# ABSOLUTELY CONTINUOUS SPECTRUM OF A CLASS OF RANDOM NON-ERGODIC SCHRÖDINGER OPERATORS

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ABSTRACT. We consider a class of random Schrödinger operators with non-decaying potentials and prove that their absolutely continuous spectrum almost surely fills the positive half-line. We establish the existence of the wave operators using the trace class scattering theory.

## 1. INTRODUCTION

We consider a class of random Schrödinger operators

$$-\Delta + V_\omega, \qquad V_\omega(x) = \sum_{j\in J} \omega_j f_j(x) \tag{1.1}$$

in $L_2(\mathbb{R}^d)$, $d \geq 2$. Here the $\omega_j$ are bounded, independent and identically distributed random variables, the $f_j$ are functions with disjoint supports and $J$ is an index set. In the usual, ergodic case the $f_j$ are translates along the lattice $J = \mathbb{Z}^d$, say, of a fixed bump function. Note that in this case the number of bumps inside the spherical layer $\{x \in \mathbb{R}^d : n-1 \leq |x| < n\}$ increases like $n^{d-1}$ as $n \to \infty$. In the present paper we consider models where this number grows like $n^{s(d-1)}$ for some parameter $s > 1$. Hence our potentials are non-ergodic and the parameter $s$ shows how close they are to ergodic ones. Note that we do not assume our potentials to decay.

Strong efforts have been devoted to the study of the point spectrum of the operator (1.1) in the ergodic case. It has been shown in [14], [21] that one-dimensional ergodic operators have purely point spectra (see also [20], [27], [9] and, for the case of a discrete distribution of random variables, [8], [25]). In higher dimensions ergodic Schrödinger operators have regions covered by purely point spectrum. The papers [12], [3] represent two different approaches to localization (see also [11], [2], [4]). We do not try to list all the important contributions but refer to the surveys [28], [26], [18] and the monograph [29]. Moreover, we would like to mention the recent developments [13], [1], [7]. On the other hand, it is expected that in the ergodic,

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multi-dimensional case the operator (1.1) has absolutely continuous spectrum for low disorder and away from the edge of the spectrum. Note that for dimension $d = 2$ the claims change from time to time. This conjecture is not proven.

However, there are other, non-ergodic models for which the existence of absolutely continuous spectrum has been established. First of all we would like to mention the papers by Bourgain [5] (see also [6]) and Denisov [10]. In their (discrete and continuous, respectively) models the supports of the functions $f_j$ in (1.1) are constant as in the ergodic case, but the height of the bumps is assumed to decay so that $|V_\omega(x)| \leq C(1 + |x|)^{-1/2-\epsilon}$. In [5] the almost sure existence and completeness of local wave operators is proven, whereas the approach of [10] is in the spirit of trace formulae. In the paper by Rodnianski, Schlag [24] operators of the form (1.1) are considered where the size of the bumps increases and the heights of the bumps decreases so that $|V_\omega(x)| \leq C(1+|x|)^{-3/4}$. The existence of modified wave operators is proven there. We would also like to mention the works by Krishna [19], Kirsch, Krishna, Obermeit [17] and Hundertmark, Kirsch [15] who essentially assume that $(\mathbb{E}[V_\omega(x)^2])^{1/2} \leq C(1 + |x|)^{-1-\epsilon}$.

In contrast to the preceeding results the absolute value of $V_\omega$ in our model can be constant for all $x$ and for all values of the random parameter $\omega$. We assume that $\mathbb{E}[\omega_j] = 0$ in our model, which for certain choices of functions $f_j$ is only a normalization. Then $V_\omega$ changes its sign almost surely, and we will prove that this implies that the interval $[0, \infty)$ is covered by absolutely continuous spectrum. The arguments in our proof depend on the dimension. In particular, they break down in dimension $d = 1$, when the potential becomes the same as in the continuous Anderson model for which the spectrum is known to be purely point.

We will not describe the details of our model in this introduction but give an intuitive picture of what is happening. Divide the plane into spherical layers $\{x \in \mathbb{R}^2 : n - 1 \leq |x| < n\}$, $n \in \mathbb{N}$, and divide each layer in (approximately) $n^s$ pieces with some fixed $s > 1$. Now paint these pieces randomly in yellow and blue colors. With the growth of $n$ the pieces become finer and finer and, outside of a large sphere, the plane will look green!

In one of our models (see Section 2) something similar happens to an electron: Yellow and blue colors correspond to positive and negative bumps of the potential. At infinity they average to zero and allow the electron to propagate.

Let us note, however, that this picture is rather naive and is not in relation with the difficulty of the problem. For example in the ergodic case $s = 1$ the average of the potential over the sphere $\{|x| = r\}$ decreases as a negative power (presumably as $r^{-(d-1)/2}$), which means that an observer running around such a sphere will see the green colour, but to prove existence of the absolutely continuous spectrum for ergodic potentials seems to be a very hard problem.

We show also that the higher order operator

$$(-\Delta)^l + V_\omega, \qquad l > d/4,$$

has absolutely continuous spectrum all over the positive real line under the condition that

$$s > 1 + 2/(d-1).$$

Thus in this model $s \to 1$ as $d \to \infty$, which gives the proper (ergodic) limit.

In Section 3 we prove an abstract result which is also applicable to operators having essential spectrum below zero. Our approach is based on a thorough use of the trace class scattering theory. In this paper we employ it in a way which is adjusted to cancelations due to the random nature of the potential.

The authors acknowledge the stimulating role of the paper [10] by S. Denisov who first applied trace formulae in the study of random operators.

## 2. A MODEL WITHOUT DECAY

For each $n \in \mathbb{N} = \{1, 2, \dots\}$ we introduce the characteristic function $\chi_n$ of the interval $[n-1, n) \subset \mathbb{R}$ and for each $j \in \mathbb{Z}^d$ we introduce the characteristic function $\xi_j$ of the cube $j + [0,1)^d \subset \mathbb{R}^d$. We put

$$f_{n,j}(x) := \chi_n(|x|)\xi_j(n^s x/|x|), \qquad n \in \mathbb{N},\ j \in \mathbb{Z}^d,$$

with a parameter $s > 1$ to be specified later. Note that $\sum_{n,j} f_{n,j} \equiv 1$. Suppose that $\omega_{n,j}$, $n \in \mathbb{N}, j \in \mathbb{Z}^d$, are bounded, independent and identically distributed random variables with zero expectations,

$$\mathbb{E}[\omega_{n,j}] = 0,$$

and consider the (bounded) random potential

$$V_\omega(x) = \sum_{n,j} \omega_{n,j} f_{n,j}(x), \qquad x \in \mathbb{R}^d.$$

A typical result of our method is

**Theorem 2.1.** *Let $d = 2, 3$ and let $s > 1 + 2/(d-1)$. Then the absolutely continuous spectrum of the operator $-\Delta + V_\omega$ contains $[0, \infty)$ almost surely.*

Actually, we will prove that the wave operators

$$W_\pm(-\Delta + V_\omega, -\Delta) = s - \lim_{t \to \pm\infty} \exp(it(-\Delta + V_\omega)) \exp(-it(-\Delta))$$

exist almost surely. This implies also that the absolutely continuous spectrum of $-\Delta + V_\omega$ is almost surely of infinite multiplicity on $[0, \infty)$.
Note that this is indeed a non-deterministic result: If $\omega_{n,j} = c$ for all but finitely many $n$, $j$, and some constant $c > 0$ then the essential spectrum of $-\Delta + V_\omega$ coincides with $[c, \infty)$.
The proof of Theorem 2.1 as well as an extension to higher dimensions will be given in Section 4.

## 3. A MORE GENERAL MODEL

3.1. **The abstract result.** Let $V_0 \in L_\infty(\mathbb{R}_+)$ and introduce the self-adjoint operators

$$H_0 := H_{00} + V_0(|x|), \qquad H_{00} := -\Delta$$

in $L_2(\mathbb{R}^d)$ with domains

$$\mathcal{D}(H_0) = \mathcal{D}(H_{00}) = H^2(\mathbb{R}^d). \tag{3.1}$$

We denote by $P_0^{(ac)}$ the projection onto the absolutely continuous subspace of $H_0$. We will consider Schrödinger operators

$$H_\omega := H_0 + V_\omega$$

with random electric potential $V_\omega$ of the form

$$V_\omega(x) = \sum_{j\in J} \omega_j f_j(x), \qquad x \in \mathbb{R}^d. \tag{3.2}$$

**Assumption 3.1.** *$\omega = (\omega_j)_{j\in J}$ is a countable family of independent, uniformly bounded random variables with*

$$\mathbb{E}[\omega_j] = 0, \qquad j \in J. \tag{3.3}$$

**Assumption 3.2.** *$(f_j)_{j\in J}$ is a family of real-valued functions such that $\sum_{j\in J} |f_j|$ is bounded and for some $\rho > 1/2$*

$$\sum_{j\in J} \mathbb{E}[\omega_j^2] \int_0^\infty \left( \int_{\mathbb{S}^{d-1}} |f_j(r\theta)|\, d\theta \right)^2 (1+r)^{2\rho} r^{d-1}\, dr < \infty.$$

It follows from Assumptions 3.1 and 3.2 that $V_\omega$ is bounded,

$$\|V_\omega\|_\infty \leq C, \tag{3.4}$$

uniformly in $\omega$. Theorem 2.1 will be deduced from the following

**Theorem 3.3.** *Let $d = 2, 3$. Under Assumptions 3.1, 3.2 the wave operators*

$$W_\pm(H_\omega, H_0) := s - \lim_{t\to\pm\infty} \exp(itH_\omega) \exp(-itH_0) P_0^{(ac)}$$

*exist almost surely.*

By general principle (see [30]) we deduce

**Corollary 3.4.** *Under the assumptions of Theorem 3.3 one has*

$$\sigma_{ac}(H_0) \subset \sigma_{ac}(H_\omega) \qquad \textit{a.s.}$$

3.2. **Preliminary material.** In this subsection we recall a simple result needed in the proof of Theorem 3.3.
For $\kappa \geq -\frac{1}{4}$ we define the operator $A_\kappa$ in $L_2(\mathbb{R}_+)$ as the Friedrichs extension of $-\frac{d^2}{dr^2} + \frac{\kappa}{r^2}$ on $C_0^\infty(\mathbb{R}_+)$. Moreover, denote by $\Lambda$ the operator of multiplication by $(1+r)$. We are interested in compactness properties of $\Lambda^{-\rho}(A_\kappa + I)^{-1}$. As usual, $\mathfrak{S}_1$ and $\mathfrak{S}_2$ denote the trace class and the Hilbert-Schmidt class, respectively.

**Lemma 3.5.** *Let* $\kappa \geq -\frac{1}{4}$. *If* $\rho > \frac{1}{2}$ *then*

$$\Lambda^{-\rho}(A_\kappa + I)^{-1} \in \mathfrak{S}_2,$$

*and if* $\rho > 1$ *then*

$$\Lambda^{-\rho}(A_\kappa + I)^{-1} \in \mathfrak{S}_1.$$

Indeed, the operator $A_\kappa$ can be diagonalized explicitly in terms of Bessel functions. The second part of the assertion follows then from Lemma 2 in [31]. By a direct estimate of the integral kernel we obtain the first part of the assertion.

3.3. **Proof of Theorem 3.3.** We denote by $Y_m$, $m \in \mathbb{N}$, a complete orthonormal system of real-valued eigenfunctions of the Laplace-Beltrami operator in $L_2(\mathbb{S}^{d-1})$ and introduce the projections $P_m$ in $L_2(\mathbb{R}^d)$,

$$(P_m u)(x) := Y_m(x/|x|) \int_{\mathbb{S}^{d-1}} u(|x|\theta) Y_m(\theta)\, d\theta, \qquad x \in \mathbb{R}^d.$$

Since $\sum_{m\in\mathbb{N}} P_m = I$ in the sense of strong convergence it suffices to prove the almost sure existence of the wave operators

$$\begin{aligned} W_\pm(H_\omega, H_0, P_m) &:= s - \lim_{t\to\pm\infty} \exp(itH_\omega)\exp(-itH_0)P_0^{(ac)}P_m = \\ &= s - \lim_{t\to\pm\infty} \exp(itH_\omega)P_m \exp(-itH_0)P_0^{(ac)}. \end{aligned} \tag{3.5}$$

For this we will prove that for every bounded interval $(a,b) \subset \mathbb{R}$

$$E_\omega(a,b)V_\omega P_m E_0(a,b) \in \mathfrak{S}_1 \qquad \text{a.s.,} \tag{3.6}$$

where $E_\omega$, $E_0$ denotes the spectral families of $H_\omega$, $H_0$, respectively. Since $P_n$ maps $\mathcal{D}(H_0) = H^2(\mathbb{R}^d)$ into $\mathcal{D}(H) = H^2(\mathbb{R}^d)$ the existence of the limits (3.5) will then follow from Theorem 6.4.9 in [30].
In view of (3.1), (3.4) the operators

$$(H_{00}+I)E_0(a,b) = \left((H_{00}+I)(H_0+\gamma I)^{-1}\right)\left((H_0+\gamma I)E_0(a,b)\right)$$

(for $\gamma$ sufficiently large) and similarly $E_\omega(a,b)(H_{00}+I)$ are bounded. Hence the relation (3.6) is a consequence of

$$(H_{00}+I)^{-1}V_\omega P_m(H_{00}+I)^{-1} \in \mathfrak{S}_1 \qquad \text{a.s.}$$

Similarly as before, we will denote by $\Lambda$ the operator of multiplication by $(1+|x|)$. With $\rho > \frac{1}{2}$ from Assumption 3.2 we factorize

$$(H_{00}+I)^{-1}V_\omega P_m(H_{00}+I)^{-1} = B_\omega^* C$$

where

$$B_\omega := \Lambda^\rho P_m V_\omega (H_{00}+I)^{-1},$$
$$C := \Lambda^{-\rho} P_m (H_{00}+I)^{-1}.$$

One easily finds that the non-trivial part of the operator $P_m H_{00}$ is unitarily equivalent to the operator $A_\kappa$ defined in the previous subsection with some $\kappa \geq 1/4$ depending on $m$ and $d$. From Lemma 3.5 we conclude that

$$C \in \mathfrak{S}_2,$$

and so it remains to show that

$$B_\omega \in \mathfrak{S}_2 \qquad \text{a.s.} \tag{3.7}$$

It follows from Assumption 3.1 that

$$\mathbb{E}[\|B_\omega\|_2^2] = \sum_{j\in J} \mathbb{E}[\omega_j^2] \|\Lambda^\rho P_m f_j (H_{00}+I)^{-1}\|_2^2. \tag{3.8}$$

Now the kernel of $\Lambda^\rho P_m f_j (H_{00}+I)^{-1}\mathcal{F}^*$, $\mathcal{F}$ being the Fourier transform, is

$$(2\pi)^{-d/2}(1+|x|)^\rho (1+|\xi|^2)^{-1} Y_m(x/|x|) \int_{\mathbb{S}^{d-1}} f_j(|x|\theta) Y_m(\theta) e^{i|x|\langle\theta,\xi\rangle}\, d\theta.$$

Since $Y_m$ is a bounded function, the absolute value of this kernel can be estimated by

$$C_1 (1+|x|)^\rho (1+|\xi|^2)^{-1} \int_{\mathbb{S}^{d-1}} |f_j(|x|\theta)|\, d\theta,$$

and we find

$$\|\Lambda^\rho P_m f_j (H_{00}+I)^{-1}\|_2^2 \leq C_2 \int_0^\infty \left( \int_{\mathbb{S}^{d-1}} |f_j(r\theta)|\, d\theta \right)^2 (1+r)^{2\rho} r^{d-1}\, dr$$

with a finite constant $C_2$ (since $d \leq 3$), independent of $j$. Combining this with (3.8) and Assumption 3.2 we conclude that

$$\mathbb{E}[\|B_\omega\|_2^2] < \infty.$$

This implies (3.7) and finishes the proof of Theorem 3.3.

*Remark* 3.6. The previous proof can be easily generalized to the case of the polyharmonic operator. Indeed, if $H_{00} = (-\Delta)^l$ and Assumptions 3.1, 3.2 continue to hold, then the assertion of Theorem 3.3 remains true for $d < 4l$.

*Remark* 3.7. Note that we do not claim that the wave operators are complete. Indeed, it follows easily from Theorem 5.4.5 in [30] that $u \in \mathcal{R}(W_\pm(H_\omega, H_0))$ iff for all $m \in \mathbb{N}$

$$P_m \exp(-itH_\omega)P_\omega^{(ac)} u \to 0 \qquad \text{as } t \to \pm\infty.$$

Here $P_\omega^{(ac)}$ denotes the projection onto the absolutely continuous subspace of $H_\omega$. We can conclude the completeness of the wave operators provided this convergence is uniform in $m$.

3.4. **Cone condition.** Here we would like to give a 'local' version of Theorem 3.3, which involves assumptions on the random potential only inside a cone.
We fix a smooth ($H^2$, say) real-valued function $\phi$ on the unit sphere $\mathbb{S}^{d-1}$ of norm one and introduce the cone

$$C := \{x \in \mathbb{R}^d : x/|x| \in \operatorname{supp}(\phi)\}.$$

Again we define $V_\omega$ by (3.2) and but now we subject the $\omega_j$ and $f_j$ only to

**Assumption 3.8.** $\omega = (\omega_j)_{j\in J}$ *is a countable family of independent, uniformly bounded random variables with*

$$\mathbb{E}[\omega_j] = 0, \qquad \forall j \in J \text{ such that } \operatorname{supp}(f_j) \cap C \neq \emptyset.$$

**Assumption 3.9.** $(f_j)_{j\in J}$ *is a family of real-valued functions such that* $\sum_{j\in J} |f_j|$ *is bounded and for some* $\rho > 1/2$

$$\sum_{j\in J} \mathbb{E}[\omega_j^2] \int_0^\infty \left( \int_{\mathbb{S}^{d-1}} |\phi(\theta) f_j(r\theta)|\, d\theta \right)^2 (1+r)^{2\rho} r^{d-1}\, dr < \infty.$$

Note that Assumptions 3.8, 3.9 impose essential conditions on $V_\omega$ only inside the cone $C$.
We choose a real-valued function $\zeta \in C^\infty(\mathbb{R}_+)$ which vanishes near zero and equals one in $\{r > 1\}$. Put

$$\psi(x) := c(|x|)\Big(\zeta(|x|)\phi(x/|x|) + (1 - \zeta(|x|))\Big), \qquad x \in \mathbb{R}^d,$$

where $c(r)$ is chosen so that the norm of $\psi(r\cdot)$ in $L_2(\mathbb{S}^{d-1})$ is one,

$$\int_{\mathbb{S}^{d-1}} |\psi(r\theta)|^2 d\theta = 1, \qquad r > 0.$$

We introduce the projection $P$ in $L_2(\mathbb{R}^d)$,

$$(Pu)(x) := \psi(x) \int_{\mathbb{S}^{d-1}} \psi(|x|\theta) u(|x|\theta)\, d\theta, \qquad x \in \mathbb{R}^d,$$

and put $P_1 := I - P$. We consider the operator

$$H_\oplus := PH_0P + P_1H_0P_1.$$

Below we shall see that $H_\oplus$ is well-defined on $\mathcal{D}(H_0)$. We denote by $P_\oplus^{(ac)}$ the projection onto its absolutely continuous subspace. Then the analogue of Theorem 3.3 is

**Theorem 3.10.** *Let $d = 2, 3$. Under Assumptions 3.8, 3.9 the wave operators*

$$W_\pm(H_\omega, H_\oplus, P) := s - \lim_{t \to \pm\infty} \exp(itH_\omega) P \exp(-itH_\oplus) P_0^{(ac)}$$

*exist almost surely.*

Before we pass to the proof we would like to deduce information about the absolutely continuous spectrum of the operator $H_\omega$. As is well-known, if the wave operators exist then the part of $H_\omega$ on the range $\mathcal{R}(W_\pm(H_\omega, H_\oplus, P))$ of the wave operators is unitarily equivalent to the part of $H_\oplus$ on

$$\mathcal{N}(W_\pm(H_\omega, H_\oplus, P))^\perp = \mathcal{R}(P_\oplus^{(ac)} P).$$

This implies

**Corollary 3.11.** *Under the assumptions of Theorem 3.10 one has*

$$\sigma_{ac}(PH_\oplus P) \subset \sigma_{ac}(H_\omega) \qquad \textit{a.s.}$$

Note that if $V_0 \equiv 0$, i.e., $H_0 = H_{00} = -\Delta$ then the absolutely continuous spectrum of $PH_\oplus P$ coincides with $[0, \infty)$.

*Proof of Theorem 3.10.* First of all we will prove that

$$(3.9) \quad (H_0 + \gamma I)(H_\oplus + \gamma I)^{-1} \in \mathfrak{B}, \qquad (H_\oplus + \gamma I)(H_0 + \gamma I)^{-1} \in \mathfrak{B}$$

for all sufficiently large $\gamma > 0$. To this end we calculate

$$(3.10) \quad H_0 - H_\oplus = 2 \,\mathrm{Re}\, PH_0P_1 = 2 \,\mathrm{Re}\, PH_{00}P_1 = -2 \,\mathrm{Re}\, [H_{00}, P]P_1$$

and note that $[H_{00}, P]$ contains only combinations of projections and first order differentiation. In particular, on $\{|x| > 1\}$ one has

$$(3.11) \qquad [H_{00}, P] = |x|^{-2} \left( (\cdot, \Delta_\theta \phi)_{L_2(\mathbb{S}^{d-1})} \phi - (\cdot, \phi)_{L_2(\mathbb{S}^{d-1})} \Delta_\theta \phi \right),$$

where $\Delta_\theta$ denotes the Laplace-Beltrami operator in $L_2(\mathbb{S}^{d-1})$. We deduce that $H_\oplus - H_0$ is $H_0$-bounded with relative bound $0$. This implies that the domains of the operators $H_0$ and $H_\oplus$ coincide and (3.9) follows by the closed

graph theorem.
In order to prove Theorem 3.10 we will establish now that

$$(H_\oplus + \gamma I)^{-1} P - P(H_\omega + \gamma I)^{-1} = \\ = (H_\oplus + \gamma I)^{-1} P(H_\omega - H_\oplus)(H_\omega + \gamma I)^{-1} \in \mathfrak{S}_1,$$

for sufficiently large $\gamma > 0$. Similar as in (3.10) we find that

$$P(H_\omega - H_\oplus) = PV_\omega - P[H_{00}, P]P_1,$$

so in view of (3.1), (3.4), (3.9) it suffices to establish the two inclusions

$$(H_\oplus + I)^{-1} PV_\omega (H_{00} + I)^{-1} \in \mathfrak{S}_1, \tag{3.12}$$

$$(H_\oplus + I)^{-1} P[H_{00}, P]P_1 (H_\oplus + I)^{-1} \in \mathfrak{S}_1. \tag{3.13}$$

We begin with the proof of (3.12) which is similar to the proof of Theorem 3.3. We will factorize the operator on the left hand side of (3.12) as a product of operators from the Hilbert-Schmidt class. Replacing Assumptions 3.1, 3.2 above by Assumptions 3.8, 3.9 one verifies that

$$\Lambda^\rho PV_\omega (H_{00} + I)^{-1} \in \mathfrak{S}_2$$

with $\rho > 1/2$ from Assumption 3.9.
Now the non-trivial part of the operator $PH_\oplus P$ is unitarily equivalent to a second-order differential operator in $L_2(\mathbb{R}_+)$ with smooth coefficients. In a neighborhood of zero it coincides with $-\frac{d^2}{dr^2} + \kappa r^{-2}$, $\kappa = (d-1)(d-3)/4$, and in $\{r > 1\}$ with $-\frac{d^2}{dr^2} + r^{-2}(\kappa + \|\nabla\phi\|^2_{L_2(\mathbb{S}^{d-1})})$. Thus it follows easily from Lemma 3.5 that

$$\Lambda^{-\rho} P(H_\oplus + I)^{-1} \in \mathfrak{S}_2,$$

which concludes the proof of (3.12).
The proof of (3.13) uses similar ideas, but now we factorize the operator on the left hand side as a product of a trace class and a bounded operator. Indeed, one can use Lemma 3.5 again to prove that

$$\Lambda^{-\alpha} P(H_\oplus + I)^{-1} \in \mathfrak{S}_1$$

provided $\alpha > 1$. Moreover, a simple analysis of the commutator $[H_{00}, P]$ (noting in particular (3.11)) shows that

$$\Lambda^\alpha P[H_{00}, P]P_1 (H_\oplus + I)^{-1} \in \mathfrak{B}$$

provided $\alpha \leq 2$. This proves (3.13) and completes the proof of the theorem. $\square$

## 4. BACK TO THE MODEL FROM SECTION 2

4.1. **Proof of Theorem 2.1.** We are in the situation of Section 3 with $V_0 \equiv 0$ and $J = \mathbb{N} \times \mathbb{Z}^d$ and have to verify Assumption 3.2. Then Theorem 2.1 will follow from Corollary 3.4.
It is elementary to see that there is a constant $C_1 > 0$ such that for all $j \in \mathbb{Z}^d$, $\rho \geq 1$

$$\int_{\mathbb{S}^{d-1}} \xi_j(\rho\,\theta)\,d\theta \leq C_1 \rho^{-(d-1)} \chi_{[|j|-\sqrt{d},|j|+\sqrt{d}]}(\rho), \tag{4.1}$$

where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$. Hence

$$\begin{aligned}
&\sum_{n\in\mathbb{N},j\in\mathbb{Z}^d} \int_0^\infty \left( \int_{\mathbb{S}^{d-1}} |f_{n,j}(r\theta)|\,d\theta \right)^2 (1+r)^{2\rho} r^{d-1}\,dr \leq \\
&\quad \leq C_1^2 \sum_{n\in\mathbb{N},j\in\mathbb{Z}^d} n^{-2s(d-1)} \chi_{[|j|-\sqrt{d},|j|+\sqrt{d}]}(n^s) \int_{n-1}^n (1+r)^{2\rho} r^{d-1}\,dr \leq \\
&\quad \leq C_2 \sum_{n\in\mathbb{N},j\in\mathbb{Z}^d} n^{-(2s-1)(d-1)+2\rho} \chi_{[|j|-\sqrt{d},|j|+\sqrt{d}]}(n^s).
\end{aligned}$$

Now the number of $j \in \mathbb{Z}^d$ for which $\chi_{[|j|-\sqrt{d},|j|+\sqrt{d}]}(n^s) \neq 0$ coincides with the number of integer points in the spherical layer $\{n^s - \sqrt{d} \leq |x| \leq n^s + \sqrt{d}\}$ and is hence bounded by $C_3 n^{s(d-1)}$. We conclude that the above sum can be estimated by

$$C_2 C_3 \sum_{n\in\mathbb{N}} n^{-(s-1)(d-1)+2\rho}.$$

Since $s > 1 + 2/(d-1)$ we can choose $\rho > 1/2$ such that $(s-1)(d-1) - 2\rho > 1$ and so Assumption 3.2 holds. This completes the proof of Theorem 2.1.

Before we move on we would like to mention some obvious generalizations of the above proof. First, note that the assumption that the $\omega_{n,j}$ are identically distributed was not used. It suffices that they are uniformly bounded (see Example 5.3 below). Second, the precise form of the functions $f_{n,j}$ is not essential. We only used the estimate (4.1) and a bound on the number of functions supported inside a given spherical layer.

4.2. **Extension to higher dimensions.** It turns out that the conclusion of Theorem 2.1 holds also for dimensions $d \geq 4$. We give the proof only in the setting of Section 2 but remark that various generalizations are possible. Finally, we believe that the critical value on $s$ can be improved.

**Theorem 4.1.** *Let $d \geq 4$ and let $s > 1 + d/(d-1)$. Then the absolutely continuous spectrum of the operator $-\Delta + V_\omega$ contains $[0, \infty)$ almost surely.*

*Proof.* We proceed as in the proof of Theorem 3.3 and factorize again

$$(-\Delta + I)^{-1} V_\omega P_m (-\Delta + I)^{-1} = B_\omega^* C$$

where

$$B_\omega := \Lambda^\rho P_m V_\omega (-\Delta + I)^{-1},$$
$$C := \Lambda^{-\rho} P_m (-\Delta + I)^{-1}.$$

However, now we choose

$$\rho > 1, \tag{4.2}$$

so that Lemma 3.5 yields $C \in \mathfrak{S}_1$. Hence it remains to show that

$$B_\omega \in \mathfrak{B} \qquad \text{a.s.} \tag{4.3}$$

We will localize in the configuration and in the momentum space. Recall that $\chi_n$ is the characteristic function of the spherical layer $\{n-1 \leq |x| < n\}$ and put $J_n := E_{00}(0, n^{2\rho})$ where $E_{00}$ is the spectral family of $-\Delta$. We have

$$\mathbb{E}[\|B_\omega\|] \leq \sum_{n\in\mathbb{N}} \mathbb{E}[\|\chi_n B_\omega (I - J_n)\|] + \sum_{n\in\mathbb{N}} \mathbb{E}[\|\chi_n B_\omega J_n\|]. \tag{4.4}$$

For high energies we use that

$$\|(-\Delta + I)^{-1}(I - J_n)\| \leq n^{-2\rho}$$

and obtain the deterministic estimate

$$\|\chi_n B_\omega (I - J_n)\| \leq C_1 n^{-\rho}.$$

Hence the first sum in (4.4) is finite in view of (4.2).
For low energies we calculate similarly as in the proof of Theorem 3.3 that

$$(\mathbb{E}[\|\chi_n B_\omega J_n\|])^2 \leq \mathbb{E}[\|\chi_n B_\omega J_n\|_2^2] = C_2 \sum_{j\in\mathbb{Z}^d} \|\Lambda^\rho P_m f_{n,j} (-\Delta + I)^{-1} J_n\|_2^2, \tag{4.5}$$

where $C_2$ is the (common) variance of the $\omega_{n,j}$. Since $Y_m$ is a bounded function, the absolute value of the kernel of $\Lambda^\rho P_m f_{n,j}(-\Delta + I)^{-1} J_n \mathcal{F}^*$ can be estimated by

$$C_3 (1 + |x|)^\rho (1 + |\xi|^2)^{-1} \chi_{[n-1,n)}(|x|) \chi_{(0,n^\rho)}(|\xi|) \int_{\mathbb{S}^{d-1}} \xi_j(n^s \theta)\, d\theta.$$

Obviously, one has

$$\int_{n-1<|x|<n} (1 + |x|)^{2\rho}\, dx \leq C_4 n^{2\rho + d - 1}$$

and, provided $d \geq 5$,

$$\int_{|\xi|<n^\rho} \frac{d\xi}{(1+|\xi|^2)^2} \leq C_5 n^{(d-4)\rho}.$$

(If $d=4$ the argument is similar, so we restrict ourselves henceforth to the case $d \geq 5$.) Combining this with the estimate (4.1) we find

$$\|\Lambda^\rho P_m f_{n,j}(-\Delta+I)^{-1} J_n\|_2^2 \leq C_6 n^{-(2s-1)(d-1)+(d-2)\rho} \chi_{[|j|-\sqrt{d},|j|+\sqrt{d}]}(n^s).$$

As in the previous subsection we estimate the number of $j \in \mathbb{Z}^d$ for which $\chi_{[|j|-\sqrt{d},|j|+\sqrt{d}]}(n^s) \neq 0$ by $C_7 n^{s(d-1)}$, and obtain so from (4.5) the bound

$$(\mathbb{E}[\|\chi_n B_\omega J_n\|])^2 \leq C_8 n^{-(s-1)(d-1)+(d-2)\rho}.$$

Hence the second sum in (4.4) is finite if we choose $\rho > 1$ such that

$$(s-1)(d-1)-(d-2)\rho > 2,$$

(which is possible since $s > 1 + d/(d-1)$). We conclude that

$$\mathbb{E}[\|B_\omega\|] < \infty.$$

This implies (4.3) and finishes the proof of Theorem 4.1. □

## 5. Further examples

Finally, we would like to show that our method is not restricted to the model from Section 2 but yields the existence of absolutely continuous spectrum also in other situations, e.g., with sparse potentials or decaying randomness. In each case the proof reduces to the verification of Assumption 3.2. We emphasize that this assumption allows to combine different mechanisms for absolutely continuous spectrum and that their influence is essentially additive.

**Example 5.1** (Additional decay)**.** Let $d = 2, 3$ and define $f_{n,j}$ as in Section 2, depending on the parameter $s > 1$. Moreover, let $(\alpha_{n,j})$ be a deterministic sequence of real numbers satisfying

$$|\alpha_{n,j}| \leq C n^{-t}, \qquad n \in \mathbb{N},\ j \in \mathbb{Z}^d,$$

with some parameter $t \geq 0$. We define now

$$V_\omega(x) = \sum_{n,j} \omega_{n,j} \alpha_{n,j} f_{n,j}(x), \qquad x \in \mathbb{R}^d,$$

where $\omega_{n,j}$ are as in Section 2.
If $t > 1$ this is a short-range potential and standard deterministic results are applicable. If $t > 1/2$ the almost sure existence of absolutely continuous spectrum is proven in [5], [10] for $s = 1$. We will show that absolutely continuous spectrum exists almost surely for all $t \geq 0$ at the expense of

increasing $s$. Put differently, decay (i.e., $t > 0$) helps to decrease the lower bound on $s$ in Theorem 2.1. Indeed, a straightforward generalization of the proof in Subsection 4.1 shows that $[0, \infty)$ is almost surely contained in the absolutely continuous spectrum of $-\Delta + V_\omega$ provided

$$s > 1 + 2(1 - t)/(d - 1).$$

**Example 5.2** (Sparse potentials)**.** Let $d = 3$ and $f \in L_\infty(\mathbb{R}^3)$ with compact support. Let $(x_m)_{m\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^3$ such that $|x_m| \to \infty$ but

$$\sum_{m\in\mathbb{N}} |x_m|^{-\sigma} < \infty$$

for some $\sigma < 1$ and put

$$W_{\tilde\omega}(x) := \sum_{m\in\mathbb{N}} \tilde\omega_m f(x - x_m), \qquad x \in \mathbb{R}^3,$$

where $\tilde\omega_m$, $m \in \mathbb{N}$, are bounded, independent and identically distributed random variables with $\mathbb{E}[\tilde\omega_m] = 0$.

We claim that $[0, \infty)$ is almost surely contained in the absolutely continuous spectrum of $-\Delta + W_\omega$. Indeed, proceeding as in Subsection 4.1 and using an analogue of the estimate (4.1) we find that

$$\sum_{m\in\mathbb{N}} \int_0^\infty \left( \int_{\mathbb{S}^2} |f(r\theta - x_m)| \, d\theta \right)^2 (1 + r)^{2\rho} r^2 \, dr \le C \sum_{m\in\mathbb{N}} |x_m|^{-2+2\rho},$$

which is finite for $\rho = 1 - \sigma/2 > 1/2$. We note that $-\Delta + W_{\tilde\omega}$ has non-empty essential spectrum below zero whenever $f$ is sufficiently negative.

The same idea can be used to construct negative essential spectrum for the operator from Section 2. Let $V_\omega$ be of the same form as there with $\omega_{n,j}$ independent from $\tilde\omega_m$. Since $V_\omega$ and $W_{\tilde\omega}$ satisfy Assumption 3.2 their sum does so as well. We conclude that $-\Delta + V_\omega + W_{\tilde\omega}$ has almost surely absolutely continuous spectrum in $[0, \infty)$ and also, if $f$ is sufficiently negative, essential spectrum below zero.

**Example 5.3** (Decaying randomness)**.** We return once more to the model from Section 2. Let $d = 2, 3$ and let $f_{n,j}$ and $V_\omega$ be defined as before, but assume now that the $\omega_{n,j}$ are *not* identically distributed. For the sake of definiteness assume that they are independent random variables assuming only the values $-1, 0, 1$ with

$$\mathbb{P}[\omega_{n,j} = 1] = \mathbb{P}[\omega_{n,j} = -1] =: p_{n,j}/2, \qquad n \in \mathbb{N},\ j \in \mathbb{Z}^d.$$

We assume that

$$|p_{n,j}| \le C n^{-\tau}, \qquad n \in \mathbb{N},\ j \in \mathbb{Z}^d, \tag{5.1}$$

with some parameter $\tau > 0$. Then the set $[0, \infty)$ is almost surely contained in the absolutely continuous spectrum of the operator $-\Delta + \alpha V_\omega$, $\alpha \in \mathbb{R}$, provided

$$s > 1 + (2 - \tau)/(d - 1). \tag{5.2}$$

This follows as in Subsection 4.1 but taking now also into account that $\mathbb{E}[\omega_{n,j}^2] = p_{n,j}$. Note that if

$$\sum_{n\in\mathbb{N}, j\in\mathbb{Z}^d} p_{n,j}\chi_{[|j|-\sqrt{d},|j|+\sqrt{d}]}(n^s) < \infty, \tag{5.3}$$

then $V_\omega$ is almost surely compactly supported by the Borel-Cantelli lemma, so the existence of absolutely continuous spectrum is trivial in this case. If, however, $d = 3$ one can choose $p_{n,j}$ such that (5.1) holds with $\tau > 2$ and simultaneously the sum in (5.3) is infinite. Then (5.2) guarantees the almost sure existence of absolutely continuous spectrum on $[0, \infty)$ in the case $s = 1$. At the same time the existence of a infinite sequence of variables $\omega_{n,j}$ assuming the value $-1$ might yield the existence of negative essential spectrum for sufficiently large $|\alpha|$. We refer to [15] for a more detailed analysis of the negative essential spectrum in related models (see also [23]).

Finally, we remark that the cone condition from Subsection 3.4 allows to construct a lot of further examples having both positive absolutely continuous and negative essential spectrum. In particular, one can consider the potential of the usual Anderson model and modify it in an arbitrary small cone in one of the described ways to produce absolutely continuous spectrum on the positive half-line.

As we mentioned before the main result of our paper does not have one-dimensional analogues simply because the unit sphere in $d = 1$ consists only of two points. However, one can consider the following elementary

**Example 5.4.** Let $\chi_{n,j}$ be the characteristic function of the interval $[n + jn^{-s}, n + (j+1)n^{-s})$ where $n \in \mathbb{Z}$, $0 \leq j < [n^s]$, and $[n^s]$ notes the integer part of $n^s$. Let $\omega_{n,j}$ be bounded, independent and identically distributed random variables with $\mathbb{E}[\omega_{n,j}] = 0$ and define

$$V_\omega(x) := \sum_n \sum_{0\leq j<[n^s]} \omega_{n,j} n^{-1/4-\epsilon} \chi_{n,j}(x), \qquad x \in \mathbb{R}.$$

Now we prove the following result.

**Theorem 5.5.** *Let $s \geq 1/2$. Then the absolutely continuous spectrum of the operator $-d^2/dx^2 + V_\omega$ is almost surely essentially supported by $[0, \infty)$, i.e., the spectral measure has positive derivative almost everywhere on $[0, \infty)$.*

One might conjecture that for $0 < s < 1/2-\epsilon$ the operator $-d^2/dx^2+V_\omega$ has almost surely dense purely point spectrum on $[0,\infty)$.

*Proof.* We consider the Fourier transform

$$\hat{V}_\omega(\xi) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-i\xi x}V_\omega(x)\,dx = \sum_n \sum_{0\leq j<[n^s]} \omega_{n,j} n^{-1/4-\epsilon}\hat{\chi}_{n,j}(\xi)$$

and note that $|\hat{\chi}_{n,j}(\xi)| \leq C_1 n^{-s}$ for all $\xi$ with a constant $C_1$ independent of $j$, $n$. Hence for any finite interval $(a,b)$

$$\mathbb{E}\left[\int_a^b |\hat{V}_\omega(\xi)|^2\,d\xi\right] = \mathbb{E}[\omega_{0,0}]\sum_n n^{-1/2-2\epsilon}\sum_{0\leq j<[n^s]}\int_a^b |\hat{\chi}_{n,j}(\xi)|^2\,d\xi \leq$$
$$\leq C_2(b-a)\sum_n\sum_{0\leq j<[n^s]} n^{-2s-1/2-2\epsilon},$$

and the sum on the right hand side converges. Since the Fourier transform of $V_\omega$ belongs almost surely to $L_{2,loc}$ and $V_\omega \in L_4$, the assertion follows from the main result of [16]. (By the trick from [22] the $L_3$-condition in [16] can be substituted by a $L_4$-condition.) □

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