h.s. of (1.3) is zero for sufficiently small $\beta > 0$. As explained in the introduction, this is an important feature of (1.3) which is not shared by the classical estimate (0.1).

*Remark* 1.4. Neither Theorem 1.1 nor Corollary 1.2 hold for $\gamma = 0$. This follows from the fact that if $V \geq 0$, $V \not\equiv 0$, then the operator $-\Delta - \frac{(d-2)^2}{4|x|^2} - \beta V$ has a negative eigenvalue for all $\beta > 0$, see Remark 8.2 in [Wei] or, for the case of a spherically symmetric $V$, our Proposition 3.2 below.

*Remark* 1.5. Our constants are explicit and given in (2.5). However, they might be strongly overblown. It would be challenging to find their sharp values.

1.2. **Schrödinger operators on the semi-axis.** Our result has a one-dimensional analogue, which is an important ingredient in the proof of Theorem 1.1, but also of independent interest. We consider the operator $-\frac{d^2}{dr^2} - \frac{1}{4r^2} - V$ in $L_2(\mathbb{R}_+)$ with Dirichlet boundary conditions at the origin and prove

**Theorem 1.6.** *Let $\gamma > 0$ and $\alpha \geq 0$ such that $\gamma + \frac{1+\alpha}{2} > 1$. Then*

$$\operatorname{tr}\left(-\frac{d^2}{dr^2} - \frac{1}{4r^2} - V\right)_-^\gamma \leq C_{\gamma,1,\alpha} \int_{\mathbb{R}_+} V(r)_+^{\gamma+\frac{1+\alpha}{2}} r^\alpha \, dr$$

*with a constant $C_{\gamma,1,\alpha}$ independent of $V$.*

Similarly as before, the precise statement involves the Friedrichs extension of the form

$$\int_{\mathbb{R}_+} \left( |f'|^2 - \frac{1}{4r^2}|f|^2 - V|f|^2 \right) dr \tag{1.4}$$

on $C_0^\infty(\mathbb{R}_+)$. We stress that $\mathbb{R}_+ = (0,\infty)$. Note also that $\frac{1}{4r^2}$ is the critical Hardy weight if $d = 1$, see (3.1) below.

We will prove this theorem in two steps. The case $\alpha \geq 1$ is dealt with in Section 2 using results of [EgKo] and the case $0 \leq \alpha \leq 1$ in Section 3 using explicit diagonalization of the operator $-\frac{d^2}{dr^2} - \frac{1}{4r^2}$.

*Remark* 1.7. If $\alpha = 0$ and $\gamma > \frac{1}{2}$ this leads to a Lieb-Thirring-type inequality in the spirit of Corollary 1.2. It would be interesting to extend the estimate to the critical case $\gamma = \frac{1}{2}$.

*Remark* 1.8. If $\alpha \geq 1$ we can take any $\gamma > 0$. However, a similar estimate for $\gamma = 0$ cannot hold due to the virtual level of $-\frac{d^2}{dr^2} - \frac{1}{4r^2}$, see Proposition 3.2 below.

*Remark* 1.9. If $\alpha = 1$ this is a Bargmann-type inequality. Recall that

$$\operatorname{tr} \left( -\frac{d^2}{dr^2} + \frac{s(s+1)}{r^2} - V \right)_-^\gamma \leq \frac{1}{(1+2s)(\gamma+1)} \int_{\mathbb{R}_+} V(r)_+^{\gamma+1} r\, dr \tag{1.5}$$

for $s > -\frac{1}{2}$ and $\gamma \geq 0$. Indeed, the proof of the Bargmann inequality in [Si2] (Theorem 7.3) applies for any $s > -\frac{1}{2}$ and $\gamma = 0$. By the argument of Aizenman-Lieb [AiLi] one extends this inequality to $\gamma \geq 0$.

When $s \to -\frac{1}{2}$ the constant in (1.5) diverges to infinity. Our Theorem 1.6 with $\alpha = 1$ yields (1.5) for $s = -\frac{1}{2}$ and $\gamma > 0$ with a finite constant. Indeed, the constant can be chosen as $2\pi L_{\gamma,2}$ with $L_{\gamma,2}$ from (0.1), see Remark 2.3.

1.3. **Aharonov-Bohm operators in $d = 2$.** If $d = 2$ then the Hardy inequality (0.2) becomes trivial. In this case it is interesting to consider the magnetic Schrödinger operator $(-\mathrm{i}\nabla - \phi\mathbf{A})^2$, where $\phi \in \mathbb{R}$ and $\mathbf{A}$ is the Aharonov-Bohm magnetic vector potential,

$$\mathbf{A}(x) = |x|^{-2}(-x_2, x_1), \qquad x \in \mathbb{R}^2 \setminus \{0\}.$$

As usual, $(-\mathrm{i}\nabla - \phi\mathbf{A})^2$ is defined as Friedrichs extensions of the corresponding differential operator on $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. By gauge invariance we can assume that $-\frac{1}{2} < \phi \leq \frac{1}{2}$. Recall (see [LaWei1]) the Hardy-type inequality

$$\phi^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\, dx \leq \int_{\mathbb{R}^2} |(-\mathrm{i}\nabla - \phi\mathbf{A})u|^2\, dx, \qquad u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}).$$

Here the constant $\phi^2$ is sharp. Our result is

**Theorem 1.10.** *Let $\gamma > 0$, $\alpha \geq 0$ and $-\frac{1}{2} < \phi \leq \frac{1}{2}$. Then*

$$\operatorname{tr}\left((-\mathrm{i}\nabla - \phi\mathbf{A})^2 - \frac{\phi^2}{|x|^2} - V\right)_-^\gamma \leq C_{\gamma,2,\alpha} \int_{\mathbb{R}^2} V(x)_+^{\gamma+\frac{2+\alpha}{2}} |x|^\alpha \, dx$$

*with a constant $C_{\gamma,2,\alpha}$ independent of $V$ and $\phi$.*

The proof is found in Section 4.

## 2. Proof of Theorem 1.1

In this section we assume, unless stated otherwise, that $d \geq 3$ and write

$$H_0 = -\Delta - \frac{(d-2)^2}{4|x|^2}.$$

The main idea in the proof of Theorem 1.1 is to consider $H_0 - V$ separately on the space of spherically symmetric functions and on its orthogonal complement. An essential ingredient in our study of the operator on the former space will be the following result from [EgKo].

**Proposition 2.1.** *Let $d \geq 2$, $\gamma > 0$ and $\alpha \geq 0$. Then*

$$\operatorname{tr}(-\Delta - V)_-^\gamma \leq C_{\gamma,d,\alpha}^{EK} \int_{\mathbb{R}^d} V(x)_+^{\gamma+\frac{d+\alpha}{2}} |x|^\alpha \, dx$$

*with a constant $C_{\gamma,d,\alpha}^{EK}$ independent of $V$.*

For the convenience of the reader we will give a proof of this proposition in the appendix.

We remark that for $\alpha = 0$ this coincides with the classical Lieb-Thirring inequality (0.1). The inclusion of the weight $|x|^\alpha$ increases the power of $V$ by $\frac{\alpha}{2}$ as compared to (0.1). That this is necessary can easily be seen by scaling of the space variables. We note that the result holds also if $d \geq 3$ and $\gamma = 0$ and if $d = 1$ and $\gamma > \frac{1+\alpha}{2}$.

From Proposition 2.1 we deduce now the first part of Theorem 1.6. Recall that we consider the operator $-\frac{d^2}{dr^2} - \frac{1}{4r^2} - V$ in $L_2(\mathbb{R}_+)$ with Dirichlet boundary condition.

**Corollary 2.2.** *Let $\gamma > 0$ and $\alpha \geq 1$. Then*

$$\operatorname{tr}\left(-\frac{d^2}{dr^2} - \frac{1}{4r^2} - V\right)_-^\gamma \leq C_{\gamma,1,\alpha} \int_{\mathbb{R}_+} V(r)_+^{\gamma+\frac{1+\alpha}{2}} r^\alpha \, dr$$

*with a constant $C_{\gamma,1,\alpha}$ independent of $V$.*

*Proof.* The operator $-\Delta - V(|\cdot|)$ in $L_2(\mathbb{R}^2)$ is unitarily equivalent to the direct sum $\oplus_{n\in\mathbb{Z}}(h_n - V)$ in $\oplus_{n\in\mathbb{Z}} L_2(\mathbb{R}_+)$, where we define

$$h_n - V := -\frac{d^2}{dr^2} - \frac{1}{4r^2} + \frac{n^2}{r^2} - V$$

as quadratic form on $C_0^\infty(\mathbb{R}_+)$. (Here we used that $C_0^\infty(\mathbb{R}^2\setminus\{0\})$ is a form core for $-\Delta - V$.) Hence Proposition 2.1 yields

$$\begin{aligned}\operatorname{tr}_{L_2(\mathbb{R}_+)}\left(-\frac{d^2}{dr^2}-\frac{1}{4r^2}-V\right)_-^\gamma &\le \operatorname{tr}_{L_2(\mathbb{R}^2)}\left(-\Delta - V(|\cdot|)\right)_-^\gamma \\ &\le C^{EK}_{\gamma,2,\alpha-1}\int_{\mathbb{R}^2} V(|x|)_+^{\gamma+\frac{1+\alpha}{2}}|x|^{\alpha-1}\,dx \\ &= 2\pi C^{EK}_{\gamma,2,\alpha-1}\int_{\mathbb{R}_+} V(r)_+^{\gamma+\frac{1+\alpha}{2}} r^\alpha\,dr\end{aligned}$$

as claimed. ☐

*Remark* 2.3. In the case $\alpha = 1$ we can apply the Lieb-Thirring inequality (0.1) instead of Proposition 2.1. This shows that the sharp value of the constant $C_{\gamma,1,1}$ is bounded from above by $2\pi L_{\gamma,2}$ with $L_{\gamma,2}$ from (0.1).

Corollary 2.2 will allow us to treat the part of $H_0 - V$ on spherically symmetric functions. On the orthogonal complement of that space one has an improved Hardy inequality.

**Lemma 2.4.** *Let $d \ge 2$. Then*

$$\frac{d^2}{4}\int_{\mathbb{R}^d}\frac{|u|^2}{|x|^2}\,dx \le \int_{\mathbb{R}^d}|\nabla u|^2\,dx$$

*for all $u \in C_0^\infty(\mathbb{R}^d)$ satisfying $\int_{\mathbb{S}^{d-1}} u(r\omega)\,d\omega = 0$ for all $r > 0$.*

This inequality appears, *e.g.*, in [BiLa]. We sketch the simple proof.

*Proof.* We substitute $u = |x|^{(2-d)/2}v$ and obtain

(2.1)

$$\begin{aligned}\int_{\mathbb{R}^d}\left(|\nabla u|^2 - \frac{(d-2)^2}{4|x|^2}|u|^2\right)dx &= \int_{\mathbb{R}^d}|\nabla v|^2|x|^{2-d}\,dx \\ &= \int_0^\infty\int_{\mathbb{S}^{d-1}}\left(\left|\frac{\partial v}{\partial r}\right|^2 + \frac{|\nabla_\theta v|^2}{r^2}\right)d\theta\, r\,dr \\ &\ge \int_0^\infty\int_{\mathbb{S}^{d-1}} r^{-1}|\nabla_\theta v|^2\,d\theta\,dr.\end{aligned}$$

For fixed $r$ the function $v(r\cdot)$ is orthogonal to constants, *i.e.*, to the first eigenfunction of the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. Since the next eigenvalue is $d-1$ we find

$$\int_{\mathbb{S}^{d-1}}|\nabla_\theta v(r\theta)|^2\,d\theta \ge (d-1)\int_{\mathbb{S}^{d-1}}|v(r\theta)|^2\,d\theta.$$

Multiplying by $r^{-1}$ and integrating yields

$$\int_0^\infty\int_{\mathbb{S}^{d-1}} r^{-1}|\nabla_\theta v(r\theta)|^2\,d\theta\,dr \ge (d-1)\int_{\mathbb{R}^d}|x|^{-2}|u|^2\,dx.$$

Combining this with (2.1) we obtain the result. ☐

Now we are in position to give the

*Proof of Theorem 1.1.* By the variational principle it suffices to prove the result for $V \geq 0$. Moreover, we will assume that the r.h.s. of (1.1) is finite and that the quadratic form (1.2) is lower semi-bounded on $C_0^\infty(\mathbb{R}^d)$. (Note that these assumptions are satisfied, *e.g.*, if $V$ is bounded and has compact support.) Since the form (1.2) is closable we can define $H_0 - V$ as the operator associated with this form. At the end of the proof we use a standard approximation argument to show that the finiteness of the r.h.s. of (1.1) implies the lower semi-boundedness.

We denote by $P$ the projection onto spherically symmetric functions,

$$(Pu)(x) := |\mathbb{S}^{d-1}|^{-1} \int_{\mathbb{S}^{d-1}} u(|x|\,\omega)\, d\omega, \qquad x \in \mathbb{R}^d,$$

and put $Q := I - P$. Note that $P$ and $Q$ commute with $H_0$. Moreover, for $u \in C_0^\infty(\mathbb{R}^d)$ one has

$$2\mathrm{Re}\,(PVQu, u) \leq 2\|V^{1/2}Qu\| \cdot \|V^{1/2}Pu\| \leq (PVPu, u) + (QVQu, u),$$

which implies the operator inequality

$$PVQ + QVP \leq PVP + QVQ.$$

It follows that

$$\begin{aligned} H_0 - V &= P\,(H_0 - V)\,P + Q\,(H_0 - V)\,Q - PVQ - QVP \\ &\geq P\,(H_0 - 2V)\,P + Q\,(H_0 - 2V)\,Q, \end{aligned}$$

and hence

$$\mathrm{tr}(H_0 - V)_-^\gamma \leq \mathrm{tr}\,(P\,(H_0 - 2V)\,P)_-^\gamma + \mathrm{tr}\,(Q\,(H_0 - 2V)\,Q)_-^\gamma\,. \tag{2.2}$$

We consider the two terms separately and begin with the second one. By Lemma 2.4 we find for all $0 < \rho \leq 1$ that

$$Q\,(H_0 - 2V)\,Q \geq \rho Q\left(-\Delta - \rho^{-1}2V\right)Q + \tfrac{1}{4}\left((1-\rho)d^2 - (d-2)^2\right)Q|x|^{-2}Q.$$

We choose $\rho$ such that $(1-\rho)d^2 = (d-2)^2$ and obtain from Proposition 2.1 (or (0.1) if $\alpha = 0$) that

$$\begin{aligned} \mathrm{tr}\,(Q\,(H_0 - 2V)\,Q)_-^\gamma &\leq \rho^\gamma\, \mathrm{tr}\left(Q\left(-\Delta - \rho^{-1}2V\right)Q\right)_-^\gamma \\ &\leq \rho^\gamma\, \mathrm{tr}\left(-\Delta - \rho^{-1}2V\right)_-^\gamma \\ &\leq \rho^{-\frac{d+\alpha}{2}} 2^{\gamma+\frac{d+\alpha}{2}} C_{\gamma,d,\alpha}^{EK} \int_{\mathbb{R}^d} V(x)^{\gamma+\frac{d+\alpha}{2}} |x|^\alpha\, dx. \end{aligned} \tag{2.3}$$

Now we turn to the term $\mathrm{tr}\,(P\,(H_0 - 2V)\,P)_-^\gamma$. We define the spherical average of $V$ by

$$\tilde{V}(r) := |\mathbb{S}^{d-1}|^{-1} \int_{\mathbb{S}^{d-1}} V(r\omega)\, d\omega, \qquad r \in \mathbb{R}_+,$$

and note that (the non-trivial part of) $P\,(H_0 - 2V)\,P$ is unitarily equivalent to the operator $-\frac{d^2}{dr^2} - \frac{1}{4r^2} - 2\tilde{V}$ in $L_2(\mathbb{R}_+)$. By Corollary 2.2 (with $\alpha$ replaced by $\alpha + d - 1$) we obtain

$$\operatorname{tr}\,(P\,(H_0 - 2V)\,P)_-^\gamma \le 2^{\gamma + \frac{d+\alpha}{2}} C_{\gamma,1,\alpha+d-1} \int_{\mathbb{R}_+} \tilde{V}(r)^{\gamma+\frac{d+\alpha}{2}} r^{\alpha+d-1}\,dr$$

Now Hölder's (or Jensen's) inequality implies that

$$\tilde{V}(r)^{\gamma+\frac{d+\alpha}{2}} \le |\mathbb{S}^{d-1}|^{-1} \int_{\mathbb{S}^{d-1}} V(r\omega)^{\gamma+\frac{d+\alpha}{2}}\,d\omega, \qquad r \in \mathbb{R}_+,$$

and hence

$$\begin{aligned}
&\operatorname{tr}\,(P\,(H_0 - 2V)\,P)_-^\gamma \\
&\quad \le 2^{\gamma+\frac{d+\alpha}{2}} |\mathbb{S}^{d-1}|^{-1} C_{\gamma,1,\alpha+d-1} \int_{\mathbb{R}_+}\int_{\mathbb{S}^{d-1}} V(r\omega)^{\gamma+\frac{d+\alpha}{2}} r^{\alpha+d-1}\,d\omega\,dr \\
&\quad = 2^{\gamma+\frac{d+\alpha}{2}} |\mathbb{S}^{d-1}|^{-1} C_{\gamma,1,\alpha+d-1} \int_{\mathbb{R}^d} V(x)^{\gamma+\frac{d+\alpha}{2}} |x|^\alpha\,dx.
\end{aligned} \tag{2.4}$$

Adding (2.3) and (2.4) we obtain the assertion in view of (2.2) with a constant satisfying

$$C_{\gamma,d,\alpha} \le 2^{\gamma+\frac{d+\alpha}{2}} \left( \rho^{-\frac{d+\alpha}{2}} C^{EK}_{\gamma,d,\alpha} + |\mathbb{S}^{d-1}|^{-1} C_{\gamma,1,\alpha+d-1} \right). \tag{2.5}$$

To complete the proof it remains that show that if $0 \le V \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$ is such that the r.h.s. of (1.1) is finite then the quadratic form (1.2) is lower semi-bounded on $C_0^\infty(\mathbb{R}^d)$. To see this, choose bounded, compactly supported functions $0 \le V_n \le V$ such that $V_n \to V$ a.e. and

$$\int_{\mathbb{R}^d} (V(x) - V_n(x))^{\gamma+\frac{d+\alpha}{2}} |x|^\alpha\,dx \to 0. \tag{2.6}$$

The operators $H_0 - V_n$ are well-defined and (1.1) holds for them. In particular, $\lambda(V_n) := \inf \sigma\,(H_0 - V_n)$ satisfies

$$\lambda(V_n) \ge -C_{\gamma,d,\alpha}^{\frac{1}{\gamma}} \left( \int_{\mathbb{R}^d} V_n(x)^{\gamma+\frac{d+\alpha}{2}} |x|^\alpha\,dx \right)^{\frac{1}{\gamma}}.$$

Hence for any $u \in C_0^\infty(\mathbb{R}^d)$ one has

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left( |\nabla u|^2 - \frac{(d-2)^2}{4|x|^2}|u|^2 - V_n|u|^2 \right) dx \\
&\qquad \ge -C_{\gamma,d,\alpha}^{\frac{1}{\gamma}} \|u\|^2 \left( \int_{\mathbb{R}^d} V_n(x)^{\gamma+\frac{d+\alpha}{2}} |x|^\alpha\,dx \right)^{\frac{1}{\gamma}}.
\end{aligned}$$

Using dominated convergence and (2.6) we can pass to the limit $n \to \infty$ and find that also the form (1.2) is bounded from below on $C_0^\infty(\mathbb{R}^d)$. This completes the proof. □

*Proof of Corollary 1.2.* Assume that the r.h.s. of (1.3) is finite. Then according to our comments after Theorem 1.1 the form

$$\int_{\mathbb{R}^d} \left( |\nabla u|^2 - \tilde{V}|u|^2 \right) dx, \tag{2.7}$$

where

$$\tilde{V}(x) := \frac{(d-2)^2}{4|x|^2} + \left( V(x) - \frac{(d-2)^2}{4|x|^2} \right)_+ ,$$

is lower semi-bounded on $C_0^\infty(\mathbb{R}^d)$ and we denote by $-\Delta - \tilde{V}$ the operator associated with its closure. Since $V \le \tilde{V}$ the form (2.7), with $\tilde{V}$ replaced by $V$, is also lower semi-bounded on $C_0^\infty(\mathbb{R}^d)$ and the associated operator satisfies $-\Delta - V \ge -\Delta - \tilde{V}$. Now Corollary 1.2 follows from Theorem 1.1 with $\alpha = 0$ by the variational principle. □

## 3. Proof of Theorem 1.6

Recall the one-dimensional Hardy inequality

$$\int_{\mathbb{R}_+} \frac{|f(r)|^2}{r^2}\, dr \le 4 \int_{\mathbb{R}_+} |f'(r)|^2\, dr, \qquad f \in C_0^\infty(\mathbb{R}_+). \tag{3.1}$$

It allows to define the non-negative operator

$$h_0 = -\frac{d^2}{dr^2} - \frac{1}{4r^2} \qquad \text{in } L_2(\mathbb{R}_+) \tag{3.2}$$

as the Friedrichs extension of the quadratic form (1.4) with $V \equiv 0$ on $C_0^\infty(\mathbb{R}_+)$. This operator can be diagonalized explicitly. Indeed, let $J_0$ be the first Bessel function of order zero (see [AbSt]). Then

$$(\mathcal{F}_0 f)(k) := \int_0^\infty \sqrt{kr}\, J_0(kr) f(r)\, dr, \qquad k \in \mathbb{R}_+,$$

initially defined for $f \in C_0^\infty(\mathbb{R}_+)$, can be extended to a unitary operator $\mathcal{F}_0 : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$. It has the property

$$(\mathcal{F}_0 h_0 f)(k) = k^2 (\mathcal{F}_0 f)(k), \qquad k \in \mathbb{R}_+, \tag{3.3}$$

for all $f \in \mathcal{D}(h_0)$. (These facts are essentially contained in Chapter 4 of [StWei].)

We denote by $N(\tau, h_0 - V)$ the number of eigenvalues less than $-\tau$, counting multiplicities, of the operator $h_0 - V$ in $L_2(\mathbb{R}_+)$. Our proof of Theorem 1.6 relies on the following

**Lemma 3.1.** *Let $q > 1$, $0 \le \alpha \le 1$ such that $2q - \alpha > 1$. Then*

$$N(\tau, h_0 - V) \le C_{\alpha,q} \tau^{-q + \frac{1+\alpha}{2}} \int_{\mathbb{R}_+} V(r)_+^q r^\alpha\, dr, \qquad \tau > 0, \tag{3.4}$$

*with a constant $C_{\alpha,q}$ independent of $V$.*

What we precisely prove is that if $V \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ and if the r.h.s. of (3.4) is finite, then the form (1.4) is closed and lower semi-bounded on $\mathcal{D}\big(h_0^{1/2}\big)$ and for the corresponding self-adjoint operator $h_0 - V$ the estimate (3.4) holds.

Before we begin the proof we recall (see [BiSo1], [Si2]) that $\mathfrak{S}_p$ denotes the class of compact operators $K$ (in a given Hilbert space, in our case in $L_2(\mathbb{R}_+)$) such that

$$\|K\|_p := \Big(\mathrm{tr}(K^*K)^{\frac{p}{2}}\Big)^{\frac{1}{p}} < \infty.$$

We will use the following fact (see [LiTh]). If $q \geq 1$ and $A$, $B$ are self-adjoint, non-negative operators such that $A^qB^q \in \mathfrak{S}_2$ then $AB \in \mathfrak{S}_{2q}$ and

$$\|AB\|_{2q}^{2q} \leq \|A^qB^q\|_2^2. \tag{3.5}$$

*Proof of Lemma 3.1.* Scaling with respect to the space variables shows that it is enough to consider the case $\tau = 1$. Moreover, by the variational principle we may assume $V \geq 0$. By the Birman-Schwinger principle and the inequality (3.5) we obtain

$$N(1, h_0 - V) \leq \big\|V^{1/2}(h_0 + I)^{-1/2}\big\|_{2q}^{2q} \leq \big\|V^{q/2}(h_0 + I)^{-q/2}\big\|_2^2.$$

It follows from this estimate that we can restrict ourselves to, say, bounded and compactly supported $V$. The general result as well as the comment we made after the lemma are derived then in a standard way.

It follows from (3.3) that the operator $V^{q/2}(h_0+I)^{-q/2}\mathcal{F}_0^*$ has the integral kernel

$$V(r)^{\frac{q}{2}}\sqrt{rk}\,J_0(rk)(k^2+1)^{-\frac{q}{2}}, \qquad r, k \in \mathbb{R}_+,$$

and therefore

$$\big\|V^{q/2}(h_0 + I)^{-q/2}\big\|_2^2 = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} rkJ_0^2(rk)(k^2+1)^{-q}V(r)^q\,dk\,dr.$$

Recall (see [AbSt]) that $J_0$ is a continuous function with $J_0(0) = 1$ and

$$J_0(x) \sim \sqrt{\frac{\pi}{2}}\,\frac{\cos(x - \pi/4)}{\sqrt{x}} \quad \text{as } x \to \infty.$$

Hence with $c_\alpha := \sup_{x>0} x^{(1-\alpha)/2}J_0(x) < \infty$ we can estimate

$$N(1, h_0 - V) \leq c_\alpha^2 \int_{\mathbb{R}_+}\int_{\mathbb{R}_+}(rk)^\alpha(k^2+1)^{-q}V(r)^q\,dk\,dr = C_{\alpha,q}\int_{\mathbb{R}_+}V(r)^q r^\alpha\,dr.$$

Here $C_{\alpha,q} := c_\alpha^2 \int_{\mathbb{R}_+} k^\alpha(1+k^2)^{-q}\,dk$ is finite in view of our assumptions. □

Given Lemma 3.1 we obtain in a standard manner the

*Proof of Theorem 1.6.* The case $\alpha \geq 1$ was already proven in Corollary 2.2. We assume now $0 \leq \alpha \leq 1$. The operator inequality

$$h_0 - V + t \geq h_0 - \left(V - \frac{t}{2}\right)_+ + \frac{t}{2}$$

implies $N(t,h_0-V)\le N\left(\frac{t}{2},h_0-(V-\frac{t}{2})_+\right)$ and hence

$$\begin{aligned}\operatorname{tr}(h_0-V)_-^\gamma &= \gamma\int_{\mathbb{R}_+} N(t,h_0-V)t^{\gamma-1}\,dt\\ &\le \gamma\int_{\mathbb{R}_+} N\left(\tfrac{t}{2},h_0-(V-\tfrac{t}{2})_+\right)t^{\gamma-1}\,dt\\ &= \gamma 2^\gamma\int_{\mathbb{R}_+} N(\tau,h_0-(V-\tau)_+)\tau^{\gamma-1}\,d\tau.\end{aligned}$$

Now we fix $1<q<\gamma+\frac{1+\alpha}{2}$ and apply Lemma 3.1 to obtain

$$\begin{aligned}\operatorname{tr}(h_0-V)_-^\gamma &\le \gamma\, 2^\gamma\, C_{\alpha,q}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}(V(r)-\tau)_+^q\tau^{\gamma-q+\frac{\alpha-1}{2}}\,d\tau\, r^\alpha\,dr\\ &= \gamma\, 2^\gamma\, C_{\alpha,q}B(\gamma+\tfrac{1+\alpha}{2}-q,q+1)\int_{\mathbb{R}_+}V(r)_+^{\gamma+\frac{1+\alpha}{2}}r^\alpha\,dr\end{aligned}$$

where $B(a,b)=\int_0^1 s^{a-1}(1-s)^{b-1}\,ds$ is the beta function. Finally, one may optimize over all $1<q<\gamma+\frac{1+\alpha}{2}$. This establishes the theorem. □

In Theorem 1.6 it is impossible to take $\gamma=0$ since the operator $h_0$ has a virtual level. More precisely, one has

**Proposition 3.2.** *Let $V$ obey $\int_{\mathbb{R}_+}|V(r)|^{1+\delta}r\,dr<\infty$ and $\int_{\mathbb{R}_+}|V(r)|(1+r^\delta)\,r\,dr<\infty$ for some $\delta>0$. Then $h_0-\beta V$ has a negative eigenvalue for all $\beta>0$ if and only if $\int_{\mathbb{R}_+}V(r)r\,dr\ge 0$, $V\not\equiv 0$. In this case, for sufficiently small $\beta$ the eigenvalue $\lambda(\beta)$ is unique and satisfies as $\beta\to 0$*

$$\begin{aligned}&\lambda(\beta)\sim -\exp\left(\left(\tfrac{\beta}{2}\textstyle\int_{\mathbb{R}_+}V(r)r\,dr\right)^{-1}\right) && \text{if } \textstyle\int_{\mathbb{R}_+}V(r)r\,dr>0,\\ &\lambda(\beta)\sim -\exp\left(-\tfrac{c}{\beta^2}\right) && \text{if } \textstyle\int_{\mathbb{R}_+}V(r)r\,dr=0\end{aligned}$$

*with a suitable constant $c=c(V)>0$.*

Here we use the notation $\lambda(\beta)\sim -\exp\left(-a\beta^{-\rho}\right)$ meaning

$$\lim_{\beta\to 0} -\beta^\rho\log(-\lambda(\beta))=a.$$

The proof uses the same idea as the proof of Corollary 2.2.

*Proof.* We recall that $-\Delta-\beta V(|\cdot|)$ in $L_2(\mathbb{R}^2)$ is unitarily equivalent to $\oplus_{n\in\mathbb{Z}}(h_n-\beta V)$ in $\oplus_{n\in\mathbb{Z}}L_2(\mathbb{R}_+)$ where $h_n-\beta V$ are similarly defined as in the proof of Corollary 2.2. Clearly $\oplus_{n\in\mathbb{Z}}(h_n-\beta V)$ has a negative eigenvalue if and only if $h_0-\beta V$ has a negative eigenvalue. The assertion follows therefore from Theorem 3.4 in [Si1]. □

## 4. Proof of Theorem 1.10

The proof of Theorem 1.10 is similar to the proof of Theorem 1.1 and we only sketch the major steps. We write

$$H_\phi := (-\mathrm{i}\nabla - \phi\mathbf{A})^2 - \frac{\phi^2}{|x|^2}.$$

With polar coordinates $(r,\theta)$ we define the projections $P_n$, $n \in \mathbb{Z}$, in $L_2(\mathbb{R}^2)$,

$$(P_n u)(r,\theta) := \frac{1}{2\pi}\int_{-\pi}^{\pi} u(r,\omega)e^{-\mathrm{i}n\omega}\,d\omega\; e^{\mathrm{i}n\theta}, \qquad r>0,\ \theta\in(-\pi,\pi).$$

The subspace $P_n L_2(\mathbb{R}^2)$ reduces $H_\phi$ and its part in this space is unitarily equivalent to the operator $-\frac{d^2}{dr^2} - \frac{1}{4r^2} + \frac{n(n-2\phi)}{r^2}$ in $L_2(\mathbb{R}_+)$, defined as quadratic form on $C_0^\infty(\mathbb{R}_+)$. Note that this operator coincides with (3.2) if $n=0$ and, if $\phi = \frac12$, also if $n=1$. This means that $H_\phi$ has one virtual level if $|\phi| < \frac12$ and two virtual levels if $\phi = \frac12$.

*Proof of Theorem 1.10.* We assume $V \geq 0$ and put $P := P_{-1} + P_0$ if $-\frac12 < \phi \leq 0$, $P := P_0 + P_1$ if $0 < \phi \leq \frac12$ and $Q := I - P$. (We emphasize that one may also take $P = P_0$ if $|\phi| < \frac12$, but then the constants below will blow up as $|\phi| \to \frac12$.) As in the proof of Theorem 1.1 one finds

$$\operatorname{tr}(H_\phi - V)_-^\gamma \leq \operatorname{tr}\left(P\left(H_\phi - 2V\right)P\right)_-^\gamma + \operatorname{tr}\left(Q\left(H_\phi - 2V\right)Q\right)_-^\gamma .$$

On $QL_2(\mathbb{R}^2)$ we use the estimate

$$QH_\phi Q \geq (1-|\phi|)Q(-\Delta)Q$$

which is easily obtained by decomposition into the subspaces $P_nL_2(\mathbb{R}^2)$. By Proposition 2.1 we conclude

$$\begin{aligned}\operatorname{tr}\left(Q\left(H_\phi - 2V\right)Q\right)_-^\gamma &\leq (1-|\phi|)^\gamma \operatorname{tr}\left(Q\left(-\Delta - 2(1-|\phi|)^{-1}V\right)Q\right)_-^\gamma \\ &\leq (1-|\phi|)^{-\frac{2+\alpha}{2}} 2^{\gamma+\frac{2+\alpha}{2}} C^{EK}_{\gamma,2,\alpha}\int_{\mathbb{R}^2} V(x)^{\gamma+\frac{2+\alpha}{2}}|x|^\alpha\,dx.\end{aligned}$$

On the orthogonal complement $PL_2(\mathbb{R}^2)$ we estimate

$$P\left(H_\phi - 2V\right)P \geq P_0\left(H_\phi - 4V\right)P_0 + P_{\mp1}\left(H_\phi - 4V\right)P_{\mp1}.$$

The latter operator is unitarily equivalent to the operator

$$\left(-\frac{d^2}{dr^2} - \frac{1}{4r^2} - 4\tilde V\right) \oplus \left(-\frac{d^2}{dr^2} - \frac{1}{4r^2} + \frac{1-2|\phi|}{r^2} - 4\tilde V\right)$$

in $L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_+)$, where

$$\tilde V(r) := \frac{1}{2\pi}\int_{-\pi}^{\pi} V(r,\theta)\,d\theta, \qquad r>0.$$

We estimate $1-2|\phi|\geq 0$ and conclude by Corollary 2.2 that

$$\begin{aligned}\operatorname{tr}\left(P\left(H_\phi-2V\right)P\right)_-^\gamma &\leq 2\operatorname{tr}\left(-\frac{d^2}{dr^2}-\frac{1}{4r^2}-4\tilde{V}\right)_-^\gamma \\ &\leq 2\cdot 4^{\gamma+\frac{2+\alpha}{2}}C_{\gamma,1,\alpha+1}\int_{\mathbb{R}_+}\tilde{V}(r)^{\gamma+\frac{2+\alpha}{2}}r^{\alpha+1}\,dr.\end{aligned}$$

It remains to use Hölder's inequality to complete the proof. Finally, we remark that the constants can be chosen independently of $\phi$. □

## Appendix A. An inequality of Egorov-Kondrat'ev

Our exposition in this appendix follows rather closely [EgKo]. Proposition 2.1 can be deduced by standard arguments (as, *e.g.*, in our proof of Theorem 1.6 in Section 3) provided we have established

**Lemma A.1.** *Let $d\geq 2$, $q>1$, $\alpha\geq 0$ such that $2q-\alpha>d$. Then*

$$N(\tau,-\Delta-V)\leq C\,\tau^{-q+\frac{d+\alpha}{2}}\int_{\mathbb{R}^d}V(x)_+^q|x|^\alpha\,dx,\qquad \tau>0, \tag{A.1}$$

*with a constant $C=C(\alpha,d,q)$ independent of $V$.*

The same result (with the same proof) holds if $d=1$, $q>1$, $\alpha\geq 0$ such that $q-\alpha>1$.

First some terminology. By a 'cube' we mean always a cube with edges parallel to the coordinate axis, and by its 'length' we mean the length of one of its edges. We need the following variant of Rozenblum's covering lemma (see [EgKo], where also an explicit value for the constant can be found).

**Lemma A.2.** *Let $d\geq 1$. Then there exists a constant $C_1>0$ such that for any $\varepsilon\in(0,1]$, any cube $Q\subset\mathbb{R}^d$ and any non-negative $f\in L_1(Q)$ there exists a finite number of cubes $Q_1,\ldots,Q_M$ with the following properties:*

1. *$Q\subset\bigcup_{j=1}^M Q_j$ and any point in $\mathbb{R}^d$ is contained in at most $C_1$ cubes,*
2. *$\int_{Q_j}f(x)\,dx\leq\varepsilon C_1\int_Q f(x)\,dx$, $\qquad j=1,\ldots,M$,*
3. *$M\leq\varepsilon^{-1}C_1$.*

**Lemma A.3.** *Let $0\leq s<2$ and $1\leq p<\frac{d-s}{d-2}$ if $d\geq 3$ and $1\leq p<\infty$ if $d=2$. Then there exists a constant $C_2>0$ such that for any cube $Q$ of length $l$ and any $u\in H^1(Q)$*

$$\left(\int_Q|u|^{2p}|x|^{-s}\,dx\right)^{\frac{1}{p}}\leq C_2\,l^{2-d+\frac{d-s}{p}}\int_Q\left(|\nabla u|^2+l^{-2}|u|^2\right)\,dx. \tag{A.2}$$

*Moreover, if $\int_Q u\,dx=0$ then*

$$\left(\int_Q|u|^{2p}|x|^{-s}\,dx\right)^{\frac{1}{p}}\leq C_2\,l^{2-d+\frac{d-s}{p}}\int_Q|\nabla u|^2\,dx. \tag{A.3}$$

*Proof.* By scaling it suffices to consider the case $l = 1$. We can choose $1 < p_1 < \infty$ such that $2pp_1 \leq \frac{2d}{d-2}$ if $d \geq 3$ and such that $sq_1 < d$ where $p_1^{-1} + q_1^{-1} = 1$. Then by Hölder's inequality

$$\int_Q |u|^{2p}|x|^{-s}\,dx \leq \left(\int_Q |u|^{2pp_1}\,dx\right)^{\frac{1}{p_1}} \left(\int_Q |x|^{-sq_1}\,dx\right)^{\frac{1}{q_1}}.$$

The latter integral is finite, uniformly for all cubes of length one, by our choice of $q_1$. Moreover, by the Sobolev embedding theorems

$$\left(\int_Q |u|^{2pp_1}\,dx\right)^{\frac{1}{pp_1}} \leq C_2 \int_Q \left(|\nabla u|^2 + |u|^2\right)\,dx$$

for some constant $C_2 = C_2(p, p_1, d)$. If $u$ has mean value zero we may use Poincaré's inequality instead (see [LiLo]). □

*Proof of Lemma A.1.* As in the proof of Lemma 3.1 we may assume $\tau = 1$ and $V \geq 0$. Fix $q > 1$, $\alpha \geq 0$ such that $2q - \alpha > d$ and note that $p$ and $s$, defined by $p^{-1} + q^{-1} = 1$ and $\alpha = s(q-1)$, satisfy the assumptions of Lemma A.3. Put

$$I := \int_{\mathbb{R}^d} V^q |x|^\alpha\,dx$$

and introduce the unit cube $Q_0 := (0,1)^d$. By Hölder's inequality and (A.2) we obtain that

(A.4)
$$\begin{aligned}\int_{\mathbb{R}^d} V|u|^2\,dx &\leq \sum_{k\in\mathbb{Z}^d} \left(\int_{Q_0+k} V^q|x|^{s(q-1)}\,dx\right)^{\frac{1}{q}} \left(\int_{Q_0+k} |u|^{2p}|x|^{-s}\,dx\right)^{\frac{1}{p}} \\ &\leq C_2 I^{\frac{1}{q}} \int_{\mathbb{R}^d} \left(|\nabla u|^2 + |u|^2\right)\,dx.\end{aligned}$$

In view of this inequality it suffices to prove the assertion only for, say, bounded and compactly supported $V$. Moreover, we conclude from this inequality that $N(1, -\Delta - V) = 0$ if $I \leq C_2^{-q}$. Hence it is enough to establish the estimate

$$N(1, -\Delta - V) \leq CI \tag{A.5}$$

under the additional condition

$$I \geq C_2^{-q}. \tag{A.6}$$

To obtain (A.5) we find a subspace $L \subset H^1(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} V|u|^2\,dx \leq \int_{\mathbb{R}^d} \left(|\nabla u|^2 + |u|^2\right)\,dx, \qquad u \in L, \tag{A.7}$$

and such that $\operatorname{codim} L \leq CI$.

For $0 < \varepsilon \le 1$ (which will be determined later) Lemma A.2 yields cubes $Q_1, \dots, Q_M$ such that $\operatorname{supp} V \subset \bigcup_{j=1}^M Q_j$, such that each point is covered by at most $C_1$ cubes,

$$\int_{Q_j} V^q |x|^\alpha \, dx \le \varepsilon \, C_1 I \tag{A.8}$$

and $M \le \varepsilon^{-1} C_1$. With $l_j$ denoting the length of $Q_j$ put

$$J_\le := \{j : \; l_j \le 1\}, \qquad J_> := \{j : \; l_j > 1\}.$$

First we consider $j \in J_>$ (*i.e.*, large cubes). We divide $Q_j = \bigcup_k Q_{j,k}$ in a finite number of non-intersecting cubes with equal length $\tilde{l}_j \in (1, 2]$. Estimating similarly as in (A.4) and using (A.8) we obtain

$$\begin{aligned} \int_{Q_j} V|u|^2 \, dx &\le \sum_k \left( \int_{Q_{j,k}} V^q |x|^{s(q-1)} \, dx \right)^{\frac{1}{q}} \left( \int_{Q_{j,k}} |u|^{2p} |x|^{-s} \, dx \right)^{\frac{1}{p}} \\ &\le C_2 2^{2-d+\frac{d-s}{p}} (\varepsilon \, C_1 I)^{\frac{1}{q}} \int_{Q_j} \left( |\nabla u|^2 + |u|^2 \right) \, dx. \end{aligned} \tag{A.9}$$

Now we consider $j \in J_\le$ (*i.e.*, small cubes). If $u \in H^1(\mathbb{R}^d)$ satisfies

$$\int_{Q_j} u \, dx = 0, \tag{A.10}$$

then a similar estimate (but using (A.3) instead of (A.2)) yields

$$\int_{Q_j} V|u|^2 \, dx \le C_2 (\varepsilon \, C_1 I)^{\frac{1}{q}} \int_{Q_j} |\nabla u|^2 \, dx. \tag{A.11}$$

Let $L$ be the space of all $u \in H^1(\mathbb{R}^d)$ such that (A.10) holds for all $j \in J_\le$. We sum (A.9) and (A.11) over all $j$ to get

$$\int_{\mathbb{R}^d} V|u|^2 \, dx \le C_3 (\varepsilon I)^{\frac{1}{q}} \int_{\mathbb{R}^d} \left( |\nabla u|^2 + |u|^2 \right) \, dx, \qquad u \in L,$$

where $C_3 := C_2 2^{2-d+(d-s)/p} C_1^{1+1/q}$. Now we choose $\varepsilon := C_3^{-q} I^{-1}$. (Note that in view of (A.6) and $C_1 \ge 1$ one has $\varepsilon \le 1$.) Moreover, with this choice of $\varepsilon$ relation (A.7) holds and

$$\operatorname{codim} L = \sharp J_\le \le M \le \varepsilon^{-1} C_1 = C_1 C_3^q I.$$

This yields (A.5) with $C = C_1 C_3^q$ and finishes the proof. □

*Remark* A.4. If $d \ge 3$ then Proposition 2.1 follows by the argument of Aizenman-Lieb [AiLi] from

$$N(0, -\Delta - V) \le C_{0,d,\alpha}^{EK} \int_{\mathbb{R}^d} V(x)_+^{\frac{d+\alpha}{2}} |x|^\alpha \, dx.$$

Different proofs of this inequality can be found in [BlReSt], [EgKo] and [BiSo2]. It would be desirable, in particular in view of constants, to find an alternative proof of Proposition 2.1 in the case $d = 2$.

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