Low lying eigenvalues of Witten Laplacians and metastability (after Helffer-Klein-Nier and Helffer-Nier)

Bernard Helffer*
Département de Mathématiques Bât. 425,
Université Paris Sud, UMR CNRS 8628,
F91405 Orsay Cedex (FRANCE).

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Abstract

The aim of this lecture is to present the recent results obtained in collaboration with M. Klein and F. Nier on the low lying eigenvalues of the Laplacian attached to the Dirichlet form :

$$C_0^\infty(\Omega) \ni v \mapsto h^2 \int_\Omega |\nabla v(x)|^2 \; e^{-2f(x)/h} \; dx \; ,$$

where f is a C^∞ Morse function on $\overline{\Omega}$ and $h > 0$. We give in particular an optimal asymptotics as $h \to 0$ of the lowest strictly positive eigenvalue, which will hold under generic assumptions. We discuss also some aspects of the proof.

1 Main goals and assumptions

We are interested in the low lying, acually exponentially small, eigenvalues of the Dirichlet realization of the semiclassical Witten Laplacian on 0-forms

$$\Delta_{f,h}^{(0)} = -h^2 \Delta + |\nabla f(x)|^2 - h \Delta f(x) \; . \tag{1.1}$$

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This question appears quite naturally when analyzing the asymptotic behavior as $t \to +\infty$ of $\exp -t\Delta^{(0)}_{f,h}$.
We would like to describe the recent results concerning the lowest strictly positive eigenvalue of this Laplacian. We will discuss briefly three cases :

- the case of $\mathbb{R}^n$, which was analyzed by Bovier-Eckhoff-Gayrard-Klein [BEGK], Bovier-Gayrard-Klein [BoGayKl], and by Helffer-Klein-Nier [HKN];
- the case of a compact riemannian manifold of dimension n ([HKN]);
- The case of a bounded set Ω in $\mathbb{R}^n$ with regular boundary treated by Helffer-Nier [HelNi2]) (in this case, we consider the Dirichlet realization of this operator).

In all these contributions, the goal is to get the optimal accuracy asymptotic formulas for the m_0 first eigenvalues of the Dirichlet realization of $\Delta^{(0)}_{f,h}$.

This question was already addressed a long time ago, via a probabilistic approach, sometimes in relation with the problem of the simulated annealing. Here we would like to mention Freidlin-Wentcel [FrWe], Holley-Kusuoka-Strook [HolKusStr], Miclo [Mic], Kolokoltsov [Kol], Bovier-Eckhoff-Gayrard-Klein [BEGK] and Bovier-Gayrard-Klein [BoGayKl], but the proof, as mentioned for example in [Kol], of the optimal accuracy (except may be for the case of dimension 1) was left open.

The Witten Laplacian $\Delta^{(0)}_{f,h}$ is associated to the Dirichlet form

$$C_0^\infty(\Omega) \ni u \mapsto \int_\Omega |(h\nabla + \nabla f)u(x)|^2 \; dx \; .$$

Note that the probabilists look equivalently at :

$$C_0^\infty(\Omega) \ni v \mapsto h^2 \int_\Omega |\nabla v(x)|^2 \; e^{-2f(x)/h} \; dx \; .$$

Let us now describe the main assumptions. In the whole paper, we assume that :

Assumption 1.1 *The function f is a C^∞- function on $\overline{\Omega}$ and a Morse function on Ω.*

In the case when $\Omega = \mathbb{R}^n$, the operator defined on C_0^∞ is essentially selfadjoint in L^2 and if

Assumption 1.2
$$\liminf_{|x|\to+\infty} |\nabla f(x)|^2 > 0 \, ,$$
and
$$|D_x^\alpha f| \leq C_\alpha (|\nabla f|^2 + 1) \, ,$$
for $|\alpha| = 2$*,*

is satisfied, Pearson's theorem permits to prove that the bottom of the essential spectrum is, for h small enough, strictly positive.

In the case with boundary, we will assume :

Assumption 1.3 *The function f has no critical points at the boundary and the function $f_{/\partial\Omega}$ is a Morse function on $\partial\Omega$.*

The last assumption, which appears to be generic, is more diffficult to describe and will be presented in the next section.

2 Saddle points and labelling

The presentation of the results involves a right definition for the saddle points (or critical points of index 1). If for a point in Ω, we take the usual definition (the index at a critical point U being defined as the number of negative eigenvalues of the Hessian of f at U), we shall say that a point U at the boundary is a critical point of index 1 if U is a local minimum of $f_{/\partial\Omega}$ **and** if the external normal derivative of f is strictly positive.
Our statements also involve a labelling of the local minima, which is inspired by previous works by probabilists [FrWe], [Mic], [BEGK] and [BoGayKl].

The existence of such a labelling is an assumption which is generically satisfied. This discussion can be done in the three cases. Let us focus on the case of a compact connected oriented Riemannian manifold $\overline{\Omega} = \Omega \cup \partial\Omega$ with boundary and that the function f satisfies Assumptions 1.1 and 1.3 and refer to the original papers [HKN, HelNi2] (and references therein) for the other cases or for details. The set of critical points with index 1 is denoted by $\mathcal{U}^{(1)}$ and its cardinal by m_1. For the definition of the saddle set (or set of saddle points), we need some notations. For any $A, B \subset \overline{\Omega}$, $H(A, B)$ denotes the least height to be reached to go continuously from A to B. When A and B are closed nonempty subsets of $\overline{\Omega}$, one can show that $H(A,B)$ is a minimum. We next need two definitions.

Definition 2.1
Let A and B be two closed subsets of $\overline{\Omega}$. We say that $Z \subset \overline{\Omega}$ is a saddle set for (A,B), if it is not empty and satisfies the following conditions :

$$Z \subset \left(\mathcal{U}^{(1)} \cap f^{-1}(\{H(A,B)\})\right) \ ,$$
$$\{C \in \ \mathit{Conn}\ (f^{-1}((-\infty, H(A,B)]) \setminus Z)\,,\ C \cap A \neq \emptyset, C \cap B \neq \emptyset\} = \emptyset\ .$$

Definition 2.2
Let A, B be closed nonempty disjoint subsets of $\overline{\Omega}$. The point $z \in \mathcal{U}^{(1)}$ is said to be a unique (one point)-saddle set[1] for the pair (A,B) if

$$(\cap_{C\in\mathcal{C}(A,B)}C)\ \cap \left[\mathcal{U}^{(1)} \cap f^{-1}(H(A,B))\right] = \{z\}\ ,$$

where $\mathcal{C}(A,B)$ denotes the set of closed connected sets $C \subset f^{-1}((-\infty, H(A,B)])$, such that $C \cap A \neq \emptyset$ and $C \cap B \neq \emptyset$.

We now give the main assumption which ensures that each exponentially small eigenvalue of $\Delta^{(0)}_{f,h}$ is simple, with a different asymptotic behavior.

Assumption 2.3
There exists a labelling of the set of the local minima of f in Ω, $\mathcal{U}^{(0)} := \left\{U_1^{(0)}, \ldots, U_{m_0}^{(0)}\right\}$, such that, by setting

$$\mathcal{C}_0 = \partial\Omega \ \ \textit{and}\ \ \mathcal{C}_k = \left\{U_k^{(0)}, \ldots, U_1^{(0)}\right\} \cup \mathcal{C}_0\ ,\ \textit{for}\ k \geq 1\ ,$$

we have :

i) *For all $k \in \{1, \ldots, m_0\}$, $U_k^{(0)}$ is the unique minimizer of*

$$H(U, \mathcal{C}_k \setminus \{U\}) - f(U), \quad U \in \mathcal{C}_k \setminus \mathcal{C}_0\ .$$

ii) *For all $k \in \{1, \ldots, m_0\}$, the pair $\left(\{U_k^{(0)}\}, \mathcal{C}_{k-1}\right)$ admits a unique (one point)-saddle set $\{z_k^*\}$.*

[1] or more shortly, a unique saddle point,

It is possible to check that this hypothesis is generically satisfied. More precisely, it is satisfied if all the critical values of f are distinct and all the quantities $f(U^{(1)}) - f(U^{(0)})$, with $U^{(1)} \in \mathcal{U}^{(1)}$ and $U^{(0)} \in \mathcal{U}^{(0)}$ are distinct. By its definition, the point z_k^* is a critical point with index 1, $z_k^* \in \mathcal{U}^{(1)}$.

Definition 2.4 *(The map j)*
If the critical points of index 1 are denoted by $U_j^{(1)}$, $j = 1, \ldots, m_1$, we define the application $k \rightarrow j(k)$ on $\{1, \ldots, m_0\}$ by :

$$U_{j(k)}^{(1)} = z_k^* \,. \tag{2.1}$$

With these definitions, one can prove :

Proposition 2.5
Under Assumption 2.3, the following properties are satisfied :
a) *The sequence $\left(f(U_{j(k)}^{(1)}) - f(U_k^{(0)})\right)_{k \in \{1,\ldots,m_0\}}$ is strictly decreasing.*
b) *The application $j : \{1, \ldots, m_0\} \rightarrow \{1, \ldots, m_1\}$ is injective.*

3 Rough semi-classical analysis of Witten Laplacians and applications to Morse theory

3.1 Previous results

It is known (see [Sim], [Wit], [HelSj2] and more recently [CL], [HelNi2]) that the Witten Laplacian on functions $\Delta_{f,h}^{(0)}$ admits in the interval $[0, h^{\frac{6}{5}}]$ and for $h > 0$ small enough exactly m_0 eigenvalues, where m_0 is the number of local minima of f in Ω.

This is easy to guess by considering, near each of the local minima $U_j^{(0)}$, of f the function

$$u_j^{wkb,(0)}(x) := \chi_j(x) \exp -\frac{f(x) - f(U_j^{(0)})}{h} \,, \tag{3.1}$$

where χ_j is a suitable cut-off function localizing near $U_j^{(0)}$ as suitable quasi-mode. This shows, via the minimax principle, that these eigenvalues are actually exponentially small as $h \to 0$.

Note that we consider the Dirichlet problem. So the first part of Assumption 1.3 implies that the eigenfunctions corresponding to low lying eigenvalues are localized far from the boundary.

3.2 Witten Laplacians on p-forms

Although we are mainly interested in the operator $\Delta_{h,f}^{(0)}$, we will also meet in the proof other Laplacians. The spectral analysis can be extended (see Simon [Sim], Witten [Wit], Helffer-Sjöstrand [HelSj1], Chang-Liu [CL]) to Laplacians on p-forms, $p \geq 1$. We recall that Witten is first introducing a dixtorted De Rham complex d :

$$d_{f,h} = \exp -\frac{f}{h}(hd) \exp \frac{f}{h} = hd + df\wedge \,, \tag{3.2}$$

The restriction of $d_{f,h}$ to p-forms is denoted by $d_{f,h}^{(p)}$. Witten then associates to this complex the Laplacian :

$$\Delta_{f,h} = (d_{f,h} + d_{f,h}^*)^2 \,, \tag{3.3}$$

where $d_{f,h}^*$ is its L^2-adjoint. By restriction to the p-forms, one gets the Witten Laplacian $\Delta_{f,h}^{(p)}$.

3.3 Morse Inequalities

In the compact case, the analysis of the low lying eigenvalues of the Witten Laplacians was the main point of the semi-classical proof suggested by Witten of the Morse inequalities [Wit, Sim, HelSj2].

Each of the Witten Laplacians is essentially selfadjoint and an analysis based on the harmonic approximation (consisting in replacing f by its quadratic approximation at a critical point of f) shows that the dimension of the eigenspace corresponding to $[0, h^{\frac{6}{5}}]$ is, for h small enough, equal to m_p the number of critical points of index p.

Note that the dimension of the kernel of $\Delta_{h,f}^{(p)}$ being equal to the Betti number b_p, this gives immediately the so called " Weak Morse Inequalities" :

$$b_p \leq m_p \,, \quad \text{for all } p \in \{0, \dots, n\} \,. \tag{3.4}$$

4 Main result in the case of $\mathbb{R}^n$

In the case of $\mathbb{R}^n$ and under Assumptions 1.1, 1.2 and 2.3, the main result is the following :

Theorem 4.1 *The first eigenvalues $\lambda_k(h)$, $k \in \{2, \dots, m_0\}$, of $\Delta^{(0)}_{f,h}$ have the following expansions :*

$$\lambda_k(h) = \frac{h}{\pi} |\widehat{\lambda}_1(U^{(1)}_{j(k)})| \sqrt{\frac{\left|\det(\mathrm{Hess} f(U^{(0)}_k))\right|}{\left|\det(\mathrm{Hess} f(U^{(1)}_{j(k)}))\right|}} \times \exp -\frac{2}{h}\left(f(U^{(1)}_{j(k)}) - f(U^{(0)}_k)\right) \times (1 + r_1(h)) \ ,$$

with $r_1(h) = o(1)$.

Here the $U^{(0)}_k$ denote the local minima of f ordered in some specific way (see Section 2), the $U^{(1)}_{j(k)}$ are "saddle points" (critical points of index 1), attached to $U^{(0)}_k$ via the map j, and $\widehat{\lambda}_1(U^{(1)}_{j(k)})$ is the negative eigenvalue of $\mathrm{Hess} f(U^{(1)}_{j(k)})$. Actually, the estimate

$$r_1(h) = \mathcal{O}(h^{\frac{1}{2}} |\log h|) \ ,$$

is obtained in [BoGayKl] (under weaker assumptions on f) and the complete asymptotics,

$$r_1(h) \sim \sum_{j \geq 1} r_{1j} h^j \ ,$$

is proved in [HKN].

In the above statement, we have left out the case $k = 1$, which leads to a specific assumption (see Assumption 1.2) in the case of $\mathbb{R}^n$ for f at ∞. This implies that $\Delta^{(0)}_{f,h}$ is essentially selfadjoint and that the bottom of the essential spectrum is bounded below by some $\epsilon_0 > 0$ (independently of $h \in]0, h_0]$, h_0 small enough). If the function $\exp -\frac{f}{h}$ is in L^2, then

$$\lambda_1(h) = 0 \ .$$

In this case, denoting by Π_0 the orthogonal projector on $\exp -\frac{f}{h}$, the main motivation for having a good control of $\lambda_2(h)$ is the estimate, for $t > 0$,

$$\exp -t\Delta^{(0)}_{h,f} - \Pi_0 ||_{\mathcal{L}(L^2(\Omega))} \leq \exp -t\lambda_2(h) \ .$$

In other words, the estimate of $\lambda_2(h)$ permits to measure the rate of the return to equilibrium.
Note finally that other examples like $f(x) = -(x^2-1)^2$ (with $n=1$) are interesting and an asymptotic of $\lambda_1(h)$ can be given for this example.

5 About the proof in the case of $\mathbb{R}^n$.

5.1 Preliminaries

The case of $\mathbb{R}^n$ requires some care (see [Jo], [He] or [HelNi1]) for controlling the problem at infinity. The approach given in [HKN] intensively uses, together with the techniques of [HelSj2], the two following facts :

- The Witten Laplacian is associated to a cohomology complex.
- The function $\exp -\frac{f(x)}{h}$ is, as a distribution, in the kernel of the Witten Laplacian on $0-$forms.

This permits to construct very easily and efficiently – and this is specific of the case of $\Delta^{(0)}_{f,h}-$ quasimodes. We note that we have between differential operators acting on C_0^∞ the relations

$$d^{(0)}_{f,h}\Delta^{(0)}_{f,h} = \Delta^{(1)}_{f,h}d^{(0)}_{f,h}\ , \tag{5.1}$$

and

$$\Delta^{(0)}_{f,h}d^{(0),*}_{f,h} = d^{(0),*}_{f,h}\Delta^{(1)}_{f,h}\ , \tag{5.2}$$

This shows in particular that if u is an eigenvector of $\Delta^{(0)}_{h,f}$ for an eigenvalue $\lambda \neq 0$, then $d_{h,f}u$ is an eigenvector of $\Delta^{(1)}_{f,h}$ for λ.

5.2 Witten complex, reduced Witten complex

It is more convenient to consider the singular values of the restricted differential $d_{f,h}\ :\ F^{(0)} \to F^{(1)}$. The space $F^{(\ell)}$ is the m_ℓ-dimensional spectral subspace of $\Delta^{(\ell)}_{f,h}$, $\ell \in \{0,1\}$,

$$F^{(\ell)} = \mathrm{Ran}\, 1_{I(h)}(\Delta^{(\ell)}_{f,h})\ ,$$

with $I(h) = [0, h^{\frac{6}{5}}]$ and the property

$$1_{I(h)}(\Delta^{(1)}_{f,h}) d_{f,h} = d_{f,h} 1_{I(h)}(\Delta^{(0)}_{f,h}) \ .$$

We will analyze :

$$\beta^{(\ell)}_{f,h} := (d^{(\ell)}_{f,h})_{/F^{(\ell)}} \ .$$

We will mainly concentrate on the case $\ell = 0$.

5.3 Singular values

In order to exploit all the information which can be extracted from well chosen quasimodes, working with singular values s_j of $\beta^{(0)}_{f,h}$ happens to be more efficient than considering their squares, that is the eigenvalues $\lambda_j = s_j^2$ of $\Delta^{(0)}_{f,h}$. The main point[2] is probably that one can choose suitable approximate well localized "almost" orthogonal basis of $F^{(0)}$ and $F^{(1)}$ **separately** and that the errors appear "multiplicatively" when computing the singular values of $\beta^{(0)}_{f,h}$. By this we mean :

$$s_j = s_j^{app}(1 + \varepsilon_1(h)) \ ,$$

instead of additively

$$\lambda_j := s_j^2 = (s_j^{app})^2 + \varepsilon_2(h) \ ,$$

as for example in [HelSj2]. Here s^j_{app} will be explicitly obtained from the WKB analysis. In the first case, it is actually enough to prove that $\varepsilon_1(h) = \mathcal{O}(h^\infty)$. In the second case, the analysis of [HelSj2] gives a control of $\varepsilon_2(h)$ in $\mathcal{O}(\exp -\frac{S}{h})$, with $S > 2 \inf_{j,k}(f(U_j^{(1)}) - f(U_k^{(0)}))$, which is enough for estimating the highest low lying eigenvalue (see [HelNi1]) but could be unsatisfactory for the lowest strictly positive eigenvalue, as soon as the number of local minima is > 2. Although it is not completely hopeless to have a better control of $\varepsilon_2(h)$ by improving the analysis of [HelSj2] and introducing a refined notion of non resonant wells, the approach developed in [HelNi2] appears to be simpler.

[2] See also [Ni] for a pedagogical discussion,

6 The main result in the case with boundary

In the case with boundary, the function $\exp -\frac{f}{h}$, which is the only distribution in the kernel of $\Delta^{(0)}_{h,f}$, does not satisfy the Dirichlet condition, so the smallest eigenvalue can not be 0. The estimate of $\lambda_1(h)$ permits consequently to measure the decay of $|| \exp -t\Delta^{(0)}_{h,f}|| \leq \exp -t\lambda_1(h)$ as $t \to +\infty$.

For this case, a starting reference is the book by Freidlin-Wentzel [FrWe], which says (in particular) that, if f has a unique critical point, corresponding to a non degenerate local minimum $U^{(0)}_{min}$, then the lowest eigenvalue $\lambda_1(h)$ of the Dirichlet realization $\Delta^{(0)}_{f,h}$ in Ω satisfies :

$$\lim_{h\to 0} -h \log \lambda_1(h) = 2 \inf_{x\in\partial\Omega} (f(x) - f(U^{(0)}_{min})) .$$

Other results are given in this book for the case of many local minima but they are again limited to the determination of logarithmic equivalents.
We have explained in Section 2 that, under Assumption 2.3, one can label the m_0 local minima and associate via the map j from the set of the local minima into the set of the m_1 (generalized) saddle points of the Morse functions in $\overline{\Omega}$ of index 1 .
The main theorem of Helffer-Nier [HelNi2] is :

Theorem 6.1 *Under Assumptions 1.1, 1.3 and 2.3, there exists h_0 such that, for $h \in (0, h_0]$, the spectrum in $[0, h^{\frac{3}{2}})$ of the Dirichlet realization of $\Delta^{(0)}_{f,h}$ in Ω, consists of m_0 eigenvalues $\lambda_1(h) < \ldots < \lambda_{m_0}(h)$ of multiplicity 1, which are exponentially small and admit the following asymptotic expansions :*

$$\begin{aligned}\lambda_k(h) = \frac{h}{\pi}|\widehat{\lambda}_1(U^{(1)}_{j(k)})| \sqrt{\frac{\left|\det(\mathrm{Hess} f(U^{(0)}_k))\right|}{\left|\det(\mathrm{Hess} f(U^{(1)}_{j(k)}))\right|}}\, (1 + hc^1_k(h)) \times \\ \times \exp -\frac{2}{h}\left(f(U^{(1)}_{j(k)}) - f(U^{(0)}_k)\right) \quad , \quad \textit{if } U^{(1)}_{j(k)} \in \Omega ,\end{aligned}$$

and

$$\begin{aligned}\lambda_k(h) = \frac{2h^{1/2}|\nabla f(U^{(1)}_{j(k)})|}{\pi^{1/2}} \sqrt{\frac{\left|\det(\mathrm{Hess} f(U^{(0)}_k))\right|}{\left|\det(\mathrm{Hess} f\big|_{\partial\Omega}(U^{(1)}_{j(k)}))\right|}}\, (1 + hc^1_k(h)) \times \\ \times \exp -\frac{2}{h}\left(f(U^{(1)}_{j(k)}) - f(U^{(0)}_k)\right) \quad , \quad \textit{if } U^{(1)}_{j(k)} \in \partial\Omega .\end{aligned}$$

Here $c_k^1(h)$ admits a complete expansion :

$$c_k^1(h) \sim \sum_{m=0}^{\infty} h^m c_{k,m} \; .$$

This theorem extends to the case with boundary the previous results of [BoGayKl] and [HKN] (see also the books [FrWe] and [Kol] and references therein).

7 About the proof in the case with boundary

As in [HelSj2], the proof is deeply connected with the analysis of the small eigenvalues of a suitable realization (which is **not** the Dirichlet realization) of the Laplacian on the 1-forms. In order to understand the strategy, three main points have to be explained.

7.1 Define the Witten complex and the associate Laplacian.

The case of a compact manifold without boundary was treated in the foundational paper of Witten [Wit]. A finer (and rigorous) analysis is then given in [HelSj2] and further developments appear in [HKN]. The case with boundary creates specific new problems.
Our starting problem being the analysis of the Dirichlet realization of the Witten Laplacian on functions, we were let to find the right realization of the Witten Laplacian on 1-forms in the case with boundary in order to extend the commutation relation (5.1) in a suitable "strong" sense (at the level of the selfadjoint realizations).

The answer was actually present in the literature [CL] in connection with the analysis of the relative cohomology and the proof of the Morse inequalities. Let us explain how we can guess the right condition by looking at the eigenvectors.

If u is eigenvector of the Dirichlet realization of $\Delta_{f,h}^{(0)}$, then by commutation relation, $d_{f,h}^{(0)}u$ (which can be identically 0) should be an eigenvector in the domain of the realization of $\Delta_{f,h}^{(1)}$. But $d_{f,h}^{(0)}u$ does not necessarily satisfy the Dirichlet condition in all its components, but only in its tangential components.

This is the natural condition that we keep in the definition of the variational domain to take for the quadratic form $\omega \mapsto ||d^{(1)}_{f,h}\omega||^2 + ||d^{(0)\,*}_{f,h}\omega||^2$. The selfadjoint realization $\Delta^{(1)\,DT}_{f,h}$ obtained as the Friedrichs extension associated to the quadratic form gives the right answer.

Observing also that $d^{(0),*}_{f,h}(d^{(0)}_{f,h}u) = \lambda u$ (with $\lambda \neq 0$), we get the second natural (Neumann type)-boundary condition saying that a one form ω in the domain of the operator $\Delta^{(1)\,DT}_{f,h}$ should satisfy

$$d^{(0),*}_{f,h}\omega_{/\partial\Omega} = 0 \; . \tag{7.1}$$

So we have shown that the natural boundary conditions for the Witten Laplacians are (7.1) together with

$$\omega_{/\partial\Omega} = 0 \; . \tag{7.2}$$

The associated cohomology is called relative (see for example [Schw]).

7.2 Rough localization of the spectrum of this Laplacian on 1-forms.

The analysis of $\Delta^{(p)}_{f,h}$ was performed in [CL], in the spirit of Witten's idea, extending the so called Harmonic approximation. But these authors, because they were interested in the Morse theory, used the possibility to add symplifying assumptions on f and the metric near the boundary. We emphasize that [HelNi2] treats the general case. May be one could understand what is going on at the boundary by analyzing the models corresponding to $f(x', x_n) = \frac{1}{2}|x'|^2 + \epsilon x_n$, with $\epsilon = \pm 1$ in $\mathbb{R}^n_+ = \{x_n > 0\}$. The analysis in this case is easily reduced to the analysis of the one dimension case on $\mathbb{R}^+$ (together with the standard analysis of $\mathbb{R}^{n-1}$). The Dirichlet Laplacian to analyze is simply :

$$-h^2 \frac{d^2}{dx^2} + 1 \; ,$$

on $\mathbb{R}^+$, which is strictly positive, but the Laplacian on 1-forms is

$$u(x)\, dx \mapsto (-h^2 u''(x) + u(x))\, dx \; ,$$

on $\mathbb{R}^+$, but with the boundary condition :

$$hu'(0) - \epsilon u(0) = 0 \; .$$

Depending on the sign of ϵ, the bottom of the spectrum is 0 if $\epsilon < 0$ or 1 if $\epsilon > 0$. This explains our definition of critical point of index 1 at the boundary.

7.3 Construction of WKB solutions attached to the critical points of index 1.

The construction of the approximate basis of $F^{(0)}$ and $F^{(1)}$ is obtainedthrough WKB constructions. The constraints are quite different in the two cases. For $F^{(0)}$, we need rather accurate quasimodes but can take advantage of their simple structure given in (3.1). The difficulty is concentrated in the choice of χ_j. For $F^{(1)}$, it is enough to construct quasimodes lecalized in a small neighborhood of a critical point of index 1. This was done in [HelSj2] for the case without boundary, as an extension of previous constructions of [HelSj1]. The new point is the construction of WKB solutions near critical points of the restriction of the Morse function at the boundary, which is done in [HelNi2] for 1-forms. Let us explain the main lines of the construction.

The construction is done locally around a local minimum U_0 of $f\big|_{\partial\Omega}$ with $\partial_n f(U_0) > 0$. The function Φ is a local solution of the eikonal equation

$$|\nabla\Phi|^2 = |\nabla f|^2 \; ,$$

which also satisfies

$$\Phi = f \text{ on } \partial\Omega$$

and

$$\partial_n \Phi = -\partial_n f \text{ on } \partial\Omega$$

and we normalize f so that $f(U_0) = f(0) = 0$.
We first consider a local solution u_0^{wkb} near the point $x = 0$ of

$$e^{\frac{\Phi}{h}} \Delta_{f,h}^{(0)} u_0^{wkb} = \mathcal{O}(h^\infty) \; ,$$

with u_0^{wkb} in the form

$$u_0^{wkb} = a(x,h) e^{-\frac{\Phi}{h}} \; ,$$

$$a(x,h) \sim \sum_{j\geq 0} a_j(x)h^j \ ,$$

and the condition at the boundary

$$a(x,h)e^{-\frac{\Phi}{h}} = e^{-\frac{f}{h}} \quad \text{on } \partial\Omega \ ,$$

which leads to the condition

$$a(x,h)\big|_{\partial\Omega} = 1 \ .$$

In order to verify locally the boundary condition for our future u_1^{wkb}, we substract $e^{-\frac{f}{h}}$ and still obtain

$$e^{\frac{\Phi}{h}} \ \Delta_f^{(0)}(u_0^{wkb} - e^{-\frac{f}{h}}) = \mathcal{O}(h^\infty) \ . \tag{7.3}$$

We now define the WKB solution u_1^{wkb} by considering :

$$u_1^{wkb} := d_{f,h}u_0^{wkb} = d_{f,h}(u_0^{wkb} - e^{-\frac{f}{h}}) \ .$$

The 1-form $u_1^{wkb} = d_{f,h}u_0^{wkb}$ satisfies locally the Dirichlet tangential condition on the boundary (7.2) and, using (7.3), (modulo $\mathcal{O}(h^\infty)$) the Neumann type condition (7.1) is satisfied. So u_1^{wkb} gives a good approximation for a ground state of a suitable realization of $\Delta_{f,h}^{(1)}$ in a neighborhood of this boundary critical point.

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