Spectral asymptotics for periodic fourth order operators

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Abstract

We consider the operator $\frac{d^4}{dt^4}+V$ on the real line with a real periodic potential V. The spectrum of this operator is absolutely continuous and consists of intervals separated by gaps. We define a Lyapunov function which is analytic on a two sheeted Riemann surface. On each sheet, the Lyapunov function has the same properties as in the scalar case, but it has branch points, which we call resonances. We prove the existence of real as well as non-real resonances for specific potentials. We determine the asymptotics of the periodic and anti-periodic spectrum and of the resonances at high energy. We show that there exist two type of gaps: 1) stable gaps, where the endpoints are periodic and anti-periodic eigenvalues, 2) unstable (resonance) gaps, where the endpoints are resonances (i.e., real branch points of the Lyapunov function above the bottom of the spectrum). We also show that the periodic and anti-periodic spectrum together determine the spectrum of our operator. Finally, we show that for small potentials $V \neq 0$ the spectrum in the lowest band has multiplicity 4 and the bottom of the spectrum is a resonance, and not a periodic (or anti-periodic) eigenvalue.

1 Introduction and main results

We consider the self-adjoint operator $\mathcal{L}=\frac{d^4}{dt^4}+V$, acting on $L^2(\mathbb{R})$, where the real 1-periodic potential V belongs to the real space $L^1_0(\mathbb{T})=\{V\in L^1(\mathbb{T}),\int_0^1 V(t)dt=0\}$, $\mathbb{T}=\mathbb{R}/\mathbb{Z}$, equipped with the norm $\|V\|=\int_0^1|V(t)|dt<\infty$. It is well known (see [DS]) that the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ is absolutely continuous and consists of non-degenerate intervals. These intervals are separated by the gaps $G_n=(E_n^-,E_n^+), n\geqslant 1$, with length $|G_n|>0$. Introduce the fundamental solutions $\varphi_j(t,\lambda), j=0,1,2,3$, of the equation

$$y''''+Vy=\lambda y, \qquad (t,\lambda)\in\mathbb{R}\times\mathbb{C}, \tag{1.1}$$

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satisfying the following conditions: $\varphi_j^{(k)}(0,\lambda)=\delta_{jk}, j,k=0,...,3$, where δ_{jk} is the standard Kronecker symbol. Here and below we use the notation $f'=\frac{\partial f}{\partial t}, f^{(k)}=\frac{\partial^k f}{\partial t^k}$. We define the monodromy 4×4-matrix M by

$$M(\lambda)=\mathcal{M}(1,\lambda), \quad \mathcal{M}(t,\lambda)=\{\mathcal{M}_{kj}(t,\lambda)\}_{j,k=0}^3=\{\varphi_j^{(k)}(t,\lambda)\}\}_{j,k=0}^3. \tag{1.2}$$

The matrix valued function M is entire. An eigenvalue of $M(\lambda)$ is called a *multiplier*. It is a root of the algebraic equation $D(\tau,\lambda)=0$, where $D(\tau,\lambda)\equiv\det(M(\lambda)-\tau I_4), \tau,\lambda\in\mathbb{C}$. Let $D_\pm(\lambda)=\frac14 D(\pm1,\lambda)$. The zeros of $D_+(\lambda)$ (or $D_-(\lambda)$) are the eigenvalues of the periodic (anti-periodic) problem for the equation $y''''+Vy=\lambda y$. Denote by $\lambda_0^+, \lambda_{2n}^\pm, n=1,2,...$ the sequence of zeros of D_+ (counted with multiplicity) such that $\lambda_0^+\leqslant\lambda_2^-\leqslant\lambda_2^+\leqslant\lambda_4^-\leqslant\lambda_4^+\leqslant\lambda_6^-\leqslant...$ Denote by $\lambda_{2n-1}^\pm, n=1,2,...$ the sequence of zeros of D_- (counted with multiplicity) such that $\lambda_1^-\leqslant\lambda_1^+\leqslant\lambda_3^-\leqslant\lambda_3^+\leqslant\lambda_5^-\leqslant\lambda_5^+\leqslant....$

A great number of papers is devoted to the inverse spectral theory for the Hill operator. We mention all papers where the inverse problem including characterization was solved: Marchenko and Ostrovski [MO], Garnett and Trubowitz [GT1-2],Kappeler [Kap], Kargaev and Korotyaev [KK1], and Korotyaev [K1-3] and for 2×2 Dirac operator [Mi1-2], [K4-5]. Recently, one of the authors [K6] extended the results of [MO], [GT1], [K1-2] of the case $-y''+uy$ to the case of distributions, i.e. $-y''+u'y$ on $L^2(\mathbb{R})$, where $u\in L^2_{loc}(\mathbb{R})$ is periodic.

There exist many papers about the periodic systems $N\geqslant 2$ (see [YS]). The basic results for the direct spectral theory for the matrix case were obtained by Lyapunov [Ly] (see also the interesting papers of Krein [Kr], Gel'fand and Lidskii [GL]). The operator $-\frac{d^2}{dt^2}+\mathcal{V}$ on the real line where $\mathcal{V}$ is a 1-periodic 2×2 matrix potential was considered in [BBK]. The following results are obtained: the Lyapunov function is constructed as an analytic function on a 2-sheeted Riemann surface and the existence of real and complex resonances are proved for some specific potentials. Recall the well-known Lyapunov Theorem, in a formulation adapted for our case (see [Ly],[YS]):

Theorem (Lyapunov) *Let $V\in L_0^1(\mathbb{T})$. Then*

$$D(\tau,\cdot)=\tau^4 D(\tau^{-1},\cdot), \quad \tau\neq 0. \tag{1.3}$$

If for some $\lambda\in\mathbb{C}$ (or $\lambda\in\mathbb{R}$) $\tau(\lambda)$ is a multiplier of multiplicity $d\geqslant 1$, then $\tau^{-1}(\lambda)$ (or $\overline{\tau}(\lambda)$) is a multiplier of multiplicity d. Moreover, each $M(\lambda),\lambda\in\mathbb{C}$ has exactly four multipliers $\tau_1^{\pm1}(\lambda),\tau_2^{\pm1}(\lambda)$. Furthermore, $\lambda\in\sigma(\mathcal{L})$ iff $|\tau_1(\lambda)|=1$ or $|\tau_2(\lambda)|=1$. If $\tau(\lambda)$ is a simple multiplier and $|\tau(\lambda)|=1$, then $\tau'(\lambda)\neq 0$.

The spectral problems for the fourth order periodic operator were the subject of many papers (see [P1-2],[PK1-2], [GO], [YS]). Firstly, we mention the papers of Papanicolaou [P1-2] devoted to the Euler-Bernoulli equation $(ay'')''=\lambda by$ with the periodic functions a,b. For this case he defines the Lyapunov function and obtains some properties of this function. In particular, it is proved that the Lyapunov function is analytic on some two sheeted Riemann surface. It is important that for this case he proved that all branch points of the Lyapunov function are real and $\leqslant 0$. Note that in our case we have the example, Proposition 1.4, where the Lyapunov function has real and non-real branch points. This the main difference between our Lyapunov function and his one. Moreover, Papanicolaou proved that if all the gaps are closed and the Lyapunov function is entire in $\sqrt{\lambda}$, then the functions a, b are constants. Secondly, Papanicolaou and Kravvaritis [PK1-2] considered some inverse problems for the Euler-Bernoulli equation. Thirdly, Galunov, Oleinik [GO] considered the operator $y^{(2n)} + \gamma\delta_{per}(t)y, \gamma \in \mathbb{R}$, on the real line with the periodic delta-potential $\delta_{per}(t)$. They study the spectrum in the lowest band. It can be seen from the results of this paper, that the spectrum in this band has multiplicity 4, for $n = 2$ and for some γ. Recall that (see Theorem 1.3) we prove a stronger result and show that the spectrum has multiplicity 4 in the lowest band for each sufficiently small potential.

We introduce the functions

$$T_m = \frac{1}{4}\,\mathrm{Tr}\, M^m, \quad m \geqslant 1, \quad \rho = \frac{T_2 + 1}{2} - T_1^2. \tag{1.4}$$

The functions T_1, T_2, ρ are real on $\mathbb{R}$ and entire. In the case $V = 0$ the corresponding functions have the forms

$$T_m^0 = \frac{\cos mz + \cosh mz}{2}, \quad m = 1, 2, \ \rho^0 = \frac{(\cos z - \cosh z)^2}{4}, \quad z = \lambda^{\frac{1}{4}}, \tag{1.5}$$

here and below $\arg z \in (-\frac{\pi}{4}, \frac{\pi}{4}]$. It is known (see [RS]) that $D(\tau, \lambda) = \sum_0^4 \xi_m(\lambda)\tau^{4-m}$, where the functions ξ_m are given by

$$\xi_0 = 1, \quad \xi_1 = -4T_1, \quad \xi_2 = -2(T_2 + T_1\xi_1), \ \ldots .$$

Then using the identity (1.3) we obtain $D(\tau, \cdot) = (\tau^4 + 1) + \xi_1(\tau^3 + \tau) + \xi_2\tau^2$, which yields

$$D(\tau, \cdot) = \Big(\tau^2 - 2(T_1 - \sqrt{\rho})\tau + 1\Big)\Big(\tau^2 - 2(T_1 + \sqrt{\rho})\tau + 1\Big). \tag{1.6}$$

We introduce the domains

$$\mathcal{D}_r = \Big\{\lambda \in \mathbb{C} : |\lambda^{1/4}| > r, \ \ |\lambda^{1/4} - (1 \pm i)\pi n| > \frac{\pi}{4}, |\lambda^{1/4} - \pi n| > \frac{\pi}{4}, n \geqslant 0\Big\}, \quad r \geqslant 0. \tag{1.7}$$

We have $\rho(\lambda) = \rho^0(\lambda)(1 + o(1))$ as $|\lambda| \to \infty, \lambda \in \mathcal{D}_1$ (see Lemma 5.1). Then we define the analytic function $\sqrt{\rho(\lambda)}, \lambda \in \mathcal{D}_r$ for some large $r > 0$, by the condition $\sqrt{\rho(\lambda)} = \sqrt{\rho^0(\lambda)}(1 + o(1))$ as $|\lambda| \to \infty, \lambda \in \mathcal{D}_r$, where $\sqrt{\rho^0(\lambda)} = (\cos z - \cosh z)/2$.

The function ρ is real on $\mathbb{R}$, then r is a root of ρ iff $\overline{r}$ is a root of ρ. By Lemma 5.1, for large integer N the function $\rho(\lambda)$ has exactly $2N + 1$ roots, counted with multiplicity, in the disk $\{\lambda : |\lambda| < 4(\pi(N + \frac{1}{2}))^4\}$ and for each $n > N$, exactly two roots, counted with multiplicity, in the domain $\{\lambda : |\lambda^{1/4} - \pi(1 + i)n| < \pi/4\}$. There are no other roots. Thus the function $\rho(\lambda)$ has an odd number $\geqslant 1$ of real zeros (counted with multiplicity) on the real interval $(-\Gamma, \Gamma) \subset \mathbb{R}, \Gamma = 4(\pi(N + \frac{1}{2}))^4$.

Let $\{r_0^-, r_n^\pm\}_1^\infty$ be the sequence of zeros of ρ in $\mathbb{C}$ (counted with multiplicity) such that:
r_0^- is the maximal real zero, and $.. \leqslant \mathrm{Re}\, r_{n+1}^+ \leqslant \mathrm{Re}\, r_n^+ \leqslant ... \leqslant \mathrm{Re}\, r_1^+$,
if $r_n^+ \in \mathbb{C}_+$, then $r_n^- = \overline{r_n^+} \in \mathbb{C}_-$,
if $r_n^+ \in \mathbb{R}$, then $r_n^- \leqslant r_n^+ \leqslant \mathrm{Re}\, r_{m-1}^-, m = 1, .., n$.

Below we will show that $r_n^\pm = -4(\pi n)^4 + 0(n^2)$ as $n \to \infty$, see Lemma 5.1. We call a zero of ρ a resonance of $\mathcal{L}$. Let $... \leqslant r_{n_j}^- \leqslant r_{n_j}^+ \leqslant ... \leqslant r_{n_1}^- \leqslant r_{n_1}^+ \leqslant r_0^-$ be the subsequence of the real zeros of ρ. Then $\rho(\lambda) < 0$ for any $\lambda \in \gamma_j^0 = (r_{n_{j+1}}^+, r_{n_j}^-), j \geqslant 1$. We call an interval $\gamma_j^0 \subset \mathbb{R}$ a resonance gap.

We construct the Riemann surface $\mathcal{R}$ for $\sqrt{\rho}$. For any $r_n^+ \in \mathbb{C}_+$ we take some curve η_n, which joins the points $r_n^+, \overline{r_n^+}$ and does not cross $\gamma^0 = \cup\gamma_n^0$. To "build" the surface $\mathcal{R}$, we take two replicas of the λ-plane cut along γ^0 and $\cup\eta_n$ and call them sheet $\mathcal{R}_1$ and sheet $\mathcal{R}_2$. The cut on each sheet has two edges; we label each edge with a + or a −. Then attach the − edge of the cut on $\mathcal{R}_1$ to the + edge of the cut on $\mathcal{R}_2$ and attach the + edge of the cut on $\mathcal{R}_1$ to the − edge of the cut on $\mathcal{R}_2$. Thus, whenever we cross the cut, we pass from one sheet to the other. There exists a unique analytic continuation of the function $\sqrt{\rho}$ from $\mathcal{D}_r$ into the two sheeted Riemann surface $\mathcal{R}$ of the function $\sqrt{\rho}$. Let below $\zeta \in \mathcal{R}$ and let $\phi(\zeta) = \lambda$ be the natural projection $\phi : \mathcal{R} \to \mathbb{C}$.

We introduce the Lyapunov function by

$$\Delta(\zeta) = T_1(\zeta) + \sqrt{\rho(\zeta)}, \qquad \zeta \in \mathcal{R}. \tag{1.8}$$

Note that $T_1(\zeta) = T_1(\lambda)$, since T_1 is entire. Let $\Delta(\zeta) = \Delta_1(\lambda)$ on the first sheet $\mathcal{R}_1$, $\Delta(\zeta) = \Delta_2(\lambda)$ on the second sheet $\mathcal{R}_2$. Then

$$\Delta_1(\lambda) = T_1(\lambda) + \sqrt{\rho(\lambda)}, \quad \Delta_2(\lambda) = T_1(\lambda) - \sqrt{\rho(\lambda)}, \quad \lambda = \phi(\zeta). \tag{1.9}$$

Now we formulate our first result about the function $\Delta(\lambda)$.

Theorem 1.1. *Let $V \in L_0^1(\mathbb{T})$. Then the function $\Delta = T_1 + \sqrt{\rho}$ is analytic on the two sheeted Riemann surface $\mathcal{R}$ and the branches Δ_m of Δ have the forms*

$$\Delta_m(\lambda) = \frac{\tau_m(\lambda) + \tau_m^{-1}(\lambda)}{2}, \quad \lambda \in \mathcal{R}_m, \quad m = 1, 2, \tag{1.10}$$

and the following properties:
i) The following identities and asymptotics are fulfilled

$$\tau_1(\lambda) = e^{\lambda^{1/4} + O(\lambda^{-3/2})}, \quad \tau_2(\lambda) = e^{i\lambda^{1/4} + O(\lambda^{-3/2})}, \tag{1.11}$$

$$\Delta_1(\lambda) = \cosh \lambda^{1/4}\Big(1 + O(\lambda^{-3/2})\Big), \quad \Delta_2(\lambda) = \cos \lambda^{1/4}\Big(1 + O(\lambda^{-3/2})\Big) \tag{1.12}$$

as $|\lambda| \to \infty, \lambda \in \mathcal{D}_1$.
ii) $\lambda \in \sigma(\mathcal{L})$ iff $\Delta_m(\lambda) \in [-1, 1]$ for some $m = 1, 2$. Moreover, if $\lambda \in \sigma(\mathcal{L})$, then $\rho(\lambda) \geqslant 0$.
iii) The spectrum of $\mathcal{L}$ on an interval $S \subset \mathbb{R}$ has multiplicity 4 iff $-1 < \Delta_m(z) < 1$ for all $m = 1, 2, \lambda \in S$, except for finite number of points.
iv) the spectrum of $\mathcal{L}$ on an interval $S \subset \mathbb{R}$ has multiplicity 2 iff $-1 < \Delta_1(\lambda) < 1, \Delta_2(\lambda) \in \mathbb{R} \setminus [-1, 1]$ or $-1 < \Delta_2(\lambda) < 1, \Delta_1(\lambda) \in \mathbb{R} \setminus [-1, 1]$ for all $\lambda \in S$,except for finite number of points.
v) Let Δ_m be real analytic on some interval $I = (\alpha_1, \alpha_2) \subset \mathbb{R}$ and $-1 < \Delta_m(\lambda) < 1$, for any $\lambda \in I$ for some $m \in \{1, 2\}$. Then $\Delta_m'(\lambda) \neq 0$ for each $\lambda \in I$ (the monotonicity property).

vi) Each gap $G_n = (E_n^-, E_n^+), n \geqslant 1$ is a bounded interval and $E_n^\pm$ are either periodic (anti-periodic) eigenvalues or real branch point of Δ_m (for some $m = 1, 2$) which is a zero of ρ (that is a resonance).

Remark. i) In the case of the Hill operator the monodromy matrix has exactly 2 eigenvalues τ, τ^{-1}. The Lyapunov function $\frac{1}{2}(\tau + \tau^{-1})$ is an entire function of the spectral parameter. It defines the band structure of the spectrum. By Theorem 1.1, the Lyapunov function for the operator $\mathcal{L}$ also defines the band structure of the spectrum, but it is an analytic function on a 2-sheeted Riemann surface. The qualitative behavior of the Lyapunov function for small potentials is shown on Fig. 1.
ii) Recall that in the scalar case the spectrum of each spectral band has multiplicity 2 with a possible exception in the end points of the bands. For $\mathcal{L}$ this is similar.
iii) In the case $V = 0$ the corresponding functions have the forms

$$\Delta_1^0(\lambda) = \cosh z, \quad \Delta_2^0(\lambda) = \cos z, \qquad m = 1, 2, \qquad z = \lambda^{\frac{1}{4}}. \tag{1.13}$$

Thus the function $\Delta^0(\lambda) = \sum_0^\infty (-1)^n \frac{(\sqrt{\lambda})^n}{(2n)!}$ is analytic on the two-sheeted Riemann surface $\mathcal{R}^0$ of the function $\sqrt{\lambda}$, where $\sqrt{1} = 1$ on the first sheet $\mathcal{R}_1^0$. We have only one resonance gap $(-\infty, 0)$, since we have only one branch point, which equals zero. We also have

$$D_\pm^0 = (\cosh z \mp 1)(\cos z \mp 1), \quad \lambda_n^{\pm 0} = (\pi n)^4, \quad r_n^{\pm 0} = -4(\pi n)^4, \quad n \geqslant 1, \quad r_0^{-0} = 0, \ \lambda_0^{+0} = 0. \tag{1.14}$$

iv) We describe the difference between the Lyapunov functions for the operators $\mathcal{L}$ and the operator $\mathcal{L}_{sys} = -\frac{d^2}{dt^2} + \mathcal{V}$ on the real line where $\mathcal{V}$ is a 1-periodic 2×2 matrix potential from [BBK]:

1) For the operator $\mathcal{L}$ the Lyapunov function Δ_1 is increasing and Δ_2 is bounded on the real line at high energy. It creates some problem to determine the asymptotics of the spectral data for $\mathcal{L}$. Moreover, this implies that the spectrum of $\mathcal{L}$ has multiplicity 2 at high energy. For the operator $\mathcal{L}_{sys}$ all Lyapunov functions are bounded on the real line.

2) The resonances for $\mathcal{L}$ go to $-\infty$ and the resonances for the operator $\mathcal{L}_{sys}$ go to $+\infty$.

We formulate our theorem about the asymptotics of the periodic and anti-periodic eigenvalues and resonances at high energy and the recovering the spectrum of $\mathcal{L}$.

Theorem 1.2. *Let $V \in L_0^1(\mathbb{T})$. Then*
i) There exists an integer $N \geqslant 0$ such that for all $n \geqslant N$ the inequalities are fulfilled:

$$\lambda_n^- \leqslant \lambda_n^+ < \lambda_{n+1}^- \leqslant \lambda_{n+1}^+ < \lambda_{n+2}^- \leqslant \lambda_{n+2}^+ < ... \tag{1.15}$$

where the intervals $[\lambda_n^+, \lambda_{n+1}^-]$ are spectral bands of multiplicity 2 in $(\lambda_n^+, \lambda_{n+1}^-)$, and the intervals $(\lambda_n^-, \lambda_n^+)$ are gaps. Moreover, the following asymptotics are fulfilled:

$$\lambda_n^\pm = (\pi n)^4 \pm |\hat{V}_n| + O(n^{-3/2}), \qquad \hat{V}_n = \int_0^1 V(t) e^{-i2\pi n t} dt, \tag{1.16}$$

$$r_n^\pm = -4(\pi n)^4 \pm \sqrt{2} |\hat{V}_n| + O(n^{-3/2}), \quad n \to +\infty. \tag{1.17}$$

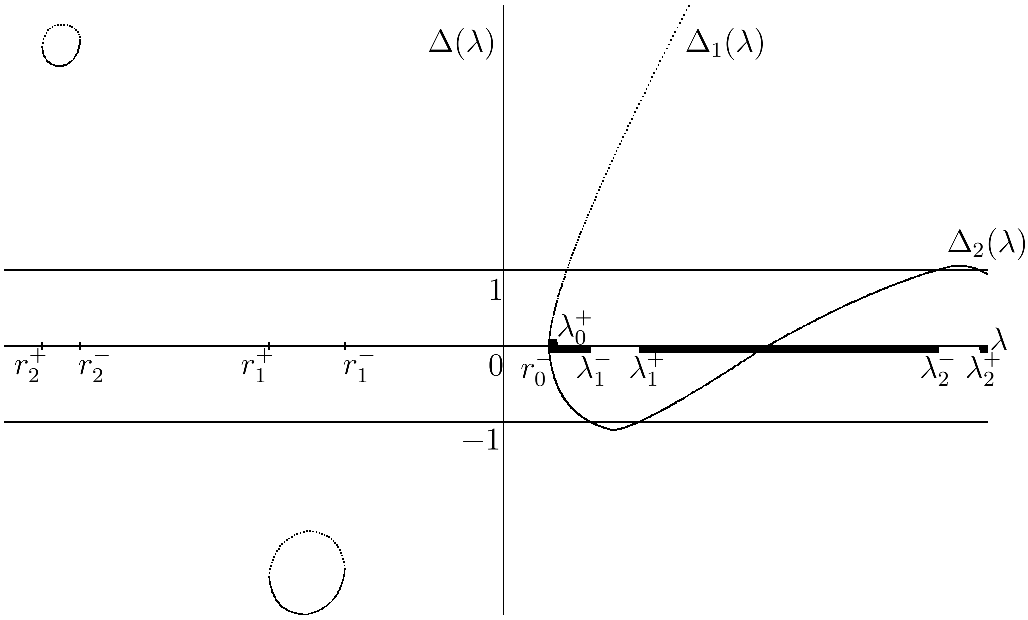


Figure 1: The function Δ for small V.

ii) The periodic spectrum and the anti-periodic spectrum recover the resonances and the spectrum of the operator $\mathcal{L}$.
iii) The periodic (anti-periodic) spectrum is recovered by the anti-periodic (periodic) spectrum and the resonances.

Remark. Assume that in Theorem 1.2 the potential V has the Fourier coefficients $\hat{V}_n = 1/n, |n| > n_*$ for some $n_* \in \mathbb{N}$. Using asymptotics (1.16) we deduce that there exist infinitely many gaps in the spectrum of $\mathcal{L}$ and infinitely many resonance gaps. Unfortunately, we can not construct a potential with a finite number of gaps in the spectrum of $\mathcal{L}$.

Consider the operator $\mathcal{L} = \frac{d^4}{dt^4} + \gamma V, V \in L_0^1(\mathbb{T})$ and real γ. We will show that for small $\gamma \neq 0$ the lowest spectral band of $\mathcal{L}$ contains an interval (r_0^-, λ_0^+) of multiplicity 4 (see Fig.1).

Theorem 1.3. *Let $\mathcal{L} = \frac{d^4}{dt^4} + \gamma V$, where $V \in L_0^1(\mathbb{T}), V \neq 0, \gamma \in \mathbb{R}$. Then there exist two real analytic functions $r_0^-(\gamma), \lambda_0^+(\gamma)$ in the disk $\{|\gamma| < \varepsilon\}$ for some $\varepsilon > 0$ such that $r_0^-(\gamma) < \lambda_0^+(\gamma)$ for all $\gamma \in (-\varepsilon, \varepsilon) \setminus \{0\}$. Here $r_0^-(\gamma)$ is a simple zero of the function $\rho(\lambda, \gamma V), r_0^-(0) = 0$ and $\lambda_0^+(\gamma)$ is a simple zero of the function $D_+(\lambda, \gamma V), \lambda_0^+(0) = 0$. Moreover, the following asymptotics are fulfilled:*

$$r_0^-(\gamma) = 2\gamma^2(4v_1 - v_2) + O(\gamma^3), \quad \lambda_0^+(\gamma) = 2\gamma^2(4v_1 - v_2) + O(\gamma^3), \tag{1.18}$$

$$\lambda_0^+(\gamma) - r_0^-(\gamma) = 4A^2\gamma^4 + O(\gamma^5), \quad A = \frac{v_2}{12} - \frac{4}{3}v_1 = \frac{5}{4}\sum_{n\neq 0} \frac{|\hat{V}_n|^2}{(2\pi n)^6} > 0, \tag{1.19}$$

as $\gamma \to 0$, where

$$\hat{V}_n = \int_0^1 V(t)e^{-i2\pi n}dt, \quad v_m = \frac{1}{144}\int_0^m dt \int_0^t v(s)v(t)(m-t+s)^3(t-s)^3 ds, \quad m = 1, 2. \tag{1.20}$$

Furthermore, the spectral interval $(r_0^-(\gamma), \lambda_0^+(\gamma))$ has multiplicity 4 for any $\gamma \in (-\varepsilon, \varepsilon) \setminus \{0\}$.

Consider the operator $\mathcal{L}^\gamma = \frac{d^4}{dt^4} + \gamma\delta_{per}, \gamma \in \mathbb{R}$, where $\delta_{per}(t) = \sum \delta(t-n)$. We prove that the function $\Delta(\lambda, \gamma\delta_{per})$ has real and as well as non-real branch points for some $\gamma > 0$.

Proposition 1.4. *There exists $N > 0$ such that for each $n \geqslant N$ there exist $z_n \in (2n\pi, (2n+1)\pi), \gamma_n \in \mathbb{R}, \varepsilon_n > 0$, and the functions $r_n^\pm(\gamma), -\varepsilon_n < \gamma - \gamma_n < \varepsilon_n$, such that $r_n^\pm(\gamma)$ are zeros of the function $\rho(\lambda, \gamma\delta_{per}), r_n^\pm(\gamma_n) = z_n^4$. Moreover, the following asymptotics are fulfilled:*

$$r_n^\pm(\gamma) = z_n^4 \pm \alpha_n z_n^3 \sqrt{\nu} + O(\nu^{\frac{3}{2}}), \quad \alpha_n > 0, \quad \nu = \gamma - \gamma_n \to 0. \tag{1.21}$$

Remark. i) Numerical experiments show that asymptotics (1.21) hold for all $n \geqslant 1$. The qualitative behavior of $\rho(\lambda, \gamma\delta_{per}), \Delta(\lambda, \gamma\delta_{per})$ at $\gamma \approx \gamma_1$ is shown by Fig. 2. ii) If $\nu > 0$, then the branch points $r_n^\pm(\gamma)$ are real. If $\nu < 0$, then the branch points $r_n^\pm(\gamma)$ are non-real.

We describe the plan of our paper. In Sect.2 we obtain the basic properties of the fundamental solutions and give a convenient representation of the functions T_m in terms of the entries of some auxiliary matrix Φ. In Sect.3 we determine the asymptotics of the matrix Φ. Using these results in Sect.4 we determine the asymptotics of T_m. We prove Theorems 1.1, 1.2 in Sect.5. In Sect.6 we consider the case of small potentials and prove Theorem 1.3. The periodic δ-potential is considered in Sect.7. The existence of non-real branch points of the Lyapunov function is shown and Proposition 1.4 is proved in Section 7.

Recall that the spectrum of the operator $-\frac{\partial^2}{\partial \mathbf{x}^2} + Q(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^3$, where Q is a real periodic potential is absolutely continuous and consists from spectral bands separated gaps. We have a conjecture that ends of these spectral bands are periodic or anti-periodic eigenvalues or some numbers similar to resonances from our paper.

2 Fundamental solutions

We begin with some notational convention. A vector $h = \{h_n\}_1^N \in \mathbb{C}^N$ has the Euclidean norm $|h|^2 = \sum_1^N |h_n|^2$, while a $N \times N$ matrix A has the operator norm given by $|A| = \sup_{|h|=1} |Ah|$. In this section we study the fundamental solutions $\varphi_j, j = 0,1,2,3,$. We introduce the fundamental solutions φ_j^0 of the unperturbed equation $y'''' = \lambda y$ given by

$$\varphi_0^0(t,\lambda) = \frac{\cosh zt + \cos zt}{2}, \quad \varphi_1^0(t,\lambda) = \frac{\sinh zt + \sin zt}{2z}, \tag{2.1}$$

$$\varphi_2^0(t,\lambda) = \frac{\cosh zt - \cos zt}{2z^2}, \quad \varphi_3^0(t,\lambda) = \frac{\sinh zt - \sin zt}{2z^3}, \quad z = \lambda^{1/4},\ \arg z \in (-\frac{\pi}{4}, \frac{\pi}{4}], \tag{2.2}$$

which are entire in $\lambda \in \mathbb{C}$. Here below we have $z = x + iy,\ x \geqslant |y|$. They satisfy

$$\partial_t^k \varphi_j^0(t,\lambda) = \varphi_{j-k}^0(t,\lambda), \quad \sum_{m=0}^3 \varphi_{j-m}^0(t,\lambda)\varphi_{m-k}^0(s,\lambda) = \varphi_{j-k}^0(t+s,\lambda), \quad 0 \leqslant k,j \leqslant 3, \tag{2.3}$$

$$\sum_{j=0}^3 \varphi_{0,j}^{(j)}(m,\lambda) = 2(\cosh mz + \cos mz), \qquad m \geqslant 1, \tag{2.4}$$

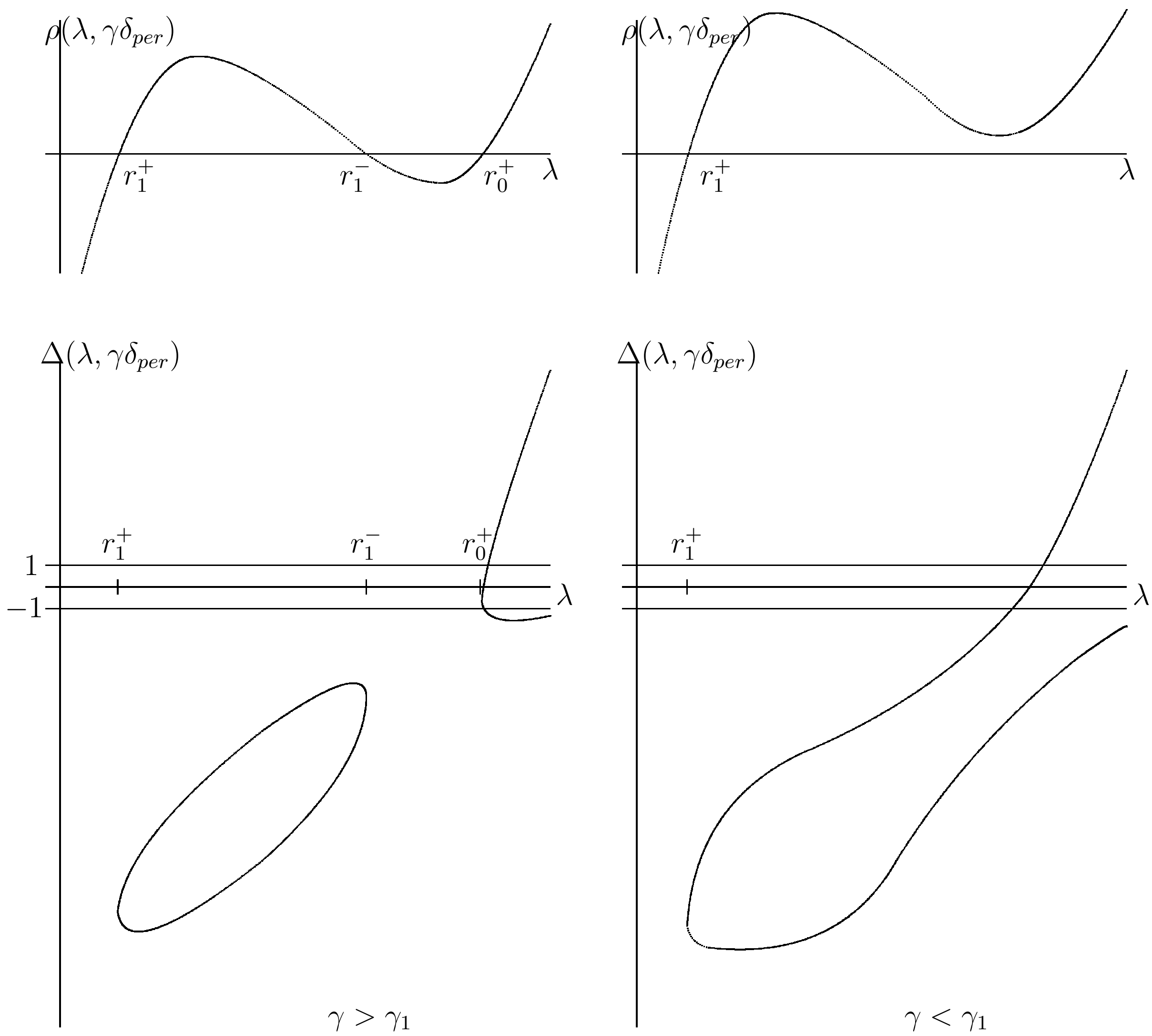


Figure 2: The qualitative behavior of the functions $\rho(\lambda, \gamma\delta_{per})$, $\Delta(\lambda, \gamma\delta_{per})$ at $\gamma \approx \gamma_1$.

$$|\varphi_j^0(t,\lambda)| \leqslant \frac{e^{tx}+e^{t|y|}}{2} \leqslant e^{xt}, \quad j=0,1,2,3. \tag{2.5}$$

The fundamental solutions $\varphi_j, j=0,1,2,3$, satisfy the following integral equations

$$\varphi_j(t,\lambda) = \varphi_j^0(t,\lambda) - \int_0^t \varphi_3^0(t-s,\lambda)V(s)\varphi_j(s,\lambda)ds, \qquad (t,\lambda)\in\mathbb{R}\times\mathbb{C}. \tag{2.6}$$

The standard iterations in (2.6) yield

$$\varphi_j(t,\lambda) = \sum_{n\geqslant 0}\varphi_{n,j}(t,\lambda), \quad \varphi_{n+1,j}(t,\lambda) = -\int_0^t \varphi_3^0(t-s,\lambda)V(s)\varphi_{n,j}(s,\lambda)ds, \tag{2.7}$$

where $\varphi_{0,j}=\varphi_j^0$. Define the functions

$$T_{m,2}(\lambda) = \frac{1}{4}\int_0^m dt\int_0^t V(s)V(t)\varphi_3^0(m-t+s,\lambda)\varphi_3^0(t-s,\lambda)ds, \quad m=1,2. \tag{2.8}$$

We prove

Lemma 2.1. *For each $(t,V)\in\mathbb{R}_+\times L_0^1(\mathbb{T})$ and $j=0,1,2,3$ the functions $\varphi_j(t,\cdot)$ are real on $\mathbb{R}$ and entire and for each $N\geqslant -1$ the following estimates are fulfilled:*

$$\max_{0\leqslant j,k\leqslant 3}\left\{\left|\lambda^{\frac{j-k}{4}}\Big(\varphi_j^{(k)}(t,\lambda) - \sum_0^N \varphi_{n,j}^{(k)}(t,\lambda)\Big)\right|\right\} \leqslant \frac{(\varkappa t)^{N+1}}{(N+1)!}e^{xt+\varkappa}, \qquad \varkappa = \frac{\|V\|}{|\lambda|_1^{3/4}}, \tag{2.9}$$

where $|\lambda|_1\equiv\max\{1,|\lambda|\}$. Moreover, $T_m, m=1,2$ is real for $\lambda\in\mathbb{R}$, entire and satisfies

$$|T_m(\lambda)|\leqslant e^{xm+\varkappa}, \quad |T_m(\lambda)-T_m^0(\lambda)|\leqslant \frac{(m\varkappa)^2}{2}e^{xm+\varkappa}, \tag{2.10}$$

$$|T_m(\lambda)-T_m^0(\lambda)-T_{m,2}(\lambda)|\leqslant \frac{(m\varkappa)^3}{3!}e^{xm+\varkappa}. \tag{2.11}$$

Proof. We estimate φ_0, the proof of other estimates is similar. (2.7) gives

$$\varphi_{n,0}(t,\lambda) = \int\limits_{0<t_n<\ldots<t_2<t_1\leqslant t_0=t}\Big(\prod_{1\leqslant k\leqslant n}\varphi_3^0(t_{k-1}-t_k,\lambda)V(t_k)\Big)\varphi_{0,0}(t_n,\lambda)dt_1dt_2\ldots dt_n. \tag{2.12}$$

Substituting estimates (2.5) into (2.12) we obtain $|\varphi_{n,0}(t,\lambda)|\leqslant\frac{(\varkappa t)^n}{n!}e^{xt}$, which shows that for any fixed $t\in[0,1]$ the formal series (2.7) converges uniformly on bounded subset of $\mathbb{C}$. Each term of this series is an entire function. Hence the sum is an entire function. Summing the majorants we obtain estimates (2.9).

The monodromy matrix is real on the real line. Then T_1, T_2 are real on $\mathbb{R}$. We will prove (2.10), (2.11). We have

$$4T_m = \operatorname{Tr} M^m(\lambda) = \operatorname{Tr} M(m,\lambda) = \sum_{j=0}^3\varphi_j^{(j)}(m,\lambda) = \sum_{n\geqslant 0}\sum_{j=0}^3\varphi_{n,j}^{(j)}(m,\lambda), \quad m=1,2. \tag{2.13}$$

The estimates $|\varphi_{n,j}^{(j)}(m,\lambda)| \leqslant \frac{(m\varkappa)^n}{n!} e^{xm+\varkappa}$ yield

$$\Big|\sum_{j=0}^{3} \varphi_{n,j}^{(j)}(m,\lambda)\Big| \leqslant 4\frac{(m\varkappa)^n}{n!} e^{xm+\varkappa}, \quad n \geqslant 0. \tag{2.14}$$

The last estimate shows that the series (2.13) converges uniformly on bounded subset of $\mathbb{C}$. Each term of this series is an entire function. Hence the sum is an entire function and T_1, T_2 are entire. Summing the majorants we obtain the first estimates in (2.10). Using (2.3), (2.7) we obtain

$$\sum_{j=0}^{3} \varphi_{1,j}^{(j)}(m,\lambda) = -\sum_{j=0}^{3} \int_0^m \varphi_{3-j}^0(m-t,\lambda)\varphi_j^0(t,\lambda)V(t)dt = -\varphi_3^0(m,\lambda)\int_0^m V(t)dt = 0,$$

and

$$\sum_{j=0}^{3} \varphi_{2,j}^{(j)}(m,\lambda) = \sum_{j=0}^{3} \int_0^m \int_0^t \varphi_{3-j}^0(m-t,\lambda)\varphi_3^0(t-s,\lambda)\varphi_j^0(s,\lambda)V(s)V(t)dsdt = T_{m,2}(\lambda),$$

where we have used (2.8). Then (2.13), (2.14) give the second estimate in (2.10) and (2.11). ■

Note that $\mathcal{M}(t,\lambda)$ is a solution of the equation

$$Y' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda - V & 0 & 0 & 0 \end{pmatrix} Y, \qquad (t,\lambda) \in [0,1] \times \mathbb{C}, \tag{2.15}$$

such that $\mathcal{M}(0,\lambda) = I_4$. Lemma 2.1 does not give asymptotics of $\mathcal{M}(1,\lambda)$. In order to determine the asymptotics of $\mathcal{M}(1,\lambda)$ we need another solution $Y(t,\lambda)$ of Eq.(2.15) with the good asymptotics at high energy, see Lemma 2.2. Note that $\mathcal{M}(t,\lambda) = Y(t,\lambda)Y^{-1}(0,\lambda)$.

We will construct the matrix Y using some special solutions of Eq.(1.1), which have "good" asymptotics at $|\lambda| \to \infty$ (see Lemma 2.2).

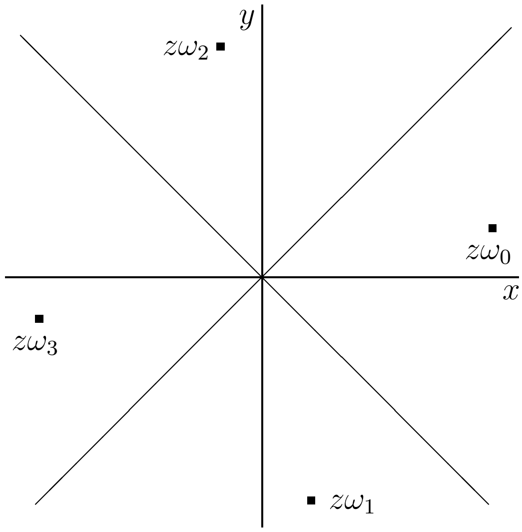
In order to determine the asymptotics of $M(\lambda)$ we need another solution $\vartheta_j, j = 0,1,2,3$. We introduce a matrix $\Omega(\lambda)$ by

$$\Omega = \Omega(\lambda) = \operatorname{diag}(\omega_0,\omega_1,\omega_2,\omega_3) = (1,-i,i,-1), \quad \lambda \in \overline{\mathbb{C}_+}, \quad \Omega(\overline{\lambda}) = \overline{\Omega(\lambda)}.$$

Here ω_0 and ω_3 are constants in $\mathbb{C}$, but $\omega_j = \omega_j(\lambda), j = 1,2$, are constants only in $\mathbb{C}_\pm$. It will imply that some of functions, which will be introduced below, they will not be analytic in whole complex plane, but will be analytic only in $\mathbb{C}_\pm$ or theirs subsets.

We define the functions $a_k(t,\lambda)$, $k = 0,1,2,3$, $(t,\lambda) \in \mathbb{R} \times \overline{\mathbb{C}_\pm}$ by

$$a_0(t,\lambda) = 0, \quad a_k(t,\lambda) = \sum_{j=0}^{k-1} \omega_j e^{zt\omega_j}, \quad k = 1,2,3, \quad t < 0, \tag{2.16}$$



$z = x + iy$-plane

Figure 3:

$$a_k(t,\lambda) = -\sum_{j=k}^{3} \omega_j e^{zt\omega_j}, \quad k = 0,1,2,3, \quad t > 0, \tag{2.17}$$

$z = \lambda^{1/4}, -\frac{\pi}{4} < \arg z < \frac{\pi}{4}$. Note that $\mathrm{Re}(z\omega_0) \geqslant \mathrm{Re}(z\omega_1) \geqslant \mathrm{Re}(z\omega_2) \geqslant \mathrm{Re}(z\omega_3)$ (see Fig.3). Then we have

$$|e^{zt\omega_j}| = e^{t\,\mathrm{Re}(z\omega_j)} \leqslant e^{t\,\mathrm{Re}(z\omega_k)}, \quad 0 \leqslant j \leqslant k, \quad t < 0, \quad |e^{zt\omega_j}| \leqslant e^{t\,\mathrm{Re}(z\omega_k)}, \quad k \leqslant j \leqslant 3, \quad t \geqslant 0.$$

Substituting these estimates into identities (2.16),(2.17) we obtain

$$|a_k(t,\lambda)| \leqslant 4e^{t\,\mathrm{Re}(z\omega_k)}, \quad k = 0,1,2,3, \quad (t,\lambda) \in \mathbb{R} \times \overline{\mathbb{C}_\pm}. \tag{2.18}$$

Identities (2.16), (2.17) show that a_k, a_k', a_k'' are continuous functions of $t \in \mathbb{R}$ and

$$a_k'''(+0,\lambda) - a_k'''(-0,\lambda) = -4z^3, \quad a_k''''(t,\lambda) - \lambda a_k(t,\lambda) = -4z^3\delta(t). \tag{2.19}$$

Let $\Lambda_r = \{\lambda \in \mathbb{C} : r\|V\| < |\lambda|^{3/4}\}$ and $\Lambda_r^\pm = \Lambda_r \cap \mathbb{C}^\pm, r > 0$. Below we need

Lemma 2.2. *Let $V \in L_0^1(\mathbb{T})$. Then for each $j = 0,1,2,3$ and $\lambda \in \overline{\Lambda_1^\pm}$ the integral equation*

$$\vartheta_j(t,\lambda) = e^{zt\omega_j} + \frac{1}{4z^3}\int_0^1 a_j(t-s,\lambda)V(s)\vartheta_j(s,\lambda)ds, \quad t \in [0,1], \tag{2.20}$$

has the unique solution $\vartheta_j(\cdot,\lambda)$ and each $\vartheta_j(t,\cdot), t \in [0,1]$ is analytic in $\Lambda_1^\pm$ and continuous in $\overline{\Lambda_1^\pm}$. Moreover, each $\vartheta_j(\cdot,\lambda), \lambda \in \overline{\Lambda_1^\pm}$ is a solution of equation $\vartheta_j'''' + V\vartheta_j = \lambda\vartheta_j$ for $t \in [0,1]$ and satisfies

$$|\vartheta_j(t,\lambda)| \leqslant \frac{e^{t\,\mathrm{Re}(z\omega_j)}}{1-\varkappa}, \quad (t,\lambda) \in [0,1] \times \overline{\Lambda_1^\pm}, \quad \varkappa = \frac{\|V\|}{|z|^3}. \tag{2.21}$$

Proof. Let $\vartheta_{j,0}(t,\lambda) = e^{zt\omega_j}$. The iterations in Eq.(2.20) provide the identities

$$\vartheta_j(t,\lambda) = \sum_0^\infty \vartheta_{j,n}(t,\lambda), \quad \vartheta_{j,n}(t,\lambda) = \frac{1}{4z^3}\int_0^1 a_j(t-s,\lambda)V(s)\vartheta_{j,n-1}(s,\lambda)ds, \quad n \geqslant 1, \tag{2.22}$$

$$\vartheta_{j,n}(t,\lambda) = \frac{1}{(4z^3)^n}\int_{[0,1]^n} a_j(t-t_n)a_j(t_n-t_{n-1})...a_j(t_2-t_1)e^{zt\omega_j}V(t_1)...V(t_n)dt_1...dt_n.$$

For $-\frac{\pi}{4} < \arg z < \frac{\pi}{4}, t \in [0,1]$ using estimates (2.18) we obtain

$$|\vartheta_{j,n}(t,\lambda)| \leqslant \frac{1}{|z|^{3n}}\int\limits_{[0,1]^n} e^{(t-t_n)\operatorname{Re}(z\omega_j)}e^{(t_n-t_{n-1})\operatorname{Re}(z\omega_j)}...e^{(t_2-t_1)\operatorname{Re}(z\omega_j)}e^{t_1\operatorname{Re}(z\omega_j)}$$

$$\times|V(t_1)|...|V(t_n)|dt_1...dt_n = \frac{e^{t\operatorname{Re}(z\omega_j)}}{|z|^{3n}}\int_{[0,1]^n}|V(t_1)|...|V(t_n)|dt_1...dt_n = \frac{\|V\|^n}{|z|^{3n}}e^{t\operatorname{Re}(z\omega_j)}.$$

Substituting the last estimate into (2.22) we get (2.21). This estimate shows that for each fixed $|z| > \|V\|_1^{1/3}, -\frac{\pi}{4} < \arg z < \frac{\pi}{4}$ series (2.22) converges uniformly on the interval $[0,1]$. Thus it gives the solution of Eq.(2.20). Suppose that there exists another solution $\widetilde{y}_j$ of this equation. Then $y = y_j - \widetilde{y}_j$ satisfies the equation $y(t,\lambda) = \frac{1}{4z^3}\int_0^1 V(s)a_j(t-s,\lambda)y(s,\lambda)ds$. The iterations of this equation give $y(t,\lambda) = 0$. For each $t \in \mathbb{R}$ the series (2.22) converges uniformly on any bounded subset of the domain $|z| > \|V\|^{1/3}, -\frac{\pi}{4} < \arg z < 0$, or $0 < \arg z < \frac{\pi}{4}$. Each term of this series is an analytic function of z in this domain. Hence the sum is an analytic function and $\vartheta_n(t,\lambda)$ is analytic of $\lambda \in \Lambda_1^\pm$ and is continuous in $\overline{\Lambda_1^\pm}$.

We will show that $\vartheta_j(\cdot,\lambda)$ is a solution of $y'''' + Vy = \lambda y$. Using (2.19) and (2.22) we obtain

$$\vartheta_j''''(t,\lambda) = \lambda\vartheta_j^0(t,\lambda) + \int_0^1\Big(\frac{\lambda a_j(t-s,\lambda)}{4z^3} - \delta(t-s)\Big)V(s)\vartheta_j(s,\lambda)ds = \lambda\vartheta_j(t,\lambda) - V(t)\vartheta_j(t,\lambda)$$

Thus $\vartheta_j(\cdot,\lambda), \lambda \in \overline{\Lambda_1^\pm}$ is a solution of $y'''' + Vy = \lambda y$, $t \in [0,1]$. ■

Let $Y(t,\lambda) = \{\vartheta_j^{(k)}(t,\lambda)\}_{k,j=0}^3$. Then $Y(t,\lambda)$ is a solution of Eq.(2.15) and hence

$$\mathcal{M}(t,\lambda) = Y(t,\lambda)Y^{-1}(0,\lambda), \quad \lambda \in \Lambda_1^\pm. \tag{2.23}$$

Differentiating Eq.(2.20) we obtain

$$\vartheta_j^{(k)}(t,\lambda) = (z\omega_j)^k e^{zt\omega_j} + \frac{1}{4z^3}\int_0^1 V(s)a_j^{(k)}(t-s,\lambda)\vartheta_j(s,\lambda)ds, \quad j,k = 0,1,2,3. \tag{2.24}$$

We introduce the matrix $\Psi = \{\psi_{kj}\}_{k,j=0}^3$, where $\psi_{kj}(t,\lambda) = z^{-k}e^{-zt\omega_j}\vartheta_j^{(k)}(t,\lambda)$. Then

$$\Psi(t,\lambda) = Z^{-1}(\lambda)Y(t,\lambda)e^{-zt\Omega}, \quad Z(\lambda) = \operatorname{diag}(1,z,z^2,z^3), \tag{2.25}$$

hence $Y(t,\lambda) = Z(\lambda)\Psi(t,\lambda)e^{zt\Omega}$. Substituting this identity into (2.23), we obtain

$$\mathcal{M}(t,\lambda) = Z(\lambda)\Psi(t,\lambda)e^{zt\Omega}\Psi^{-1}(0,\lambda)Z^{-1}(\lambda), \qquad t \in [0,1], \tag{2.26}$$

and then

$$\operatorname{Tr} M(\lambda) = \operatorname{Tr}(\Phi(\lambda)e^{z\Omega}), \quad \operatorname{Tr} M^2(\lambda) = \operatorname{Tr}(\Phi(\lambda)e^{z\Omega})^2, \quad \Phi(\lambda) = \Psi^{-1}(0,\lambda)\Psi(1,\lambda), \tag{2.27}$$

for $\lambda \in \overline{\Lambda_1^{\pm}}$. Note that $\operatorname{Tr} M$, $\operatorname{Tr} M^2$ are entire functions, but the functions $\Psi(t,\cdot)$, $t \in \mathbb{R}$, and Φ, are not entire.

3 Asymptotics of the matrices Ψ and Φ as $|\lambda| \to \infty$

We rewrite identities (2.24) in the form

$$\psi_{kj}(t,\lambda) = \omega_j^k + \frac{1}{4z^3}\int_0^1 a_{kj}(t-s,\lambda)V(s)\psi_{0j}(s,\lambda)ds, \quad a_{kj}(t,\lambda) = z^{-k}e^{-z\omega_j t}a_j^{(k)}(t,\lambda), \tag{3.1}$$

$j,k = 0,1,2,3$. The last identities contain the equations for the functions $\psi_{0j}(t,\lambda)$:

$$\psi_{0j}(t,\lambda) = 1 + \frac{1}{4z^3}\int_0^1 a_{0j}(t-s,\lambda)V(s)\psi_{0j}(s,\lambda)ds, \quad j = 0,1,2,3. \tag{3.2}$$

We rewrite (3.2) in terms of the matrix $\psi = \operatorname{diag}(\psi_{00},\psi_{01},\psi_{02},\psi_{03})$ by

$$\psi(t,\lambda) = I_4 + \frac{1}{4z^3}\int_0^1 a(t-s,\lambda)V(s)\psi(s,\lambda)ds, \quad a = \operatorname{diag}(a_{00},a_{01},a_{02},a_{03}). \tag{3.3}$$

Define the matrix $A = \{a_{kj}\}_{k,j=0}^3$ and rewrite (3.1) in the form

$$\Psi(t,\lambda) = \Psi_0 + \frac{1}{4z^3}\int_0^1 A(t-s,\lambda)V(s)\psi(s,\lambda)ds, \qquad \Psi_0 = \{\Psi_{0kj}\}_{k,j=0}^3 = \{\omega_j^k\}_{k,j=0}^3. \tag{3.4}$$

Note that $\Psi_0 = \Psi_0(\lambda)$ is constant in $\overline{\mathbb{C}_\pm}$ and satisfies the identity $\Psi_0\Psi_0^* = 4I_4$, hence $\frac{1}{2}\Psi_0$ is an unitary matrix. Identities (2.16) yield

$$a_j^{(k)}(t,\lambda) = \sum_{p=0}^{j-1}\omega_p(z\omega_p)^k e^{zt\omega_p}, \quad t<0, \quad \text{and} \quad a_j^{(k)}(t,\lambda) = -\sum_{p=j}^{3}\omega_p(z\omega_p)^k e^{zt\omega_p}, \quad t>0.$$

Then (3.1) gives

$$a_{kj}(t,\lambda) = \sum_{p=0}^{j-1}\omega_p\omega_p^k e^{zt(\omega_p-\omega_j)}, \quad t<0, \qquad \text{and} \quad a_{kj}(t,\lambda) = -\sum_{p=j}^{3}\omega_p\omega_p^k e^{zt(\omega_p-\omega_j)}, \quad t>0.$$

Thus the following identities are fulfilled:

$$A(t,\lambda)=\Psi_0\Omega H(t,\lambda),\quad \Omega=\operatorname{diag}(\omega_0,\omega_1,\omega_2,\omega_3),\qquad H=\{h_{jk}\}_{j,k=0}^3, \tag{3.5}$$

where

$$h_{jk}(t,\lambda)=\begin{cases}0 & ,\ j\geqslant k\\ e^{zt(\omega_j-\omega_k)} & ,\ j<k\end{cases},\quad t<0,\quad h_{jk}(t,\lambda)=\begin{cases}-e^{zt(\omega_j-\omega_k)} & ,\ j\geqslant k\\ 0 & ,\ j<k\end{cases},\quad t\geqslant 0. \tag{3.6}$$

Iterations in Eq. (3.3) yield

$$\psi(t,\lambda)=\sum_0^\infty \psi_n(t,\lambda),\qquad \psi_n(t,\lambda)=\frac{1}{4z^3}\int_0^1 a(t-s,\lambda)V(s)\psi_{n-1}(s,\lambda)ds,\quad \psi_0=I_4. \tag{3.7}$$

Substituting the series (3.7) into (3.4) we obtain

$$\Psi(t,\lambda)=\sum_0^\infty \Psi_n(t,\lambda),\qquad \Psi_n(t,\lambda)=\frac{1}{4z^3}\int_0^1 A(t-s,\lambda)V(s)\psi_{n-1}(s,\lambda)ds. \tag{3.8}$$

We need the result about the matrix functions $\Psi,\ \Phi(\lambda)=\Psi^{-1}(0,\lambda)\Psi(1,\lambda),\ \lambda\in\Lambda_1^\pm$.

Lemma 3.1. *Let $V\in L_0^1(\mathbb{T})$ and let $\varkappa=\frac{\|V\|}{|\lambda|^{3/4}}$. Then*
i) Each matrix function $\Psi(t,\cdot),t\in[0,1]$ is analytic in the domain $\Lambda_2^\pm$ and continuous in $\overline{\Lambda_2^\pm}$ and satisfies

$$|\Psi_0|=2,\quad |\Psi_n(t,\lambda)|\leqslant 2\varkappa^n,\quad n\geqslant 1, \tag{3.9}$$

$$|\Psi(t,\lambda)|\leqslant 4,\quad \Big|\Psi(t,\lambda)-\sum_0^N\Psi_n(t,\lambda)\Big|\leqslant 4\varkappa^{N+1},\quad N\geqslant 0,\quad \lambda\in\overline{\Lambda_2^\pm}. \tag{3.10}$$

ii) The matrix function $\Phi(\lambda)=\Psi^{-1}(0,\lambda)\Psi(1,\lambda)$ is analytic in $\lambda\in\Lambda_4^\pm$ and continuous in $\overline{\Lambda_4^\pm}$ and satisfies

$$\Phi=I_4+\Phi_1+\Phi_2+\widetilde{\Phi},\quad \Phi_1(\lambda)=\frac{\Omega}{4z^3}\int_0^1 V(s)\Big(H(1-s,\lambda)-H(-s,\lambda)\Big)ds, \tag{3.11}$$

$$\Phi_2(\lambda)=\frac{\Omega}{16z^6}\int_0^1 du\int_0^1 V(u)V(s)\Big(F(0,u,s,\lambda)-F(1,u,s,\lambda)\Big)ds, \tag{3.12}$$

$$F(t,u,s,\lambda)=H(-u,\lambda)\Omega H(t-s,\lambda)-H(t-u,\lambda)a(u-s,\lambda),$$

$$|\Phi(\lambda)|\leqslant 4,\quad |\Phi_1(\lambda)|\leqslant 2\varkappa,\quad |\Phi_2(\lambda)|\leqslant 4\varkappa^2,\qquad |\widetilde{\Phi}(\lambda)|\leqslant 60\varkappa^3,\quad \lambda\in\overline{\Lambda_4^\pm}. \tag{3.13}$$

Proof. i) Recall that $\frac12\Psi_0$ is a unitary matrix. Then $|\Psi_0|=2$. Let $a(t)=a(t,\lambda)$. Identity (3.7) for ψ_n gives

$$\psi_n(t,\lambda)=\frac{1}{(4z^3)^n}\int_{[0,1]^n} a(t-t_1)a(t_1-t_2)...a(t_{n-1}-t_n)V(t_1)...V(t_n)dt_1...dt_n,\ n\geqslant 1.$$

Identity (3.1) for a_{kj} together with estimate (2.18) yields $|a_{0j}(t)| \leqslant 4$. Then $|a(t,\lambda)| \leqslant 4$ for $(t,\lambda) \in \mathbb{R} \times \overline{\mathbb{C}_\pm}$. Using the last estimate we obtain

$$|\psi_n(t,\lambda)| \leqslant \varkappa^n, \quad n \geqslant 0, \quad (t,\lambda) \in \ [0,1] \times \overline{\mathbb{C}_\pm}. \tag{3.14}$$

Identities (3.5)-(3.6) imply $|H(t,\lambda)| \leqslant 4, |\Omega| = 1$. Identity (3.5) for A gives $|A(t,\lambda)| \leqslant 8$. Substituting this estimate and estimates (3.14) into identity (3.8) for Ψ_n we get (3.9). Estimate (3.9) shows that for each $t \in [0,1]$ the series (3.8) converges uniformly on any bounded domain in $\Lambda_1^\pm$. Each term in this series is an analytic function in $\Lambda_1^\pm$. Hence $\Psi(t,\lambda)$ is also analytic function in $\Lambda_1^\pm$. Summing the majorants we obtain

$$|\sum_{N+1}^{\infty} \Psi_n(t,\lambda)| \leqslant 2 \sum_{N+1}^{\infty} \varkappa^n = \frac{2\varkappa^{N+1}}{1-\varkappa} \leqslant 4\varkappa^{N+1}, \quad \lambda \in \overline{\Lambda_2^\pm}, \quad N \geqslant -1,$$

since $\varkappa \leqslant \frac{1}{2}, \lambda \in \overline{\Lambda_2^\pm}$. Then we obtain (3.10).
ii) For the case $N = 2$ (3.10) yields

$$\Psi(t,\lambda) = \Psi_0 + \Psi_1(t,\lambda) + \Psi_2(t,\lambda) + \widetilde{\Psi}(t,\lambda), \quad |\widetilde{\Psi}| \leqslant 4\varkappa^3, \quad (t,\lambda) \in [0,1] \times \overline{\Lambda_2^\pm}. \tag{3.15}$$

We introduce the matrices $\Psi^0 = \Psi(0,\cdot), R = (\Psi^0)^{-1}, \Psi_n^0 = \Psi_n(0,\cdot), \widetilde{\Psi}^0 = \widetilde{\Psi}(0,\cdot)$. We will prove that the matrix $R(\lambda)$ is analytic in the domains $\Lambda_4^\pm$ and satisfies

$$R = R_0 + R_1 + R_2 + \widetilde{R}, \quad R_0 = \Psi_0^{-1}, \quad R_1 = -R_0\Psi_1^0 R_0, \quad R_2 = -R_0\Psi_2^0 R_0 + R_0(\Psi_1^0 R_0)^2, \tag{3.16}$$

$$\widetilde{R} = -R_0\widetilde{\Psi}^0 R_0 + R_0(\Psi_2^0 + \widetilde{\Psi}^0)R_0(\Psi^0 - \Psi_0)R_0 + R_0\Psi_1^0 R_0(\Psi_2^0 + \widetilde{\Psi}^0)R_0 - R((\Psi^0 - \Psi_0)R_0)^3, \tag{3.17}$$

$$|R| \leqslant 1, \quad |R_0| = \frac{1}{2}, \quad |R_1| \leqslant \frac{\varkappa}{2}, \quad |R_2| \leqslant \varkappa^2, \quad |\widetilde{R}| \leqslant 13\varkappa^3, \quad \lambda \in \overline{\Lambda_4^\pm}. \tag{3.18}$$

The matrix $\frac{1}{2}\Psi_0$ is unitary, then $|R_0| = |\Psi_0^{-1}| = |\frac{1}{4}\Psi_0^*| = \frac{1}{2}$. We have $\Psi^0 = \Psi_0 + (\Psi^0 - \Psi_0)$. Recall $|\Psi^0| = 2$. If $\lambda \in \Lambda_4$, then $\varkappa < \frac{1}{4}$ and (3.10) yields $|\Psi^0 - \Psi_0| \leqslant 4\varkappa < 1$. Hence 0 is not an eigenvalue of Ψ^0 (see [Ka]), Ψ^0 is invertible and R is analytic in $\Lambda_4^\pm$. Substituting estimates (3.9) into (3.16) we have estimates of $|R_1|, |R_2|$ in (3.18). Using the standard identity $A^{-1} - B^{-1} = -A^{-1}(A-B)B^{-1}$ for the matrices $A = \Psi^0, B = \Psi_0$ we obtain

$$R = R_0 - R(\Psi^0 - \Psi_0)R_0 = R_0 - R_0(\Psi^0 - \Psi_0)R_0 + R_0\Big((\Psi^0 - \Psi_0)R_0\Big)^2 - R\Big((\Psi^0 - \Psi_0)R_0\Big)^3 \tag{3.19}$$

which yields (3.16), (3.17). Using the first identity in (3.19) we obtain

$$|R| \leqslant \frac{|R_0|}{1 - |\Psi^0 - \Psi_0||R_0|} \leqslant 1, \quad \lambda \in \overline{\Lambda_4^\pm},$$

which yields estimate of $|R|$ in (3.18). Substituting this estimate and (3.9), (3.10), (3.15) into (3.17) we obtain estimate for $|\widetilde{R}|$ in (3.18). Thus relations (3.16)-(3.18) have been proved.

Now we prove (3.11)-(3.13). Estimates (3.10) and (3.18) give (3.13) for Φ. Let $\Psi = \Psi(1,\lambda), \Psi_n = \Psi_n(1,\lambda)$ and $R = R(\lambda), R_n = R_n(\lambda)$. Identities (3.15) and (3.16) give (3.11), where

$$\Phi_1 = R_0\Psi_1 + R_1\Psi_0, \quad \Phi_2 = R_2\Psi_0 + R_1\Psi_1 + R_0\Psi_2, \quad \widetilde{\Phi} = R_0\widetilde{\Psi} + R_1(\Psi_2 + \widetilde{\Psi}) + R_2(\Psi - \Psi_0) + \widetilde{R}\Psi.$$

Using estimates (3.9), (3.10), (3.18) we get (3.13).

Let $A(t) = A(t,\lambda), H(t) = H(t,\lambda), a(t) = a(t,\lambda)$. Using $R_1 = -R_0\Psi_1^0\Psi_0^{-1}$ and (3.8) we get

$$\Phi_1 = R_0\Psi_1 + R_1\Psi_0 = R_0(\Psi_1(1,\lambda) - \Psi_1(0,\lambda)) = \frac{R_0}{4z^3}\int_0^1 V(s)(A(1-s) - A(-s))ds.$$

Substituting (3.5) for A into the last identity we obtain (3.11).

We will prove (3.12). Recall $\Phi_2 = R_2\Psi_0 + R_1\Psi_1 + R_0\Psi_2$. Identities (3.16) give $R_2 = ((R_0\Psi_1^0)^2 - R_0\Psi_2^0)R_0$. Then

$$R_2\Psi_0 = (R_0\Psi_1^0)^2 - R_0\Psi_2^0 = R_0(\Psi_1^0 R_0\Psi_1^0 - \Psi_2^0).$$

Substituting Ψ_1, Ψ_2 from (3.8) and ψ_0, ψ_1 from (3.7) into the last identity we obtain

$$R_2\Psi_0 = \frac{R_0}{16z^6}\iint_{[0,1]^2} V(u)V(s)A(-u)\Big(R_0A(-s) - a(u-s)\Big)duds.$$

Using identity (3.5) for A we have

$$R_2\Psi_0 = \frac{\Omega}{16z^6}\iint_{[0,1]^2} V(u)V(s)\Big(H(-u)\Omega H(-s) - H(-u)a(u-s)\Big)duds.$$

Recall that $R_1 = -R_0\Psi_1^0 R_0$. Then (3.8) implies

$$R_1\Psi_1 = \frac{-R_0}{16z^6}\int\int_{[0,1]^2} V(u)V(s)A(-u)R_0A(1-s)duds = \frac{-\Omega}{16z^6}\int\int_{[0,1]^2} V(u)V(s)H(-u)\Omega H(1-s)duds.$$

(3.8) gives

$$R_0\Psi_2 = \frac{R_0}{16z^6}\iint_{[0,1]^2} V(u)V(s)A(1-u)a(u-s)duds = \frac{\Omega}{16z^6}\iint_{[0,1]^2} V(u)V(s)H(1-u)a(u-s)duds.$$

The last three identities yield (3.12). ■

4 Asymptotics of the trace of the monodromy matrix

We introduce the functions $b_{jk}, c_{jk}, \alpha_j, \beta_{jk}$ in $\overline{\mathbb{C}_\pm}$ by

$$b_{jk} = \frac{\omega_j\omega_k}{16z^6}\int_0^1 du \int_0^u V(u)V(s)e^{z(u-s)(\omega_j-\omega_k)}ds, \quad c_{jk} = e^{z(\omega_j-\omega_k)}(b_{jk} + b_{kj}), \tag{4.1}$$

$$\alpha_0=\sum_{k=1}^3(b_{k0}+c_{k0}),\quad \alpha_j=\sum_{k=j+1}^3(b_{kj}+c_{kj})-\sum_{k=0}^{j-1}b_{jk},\quad j=1,2,\quad \alpha_3=-\sum_{k=0}^2 b_{3k}, \tag{4.2}$$

$$\beta_{jk}=\alpha_j+\alpha_k-c_{kj}. \tag{4.3}$$

which are analytic in $\mathbb{C}_\pm$ and are continuous in $\overline{\mathbb{C}}_\pm$. We define the function

$$T=4T_1^2-T_2,\qquad T^0=4(T_1^0)^2-T_2^0=1+2\cosh z\cos z.$$

In order to determine asymptotics of T_1, T, we will show the following identities

$$T_1=\frac14\sum_0^3\phi_{kk}e^{z\omega_k},\quad T=\frac12\sum_{0\leqslant j<k\leqslant 3}v_{jk}e^{z(\omega_j+\omega_k)},\quad v_{jk}=\phi_{jj}\phi_{kk}-\phi_{jk}\phi_{kj}, \tag{4.4}$$

for $\lambda\in\overline{\Lambda_4^\pm}$ where $\Phi=\{\phi_{jk}\}_{j,k=0}^3$. Identity (2.27) yields the first identity in (4.4). Due to (2.27) we have

$$T_2=\frac14\operatorname{Tr}M^2=\frac14\operatorname{Tr}(\Phi e^{z\Omega})^2=\frac14\Big(\sum_0^3\phi_{jj}^2e^{2z\omega_j}+2\sum_{0\leqslant j<k\leqslant 3}\phi_{jk}\phi_{kj}e^{z(\omega_j+\omega_k)}\Big).$$

Substituting the last identity and the first identity in (4.4) into $T=4T_1^2-T_2$, we obtain the second identity in (4.4). Recall $z=\lambda^{1/4}=x+iy, |y|\leqslant x, \lambda\in\mathbb{C}$.

Lemma 4.1. *Let $V\in L_0^1(\mathbb{T})$ and let $\Phi=\Psi^{-1}(0,\cdot)\Psi(1,\cdot)=\{\phi_{jk}\}_{j,k=0}^3$. Then*
i) For each $\lambda\in\overline{\Lambda_4^\pm}$ the following identities and estimates are fulfilled

$$\phi_{kk}(\lambda)=1+\alpha_k(\lambda)+\widetilde\phi_{kk}(\lambda),\quad |\alpha_k(\lambda)|\leqslant\frac38\varkappa^2,\quad |\widetilde\phi_{kk}(\lambda)|\leqslant 120\varkappa^3,\quad k=0,1,2,3, \tag{4.5}$$

$$v_{jk}(\lambda)=1+\beta_{jk}(\lambda)+\widetilde v_{jk}(\lambda),\quad |\beta_{jk}(\lambda)|\leqslant\frac78\varkappa^2,\quad |\widetilde v_{jk}(\lambda)|\leqslant 1701\varkappa^3,\quad 0\leqslant j<k\leqslant 3. \tag{4.6}$$

ii) The following estimates and asymptotics are fulfilled

$$|T_1(\lambda)-T_1^0(\lambda)|\leqslant 31\varkappa^2e^x,\quad |T(\lambda)-T^0(\lambda)|\leqslant 1281\varkappa^2e^{x+|y|},\quad \lambda\in\overline{\Lambda_4^\pm}, \tag{4.7}$$

$$T_1(\lambda)=T_1^0(\lambda)\Big(1+O(z^{-6})\Big),\quad T(\lambda)=T^0(\lambda)\Big(1+O(z^{-6})\Big),\quad |\lambda|\to\infty,\quad \lambda\in\mathcal{D}_1, \tag{4.8}$$

$$T_1(\lambda)=\frac{e^z}{4}\Big(1+\alpha_0(\lambda)+O(z^{-9})\Big),\quad T(\lambda)=\frac{e^z}{2}\Big(2\cos z+e^{\omega_1 z}\beta_{01}(\lambda)+e^{\omega_2 z}\beta_{02}(\lambda)+O(z^{-9})\Big) \tag{4.9}$$

as $|\lambda|\to\infty, |y|<\pi$, and

$$T_1(\lambda)=\frac{e^{(1+\omega_1)\frac z2}}{4}\Big(2\cosh\frac{(1-\omega_1)z}{2}+e^{(1-\omega_1)\frac z2}\alpha_0(\lambda)+e^{-(1-\omega_1)\frac z2}\alpha_1(\lambda)+O(z^{-9})\Big), \tag{4.10}$$

$$T(\lambda)=\frac{e^{(1+\omega_1)z}}{2}\Big(1+\beta_{01}(\lambda)+O(z^{-9})\Big) \tag{4.11}$$

as $|\lambda|\to\infty, x-|y|<\pi$.

Remark. i) Using the estimates of the functions $\beta_{jk}, \widetilde{\phi}_{kk}, \widetilde{v}_{jk}$ in $\overline{\Lambda_4^\pm}$ we obtain the estimates of the entire functions T_1, T in $\mathbb{C}$.
ii) The conditions $|\lambda| \to \infty, x - |y| < \pi$ imply $\operatorname{Re}\lambda \to -\infty$ and $|\operatorname{Im}\lambda| \to 0$. Note that $|e^{(1+\omega_1)z}| \to +\infty$ and $|e^{(1-\omega_1)z}|$ is bounded in (4.10), (4.11).
Proof. i) Identity (4.1) yields

$$|b_{jk}(\lambda)| \leqslant \frac{\|V\|^2}{32|z|^6}, \quad |c_{kj}(\lambda))| \leqslant \frac{\|V\|^2}{16|z|^6}, \quad \lambda \in \mathbb{C}_\pm, \quad 0 \leqslant k \leqslant j \leqslant 3.$$

Substituting these estimates into (4.2)-(4.3) we have estimates of α_j, β_{jk} in (4.5), (4.6).

We will prove estimate of $\widetilde{\phi}_{kk}$ in (4.5). Recall that (3.11) give $\Phi = I_4 + \Phi_1 + \Phi_2 + \widetilde{\Phi}$ where $\Phi_s(\lambda) = \Phi_s = \{\phi_{s,jk}\}_{j,k=0}^3, s = 1,2$ and $\widetilde{\Phi} = \{\widetilde{\phi}_{jk}\}_{j,k=0}^3$. Now we will prove that

$$\phi_{jk} = \delta_{jk} + \phi_{1,jk} + \phi_{2,jk} + \widetilde{\phi}_{jk}, \quad |\widetilde{\phi}_{jk}| \leqslant 120\varkappa^3, \quad \lambda \in \overline{\Lambda_4^\pm}, \quad j,k = 0,1,2,3. \qquad (4.12)$$

We need some simple estimate from the matrix theory. Let $A = \{a_{ij}\}_{i,j=0}^3$ be a 4×4-matrix with the usual matrix norm $|A|$. We prove that

$$\max_{0 \leqslant i,j \leqslant 3} |a_{ij}| \leqslant 2|A|. \qquad (4.13)$$

For each vector $x = \{x_k\}_{k=0}^3 \in \mathbb{R}^4$ estimates $|x|_\infty \leqslant |x| \leqslant 2|x|_\infty$ hold, where we denote $|x| = (\sum |x_k|^2)^{1/2}$ and $|x|_\infty = \max |x_k|$. Then

$$|Ax|_\infty \leqslant |Ax| \leqslant 2|Ax|_\infty \leqslant 2|A| \sum |x_k|. \qquad (4.14)$$

Let $|a_{pq}| = \max_{0 \leqslant i,j \leqslant 3} |a_{ij}|$. We take $x_k = \delta_{kq}$ in (4.14). Then $\sum_k a_{ik}x_k = a_{iq}$ and we obtain $\max_i |a_{iq}| = |a_{pq}| \leqslant 2|A|$, which yields (4.13). Estimate (4.13) together with (3.13) gives the last estimate (4.12).

Substituting (3.6) into (3.11) we obtain

$$\phi_{1,jk} = -\frac{\omega_j}{4z^3} \int_0^1 ds V(s) \begin{cases} e^{z(1-s)(\omega_j - \omega_k)} & , \quad j \geqslant k \\ e^{-zs(\omega_j - \omega_k)} & , \quad j < k \end{cases}, \qquad (4.15)$$

and the identity $\int_0^1 V(t)dt = 0$ yields $\phi_{1,jj} = 0, \quad j = 0,1,2,3$.

In order to complete the proof of (4.5) we have to prove the identity $\phi_{2,jj} = \alpha_j$, where α_j are defined by (4.2). Recall that

$$\Phi_2 = \frac{\Omega}{16z^6} \int_0^1 du \int_0^1 V(u)V(s)\Big(F(0,u,s) - F(1,u,s)\Big)ds, \qquad (4.16)$$

$$F(t,u,s) = H(-u)\Omega H(t-s) - H(t-u)a(u-s), \quad H = \{h_{jk}\}_{j,k=0}^3,$$
$$\Omega = \operatorname{diag}(\omega_0, \omega_1, \omega_2, \omega_3), \qquad a = \operatorname{diag}(a_{00}, a_{01}, a_{02}, a_{03}).$$

Then the diagonal entries of the matrix $F = \{f_{jk}\}$ are given by

$$f_{jj}(t,u,s) = \sum_{k=0}^{3} \omega_k h_{jk}(-u) h_{kj}(t-s) - h_{jj}(t-u) a_{0j}(u-s). \tag{4.17}$$

In particular,

$$f_{jj}(0,u,s) = \sum_{k=0}^{3} \omega_k h_{jk}(-u) h_{kj}(-s) - h_{jj}(-u) a_{0j}(u-s).$$

Recall the identities (3.6):

$$h_{jk}(t) = \begin{cases} 0 & , \ j \geqslant k \\ e^{zt(\omega_j - \omega_k)} & , \ j < k \end{cases}, \quad t < 0, \quad h_{jk}(t) = \begin{cases} -e^{zt(\omega_j - \omega_k)} & , \ j \geqslant k \\ 0 & , \ j < k \end{cases}, \quad t \geqslant 0. \tag{4.18}$$

Then $f_{jj}(0,u,s) = 0$ and we have

$$\phi_{2,jj} = -\frac{\omega_j}{16z^6} \iint_{[0,1]^2} V(u)V(s) f_{jj}(1,u,s)\,duds, \quad j = 0,1,2,3. \tag{4.19}$$

Let $0 \leqslant u, s \leqslant 1$. Substituting (4.18) into (4.17) we obtain

$$f_{jj}(1,u,s) = -\sum_{k=j+1}^{3} \omega_k e^{z(1-s+u)(\omega_k - \omega_j)} + a_{0j}(u-s), \quad j = 0,1,2, \quad f_{33}(1,u,s) = a_{03}(u-s).$$

Identity (4.19) gives

$$\phi_{2,jj} = -\frac{\omega_j}{16z^6} \iint_{[0,1]^2} V(u)V(s)\Big(a_{0j}(u-s) - \sum_{k=j+1}^{3} \omega_k e^{z(1-s+u)(\omega_k - \omega_j)}\Big) duds$$

$$= \frac{\omega_j}{16z^6}\Big(-\iint_{[0,1]^2} V(u)V(s) a_{0j}(u-s)\,duds + \sum_{k=j+1}^{3} \omega_k e^{z(\omega_k - \omega_j)}(b_{jk} + b_{kj})\Big), \quad j = 0,1,2,$$

$$\phi_{2,33} = -\frac{\omega_3}{16z^6} \iint_{[0,1]^2} V(u)V(s) a_{03}(u-s)\,duds.$$

Moreover, substituting (2.16), (2.17) into (3.1) we obtain

$$a_{0j}(t) = e^{-z\omega_j t} a_j(t) = \sum_{k=0}^{j-1} \omega_k e^{zt(\omega_k - \omega_j)}, \quad j = 1,2,3, \quad a_{00}(t) = 0, \quad t < 0,$$

$$a_{0j}(t) = -\sum_{k=j}^{3} \omega_k e^{zt(\omega_k - \omega_j)}, \qquad j = 0,1,2,3, \qquad t \geqslant 0.$$

Then we obtain $\phi_{2,jj} = \alpha_j$, which yields (4.5).

We will prove (4.6). Recall the identity (see (4.4),(4.12)): $v_{jk} = \phi_{jj}\phi_{kk} - \phi_{jk}\phi_{kj}$ and $\phi_{jk} = \delta_{jk} + \phi_{1,jk} + \phi_{2,jk} + \widetilde{\phi}_{jk}$. The last identities and $\phi_{1,kk} = 0, \phi_{2,kk} = \alpha_k$ give

$$v_{jk} = (1+\phi_{1,jj}+\phi_{2,jj}+\widetilde{\phi}_{jj})(1+\phi_{1,kk}+\phi_{2,kk}+\widetilde{\phi}_{kk}) - (\phi_{1,jk}+\phi_{2,jk}+\widetilde{\phi}_{jk})(\phi_{1,kj}+\phi_{2,kj}+\widetilde{\phi}_{kj}).$$

$$= (1+\alpha_j+\widetilde{\phi}_{jj})(1+\alpha_k+\widetilde{\phi}_{kk}) - (\phi_{1,jk}+\phi_{2,jk}+\widetilde{\phi}_{jk})(\phi_{1,kj}+\phi_{2,kj}+\widetilde{\phi}_{kj}),$$

which yields

$$v_{jk} = 1 + v_{2,jk} + \widetilde{v}_{jk}, \quad v_{2,jk} = \alpha_j + \alpha_k - \phi_{1,jk}\phi_{1,kj}, \tag{4.20}$$

where

$$\widetilde{v}_{jk} = \phi_{2,jj}(\phi_{2,kk}+\widetilde{\phi}_{kk}) + \widetilde{\phi}_{jj}\phi_{kk} - \phi_{1,jk}(\phi_{2,kj}+\widetilde{\phi}_{kj}) - (\phi_{2,jk}+\widetilde{\phi}_{jk})(\phi_{1,kj}+\phi_{2,kj}+\widetilde{\phi}_{kj}). \tag{4.21}$$

Recall the estimates (3.13): $|\Phi(\lambda)| \leqslant 4, |\Phi_1(\lambda)| \leqslant 2\varkappa, |\Phi_2(\lambda)| \leqslant 4\varkappa^2, |\widetilde{\Phi}(\lambda)| \leqslant 60\varkappa^3$. Using (4.13) we have

$$|\phi_{kj}| \leqslant 2|\Phi(\lambda)| \leqslant 8, \quad |\phi_{1,kj}| \leqslant 4\varkappa, \quad |\phi_{2,kj}| \leqslant 8\varkappa^2, \quad |\widetilde{\phi}_{kj}| \leqslant 120\varkappa^3, \quad 0 \leqslant k,j \leqslant 3.$$

Substituting these estimates into (4.21) we obtain estimate (4.6).

We will prove $v_{2,jk} = \beta_{jk}$. Identity (4.15) provides

$$\phi_{1,jk} = -\frac{\omega_j}{4z^3}\int_0^1 e^{-zt(\omega_j-\omega_k)}V(t)dt, \quad \phi_{1,kj} = -\frac{\omega_k}{4z^3}\int_0^1 e^{z(1-t)(\omega_k-\omega_j)}V(t)dt, \tag{4.22}$$

$0 \leqslant j < k \leqslant 3$. Then

$$\phi_{1,jk}\phi_{1,kj} = \frac{\omega_j\omega_k}{16z^6}\int_0^1 e^{-zu(\omega_j-\omega_k)}V(u)du\int_0^1 e^{z(1-s)(\omega_k-\omega_j)}V(s)ds = c_{kj}, \tag{4.23}$$

where we have used (4.1). Substituting (4.23) into (4.20) we obtain $v_{2,jk} = \alpha_j + \alpha_k - c_{kj}$, which yields $v_{2,jk} = \beta_{jk}$. Then (4.20) gives the first identity (4.6).

ii) Let $\lambda \in \overline{\Lambda_8^\pm}$. Then (4.4)-(4.6) imply

$$|T_1(\lambda) - T_1^0(\lambda)| \leqslant \frac{1}{4}\sum_0^3 |e^{z\omega_k}||\alpha_k(\lambda)+\widetilde{\phi}_{kk}(\lambda)| \leqslant e^x \max_k |\alpha_k(\lambda)+\widetilde{\phi}_{kk}(\lambda)|,$$

$$|T(\lambda) - T^0(\lambda)| \leqslant \frac{1}{2}\sum_{0\leqslant j<k\leqslant 3} |e^{z(\omega_j+\omega_k)}||\beta_{jk}(\lambda)+\widetilde{v}_{jk}(\lambda)| \leqslant 3e^{x+|y|}\max_{0\leqslant j<k\leqslant 3}|\beta_{jk}(\lambda)+\widetilde{v}_{jk}(\lambda)|,$$

which yield (4.7). Asymptotics (4.8) follows from (4.7).

We will prove (4.9), (4.11) for $y \geqslant 0$. The proof for $y < 0$ is similar. Let $|\lambda| \to \infty, 0 \leqslant y < C$. Then (4.4) gives

$$T_1(\lambda) = \frac{e^z}{4}\Big(\phi_{00}(\lambda) + O(e^{-x})\Big), \quad T(\lambda) = \frac{e^z}{2}\Big(e^{-iz}v_{01}(\lambda) + e^{iz}v_{02}(\lambda) + O(e^{-x})\Big)$$

and (4.5), (4.6) yield (4.9). Let $|\lambda|\to\infty, x-y<\pi$. Then (4.4) gives

$$T_1=\frac{1}{4}\Big(\phi_{00}e^{z}+\phi_{01}e^{-iz}+O(e^{-x})\Big),\quad T(\lambda)=\frac{1}{2}\Big(e^{(1-i)z}v_{01}(\lambda)+O(1)\Big)$$

and (4.5), (4.6) yield (4.11) and

$$T_1(\lambda)=\frac{e^{z}}{4}\Big(1+\alpha_0(\lambda)+O(z^{-9})\Big)+\frac{e^{\omega_1 z}}{4}\Big(1+\alpha_1(\lambda)+O(z^{-9})\Big),$$

which implies (4.10). ■

5 Proof of the main theorems

Using the definitions of $\Delta_1,\Delta_2,T_1,..$ we obtain following identities

$$\Delta_1^2+\Delta_2^2=1+T_2,\quad \Delta_1\Delta_2=2T_1^2-\frac{T_2+1}{2}=\frac{T-1}{2},\quad \rho=\frac{1-T}{2}+T_1^2, \tag{5.1}$$

$$D_\pm=(T_1\mp1)^2-\rho=\frac{(2T_1\mp1)^2-T_2}{2}=\frac{T\mp4T_1+1}{2},\quad D_+-D_-=-4T_1. \tag{5.2}$$

Then by Lemma 2.1, the functions $\Delta_1+\Delta_2,\Delta_1\Delta_2,D_\pm,\rho$ are entire and are real on the real line. We need the results about the function ρ. Recall $\rho^0=(\frac{\cosh z-\cos z}{2})^2, z=\lambda^{1/4}=x+iy$.

Lemma 5.1. *i) For each $V\in L^1_0(\mathbb{T})$ the function $\rho=\frac{1}{2}(T_2+1)-T_1^2$ satisfies*

$$|\rho(\lambda)-\rho^0(\lambda)|\leqslant 3\varkappa^2e^{2x+\varkappa},\quad \lambda\in\mathbb{C}, \tag{5.3}$$

$$|\rho^0(\lambda)|>\frac{e^{2x}}{4^2},\quad \text{if}\quad |\lambda^{1/4}-(1\pm i)\pi n|\geqslant\frac{\pi}{2\sqrt{2}},\quad n\in\mathbb{Z}, \tag{5.4}$$

$$\rho(\lambda)=\rho^0(\lambda)\Big(1+O(\lambda^{-3/2})\Big),\quad |\lambda|\to\infty,\quad \lambda\in\mathcal{D}_1, \tag{5.5}$$

$$\rho(\lambda)=\frac{e^{2z}}{16}\Big(1+2\alpha_0(\lambda)+O(z^{-9})\Big),\quad |\lambda|\to\infty,\quad |y|<\pi, \tag{5.6}$$

$$\rho(\lambda)=\frac{e^{(1+\omega_1)z}}{8}\Big(-1+\cosh(1-\omega_1)z+\alpha(\lambda)+O(z^{-9})\Big),\quad |\lambda|\to\infty,\quad x-|y|<\pi, \tag{5.7}$$

where

$$\alpha=(1+e^{(1-\omega_1)z})\alpha_0+(1+e^{-(1-\omega_1)z})\alpha_1-2\beta_{01},\quad \alpha(-4(\pi n)^4)=\frac{|\hat V_n|^2}{(2\pi n)^6}. \tag{5.8}$$

ii) Let $V\in L^1_0(\mathbb{T})$. Then for each integer $N>\|V\|^{1/3}$ the function $\rho(\lambda)$ has exactly $2N+1$ roots, counted with multiplicity, in the disk $\{\lambda:|\lambda|<4(\pi(N+\frac{1}{2}))^4\}$ and for each $n>N$, exactly two roots, counted with multiplicity, in the domain $\{\lambda:|\lambda^{1/4}-\pi(1+i)n|<\pi/4\}$. There are no other roots.

Proof. i) By Lemma 2.2, ρ is entire and real analytic. The identity $\rho = \frac{T_2+1}{2} - T_1^2$ yields

$$|\rho - \rho^0| = \Big|\frac{T_2 - T_2^0}{2} - T_1^2 + (T_1^0)^2\Big| \leqslant \frac{1}{2}\Big|T_2 - T_2^0\Big| + \Big|T_1 - T_1^0\Big|\Big|T_1 + T_1^0\Big|.$$

Then estimates (2.5), (2.10) provide (5.3).

Using the identity $\rho^0(\lambda) = -\sinh^2 \frac{(1-i)z}{2} \sin^2 \frac{(1-i)z}{2}$, and the estimate $e^{|y|} < 4|\sin z|, |z - \pi n| \geqslant \frac{\pi}{4}, n \in \mathbb{Z}$ (see [PT]), we obtain

$$|\rho^0(\lambda)| > \frac{1}{16} e^{2|\operatorname{Im} \frac{(1-i)z}{2}| + 2|\operatorname{Im} \frac{i(1-i)z}{2}|} = \frac{1}{16} e^{|y+x|+|y-x|} = \frac{e^{2x}}{16},$$

which yields (5.4). Asymptotics (5.5) follows immediately from (5.3),(5.4).

Substituting (4.9) into (5.1) we obtain (5.6). Substituting (4.10), (4.11) into (5.1) we obtain (5.7). We prove the second identity (5.8). Then the first identity (5.8) and (4.1),(4.3) give at $\lambda = -4(\pi n)^4$, i.e. $z = (1+i)\pi n$,

$$\alpha(\lambda) = 2(\alpha_0(\lambda) + \alpha_1(\lambda) - \beta_{01}(\lambda)) = 2c_{10}(\lambda) = \frac{-i}{8z^6} e^{z(-i-1)}(b_{01}(\lambda) + b_{10}(\lambda)) = \frac{|\hat{V}_n|^2}{(2\pi n)^6}.$$

ii) Introduce the contour $C_n(r) = \{\lambda : |\lambda^{1/4} - \pi(1+i)n| = \pi r\}$. Let $N_1 > N$ be another integer. Consider the contours $C_0(N + \frac{1}{2}), C_0(N_1 + \frac{1}{2}), C_n(\frac{1}{4}), n > N$. Note that we have $|\lambda| \geqslant 4\pi^4 \|V\|_1^{4/3}, \quad \varkappa \leqslant \frac{1}{2\sqrt{2\pi^3}}$ on all contours. Then (5.3), (5.4) yield on all contours

$$|\rho(\lambda) - \rho^0(\lambda)| \leqslant 3\varkappa^2 e^{2x+\varkappa} \leqslant \frac{e^{2x}}{16} < |\rho^0(\lambda)|.$$

Hence, by the Rouché theorem, ρ has as many roots, counted with multiplicity, as ρ^0 in each of the bounded domains and the remaining unbounded domain. Since $\rho^0(\lambda)$ has exactly one simple root at $\lambda = 0$ and exactly one root of multiplicity 2 at $-4(\pi n)^4, n \geqslant 1$, and since $N_1 > N$ can be chosen arbitrarily large, the point ii) follows. ■

Recall that the set $\{\lambda : D_+(\lambda) = 0\}$ is a periodic spectrum and the set $\{\lambda : D_-(\lambda) = 0\}$ is an anti-periodic spectrum. Now we prove a result about the number of periodic and anti-periodic eigenvalues in the large disc. Recall $D_\pm^0 = (\cos z \mp 1)(\cosh z \mp 1)$.

Lemma 5.2. *Let $V \in L_0^1(\mathbb{T})$. Then the following estimates and properties are fulfilled:*

$$|D_\pm(\lambda) - D_\pm^0(\lambda)| \leqslant 866\varkappa^2 e^{x+|y|}, \quad \lambda \in \overline{\Lambda_4^\pm}. \tag{5.9}$$

i) For each integer $N > \|V\|^{1/3}$ the function D_+ has exactly $2N+1$ roots, counted with multiplicity, in the domain $\{|\lambda|^{1/4} < 2\pi(N + \frac{1}{2})\}$ and for each $n > N$, exactly two roots, counted with multiplicity, in the domain $\{|\lambda^{1/4} - 2\pi n| < \frac{\pi}{2}\}$. There are no other roots.
ii) For each integer $N > \|V\|^{1/3}$ the function D_- has exactly $2N$ roots, counted with multiplicity, in the domain $\{|\lambda|^{1/4} < 2\pi N\}$ and for each $n > N$, exactly two roots, counted with multiplicity, in the domain $\{|\lambda^{1/4} - \pi(2n+1)| < \frac{\pi}{2}\}$. There are no other roots.

Proof. Identities (5.2) give

$$|D_\pm - D_\pm^0| \leqslant \frac{|T - T^0| + 4|T_1 - T_1^0|}{2} \leqslant \varkappa^2 \frac{1701 e^{x+|y|} + 4 \cdot 31 e^x}{2},$$

which yields (5.9).

i) Let $N' > N$ be another integer. Let λ belong to the contours $C_0(2N+1), C_0(2N'+1), C_{2n}(\frac{1}{2}), |n| > N$, where $C_n(r) = \{\lambda : |\lambda^{1/4} - \pi n| = \pi r\}, r > 0$. Note that $\varkappa \leqslant \frac{1}{(2\pi)^3}$ and $e^{\frac{1}{2}|y|} < 4|\sin\frac{z}{2}|, e^{\frac{1}{2}x} < 4|\sinh\frac{z}{2}|, z = \lambda^{1/4}$, on all contours. Then $e^{\frac{1}{2}(x+|y|)} < 16|\sin\frac{z}{2}\sinh\frac{z}{2}|$ and (5.9) on all contours yield

$$\Big|D_+(\lambda) - 4\sin^2\frac{z}{2}\sinh^2\frac{z}{2}\Big| \leqslant (30\varkappa)^2 e^{x+|y|} < (15\varkappa)^2 \Big|4\sin\frac{z}{2}\sinh\frac{z}{2}\Big|^2 < \Big|4\sin^2\frac{z}{2}\sinh^2\frac{z}{2}\Big|.$$

Hence, by Rouché's theorem, $D_+(\lambda)$ has as many roots, counted with multiplicities, as $\sin^2\frac{z}{2}\sinh^2\frac{z}{2}$ in each of the bounded domains and the remaining unbounded domain. Since $\sin^2\frac{z}{2}\sinh^2\frac{z}{2}$ has exactly one simple root at $\lambda = 0$ and exactly one root of the multiplicity two at $(2\pi n)^4, n \geqslant 1$, and since $N' > N$ can be chosen arbitrarily large, the point i) follows. The proof for D_- is similar. ■

Now we prove our first result about the Lyapunov function $\Delta = T_1 + \sqrt{\rho}$.

Proof of Theorem 1.1. By Lemma 5.1 ii, the function $\sqrt{\rho}$ is an analytic function in the domain $\mathcal{D}_r, r = 4\pi^4\|V\|_1^{4/3}$ and it has an analytic continuation onto the two-sheeted Riemann surface. The function Δ is analytic on the Riemann surface of the function $\sqrt{\rho}$. Due to identity (1.6) all branches of Δ have the forms $\Delta_m(z) = \frac{1}{2}(\tau_m(z) + \tau_m^{-1}(z)),\ m = 1, 2$.

i) We prove (1.12). Substituting (4.8), (5.5) into (1.8) we obtain the first asymptotics (1.12). Substituting (4.8) and the first asymptotics (1.12) into the identity $\Delta_2 = \frac{T-1}{2\Delta_1}$ (see (5.1)) we obtain the second asymptotics in (1.12).

By (1.6), the matrix $M(\lambda), \lambda \in \mathcal{D}_r$, for large $r > 0$, has the eigenvalues $\tau_m(\lambda)$ satisfying the identities $\tau_m(\lambda) + \tau_m(\lambda)^{-1} = 2\Delta_m(\lambda)$. Then $\tau_m(\lambda)$ has the form $\tau_m(\lambda) = \Delta_m(\lambda) + \sqrt{\Delta_m^2(\lambda) - 1}$, where $\sqrt{1} = 1$. Asymptotics (1.12) give

$$\tau_1(\lambda) = \cosh z(1 + O(z^{-6})) + \sqrt{\cosh^2 z(1 + O(z^{-6})) - 1} = e^z(1 + O(z^{-6})),$$

$$\tau_2(\lambda) = \cos z(1 + O(z^{-6})) + \sqrt{\cos^2 z(1 + O(z^{-6})) - 1} = e^{iz}(1 + O(z^{-6})),$$

$|\lambda| \to \infty, \lambda \in \mathcal{D}_1$, which yields asymptotics (1.11).

ii) By the Lyapunov Theorem (see Sect.1), $\lambda \in \sigma(\mathcal{L})$ iff $\Delta_m(\lambda) \in [-1, 1]$ for some $m = 1, 2$. If $\lambda \in \sigma(T)$, then $T_1(\lambda)$ is real. By ii), $\Delta(\lambda)$ is also real. Hence by (1.8), $\sqrt{\rho(\lambda)}$ is real and $\rho(\lambda) \geqslant 0$.

iii) Asymptotics from i) yield $\Delta_1 \neq D_2, \tau_1 \neq t_2$. Then we have the statements iii) and iv).

v) We have $\Delta_m' = \frac{1}{2}(\tau_m + \tau_m^{-1})' = \frac{1}{2}(1 - \tau_m^{-2})\tau_m' \neq 0, m = 1, 2$, since $\tau_m \neq 1, \tau_m' \neq 0$.

vi) Let $G_n = (E_n^-, E_n^+) \neq \emptyset$ for some $n \geqslant 1$. It is possible that $E_n^\pm$ is a periodic or anti-periodic eigenvalue. Assume that E_n^+ is not a periodic or anti-periodic eigenvalue. Then $\Delta_m(E_n^+) \in (-1, 1)$ for some $m = 1, 2$. If E_n^+ is not a branch point, then we have a contradiction. ■

We determine the asymptotics of the Lyapunov function near the positive semi-axis.

Lemma 5.3. *Let $V \in L_0^1(\mathbb{T})$. Then the following asymptotics are fulfilled*

$$\Delta_1(\lambda) = \frac{e^z}{2}\Big(1 + \alpha_0(\lambda) + O(z^{-9})\Big), \quad \Delta_2(\lambda) = \cos z + \frac{\beta(\lambda)}{2} + O(z^{-9}), \tag{5.10}$$

as $|\lambda| \to \infty, \quad |y| < \pi$, *where*

$$\beta = e^{\omega_1 z}\beta_{01} + e^{\omega_2 z}\beta_{02} - 2\alpha_0 \cos z, \quad \beta((\pi n)^4) = \frac{|\hat{V}_n|^2}{16(\pi n)^6}. \tag{5.11}$$

Proof. Substituting (4.9), (5.6) into the identity $\Delta_1 = T_1 + \sqrt{\rho}$ we have asymptotics (5.10) for Δ_1. Using identity $\Delta_2 = \frac{T-1}{2\Delta_1}$ and (4.9) we obtain

$$\Delta_2 = \frac{2\cos z + e^{\omega_1 z}\beta_{01} + e^{\omega_2 z}\beta_{02} + O(z^{-9})}{2(1 + \alpha_0 + O(z^{-9}))}$$

which yields asymptotics (5.10) for Δ_2. We prove the second identity in (5.11). If $\lambda = (\pi n)^4$, then $z = \pi n$ and we write $\beta = \beta((\pi n)^4), b_{jk} = b_{jk}((\pi n)^4), ...$ We have

$$\beta = (-1)^n(\beta_{02} + \beta_{01} - 2\alpha_0) = (-1)^n(\alpha_1 + \alpha_2 - c_{10} - c_{20}).$$

Identities (4.1)-(4.3) give

$$\beta = (-1)^n(b_{21} + c_{21} + b_{31} + c_{31} - b_{10} + b_{32} + c_{32} - b_{20} - b_{21} - c_{10} - c_{20})$$

$$= (-1)^n(b_{21} + b_{12} + b_{31} - b_{10} + b_{32} - b_{20}) + e^{-\pi n}(b_{31} + b_{13} + b_{32} + b_{23} - b_{10} - b_{01} - b_{20} - b_{02}).$$

We have

$$b_{21} + b_{12} = \frac{|\hat{V}_n|^2}{16(\pi n)^6}, \quad b_{20} + b_{10} = b_{31} + b_{32}, \quad b_{20} + b_{02} + b_{10} + b_{01} = b_{31} + b_{13} + b_{32} + b_{23}.$$

The last identities give the second identity in (5.11). ■

Now we prove the result about the asymptotics of the gaps and the resonance gaps.

Proof of Theorem 1.2. i) Recall that $\{\lambda_0^+, \lambda_{2n}^\pm, n \geqslant 1\}$ is the sequence of zeros of D_+ (counted with multiplicity) such that $\lambda_0^+ \leqslant \lambda_2^- \leqslant \lambda_2^+ \leqslant \lambda_4^- \leqslant \lambda_4^+ \leqslant \lambda_6^- \leqslant ....$ And $\{\lambda_{2n-1}^\pm, n \geqslant 1\}$ is the sequence of zeros of D_- (counted with multiplicity) such that $\lambda_1^- \leqslant \lambda_1^+ \leqslant \lambda_3^- \leqslant \lambda_3^+ \leqslant \lambda_5^- \leqslant \lambda_5^+ \leqslant ....$ Lemma 5.2 gives that $|(\lambda_n^\pm)^{1/4} - \pi n| < \frac{\pi}{2}, n > N$ for some $N > 0$. Furthermore, $\lambda_n^\pm$ are roots of $\Delta_j^2 - 1$ for some $j = 1, 2$. Asymptotics (5.10) of Δ_1 shows that $\Delta_1(\lambda) > 1$ for large $\lambda > 0$. Hence for such λ the spectrum of $\mathcal{L}$ has multiplicity 2 or 0, and the points $\lambda_n^\pm$ are roots of $\Delta_2^2 - 1$ for $n > N$.

We determine (1.16). Lemma 5.2.ii yields $\lambda_n^{\pm 1/4} = \pi n + \varepsilon_n, |\varepsilon_n| < \frac{\pi}{2}$ for $n > N$. Asymptotics (5.10) gives $\Delta_2(\lambda_n^\pm) = (-1)^n \cos\varepsilon_n + O(n^{-6})$. Then the identity $\Delta_2(\lambda_n^\pm) = (-1)^n$ gives $\varepsilon_n = O(n^{-3})$.

Now we will improve the asymptotics of ε_n. Using again (5.10) we have $(-1)^n\Delta_2(\lambda_n^\pm) = \cos\varepsilon_n + \frac{\beta(\lambda_n^\pm)}{2} + O(n^{-9})$. Note that

$$\alpha_j(\lambda_n^\pm) = \alpha_j((\pi n)^4) + O(1)\max_{|z-\pi n|\leqslant\varepsilon_n}|\alpha_j'(\lambda)| = \alpha_j((\pi n)^4) + O(n^{-9}), \quad n \to \infty.$$

since $\alpha_j'(\lambda) = \frac{d\alpha_j}{dz} O(n^{-3})$ and, by (4.1), $\frac{d\alpha_j}{dz} = O(n^{-3}), |z - \pi n| \leqslant 1, n \to +\infty$. The functions β_{jk}, c_{jk} have similar asymptotics. Then $\beta(\lambda_n^{\pm}) = \beta((\pi n)^4) + O(n^{-9})$ and

$$\Delta_2(\lambda_n^{\pm}) = (-1)^n \Big(1 - \frac{\varepsilon_n^2}{2} + \frac{\beta((\pi n)^4)}{2} + O(n^{-9})\Big), \quad \beta((\pi n)^4) = \frac{|\hat{V}_n|^2}{16(\pi n)^6},$$

see (5.11). Then the identity $\Delta_2(\lambda_n^{\pm}) = (-1)^n$ gives $\varepsilon_n^2 = \beta((\pi n)^4) + O(n^{-9})$ and we obtain

$$4(\pi n)^3 \varepsilon_n = \pm\sqrt{|\hat{V}_n|^2 + O(n^{-3})} = \pm|\hat{V}_n| + O(n^{-\frac{3}{2}}),$$

$$\lambda_n^{\pm} = (\pi n + \varepsilon_n)^4 = (\pi n)^4 + 4(\pi n)^3 \varepsilon_n + O(n^{-4}),$$

which implies (1.16).

Asymptotics (5.10), (1.16) provide $-1 < \Delta_2(\lambda) < 1$, as $\lambda \in (\lambda_n^+, \lambda_{n+1}^-)$ and $\Delta_2(\lambda) \notin [-1, 1]$, as $\lambda \in (\lambda_n^-, \lambda_n^+), n > N$. Then each interval $[\lambda_n^+, \lambda_{n+1}^-], n > N$ is a spectral band with multiplicity 2 and each interval $(\lambda_n^-, \lambda_n^+), n > N$ is a gaps.

We will prove (1.17). We consider the case $\operatorname{Im} r_n^{\pm} \geqslant 0$. The proof for $\operatorname{Im} r_n^{\pm} < 0$ is similar. Lemma 5.1. ii implies $z = \lambda^{1/4} = (1+i)\pi n + \delta_n, |\delta_n| < 1$ for $\lambda = r_n^{\pm}, n > N$. Then (5.7) gives $\rho(r_n^{\pm}) = \frac{e^{2\pi n}}{8}(\cosh(1+i)\delta_n - 1 + O(n^{-6}))$. The condition $\rho(r_n^{\pm}) = 0$ yields $\delta_n = O(n^{-3})$.

Using (5.7) again and the asymptotics $\alpha(r_n^{\pm}) = \alpha(-4(\pi n)^4) + O(n^{-9})$ we obtain

$$\rho(r_n^{\pm}) = \frac{e^{2\pi n}}{8}\Big(i\delta_n^2 + \alpha(-4(\pi n)^4) + O(n^{-9})\Big).$$

The condition $\rho(r_n^{\pm}) = 0$ yields $\delta_n^2 = i\alpha(-4(\pi n)^4) + O(n^{-9})$. Identities (5.8) give

$$\delta_n^2 = \frac{i|\hat{V}_n|^2 + O(n^{-3})}{(2\pi n)^6}, \qquad \delta_n = \pm\frac{(1+i)|\hat{V}_n| + O(n^{-2/3})}{\sqrt{2}(2\pi n)^3},$$

and then

$$r_n^{\pm} = \Big((1+i)\pi n + \delta_n\Big)^4 = -4(\pi n)^4 - 8(1-i)(\pi n)^3\delta_n + O(n^{-4}),$$

which yields (1.17).

ii) Assume that we have the periodic spectrum $\lambda_0, \lambda_{2n}^{\pm}, n \geqslant 1$. Using the asymptotics (1.16) and repeating the standard arguments (see [PT,pp.39-40]) we obtain the Hadamard factorization

$$D_+(\lambda) = -\frac{\lambda - \lambda_0}{4} \prod_{n \geqslant 1} \frac{(\lambda_{2n}^+ - \lambda)(\lambda_{2n}^- - \lambda)}{(2\pi n)^8}.$$

By the similar way, we determine D_- by the anti-periodic spectrum. Using (5.2) we have ρ. Thus, we recover the resonances.

iii) Suppose, that we have the periodic spectrum and the set of the resonances. Then we determine the functions ρ by the resonances, and D_+ by the periodic spectrum. Using (5.2) we get T_1, T_2 and then D_-. Thus, we recover the anti-periodic spectrum. The proof of another case is similar. ■

6 The spectrum for the small potential

Proof of Theorem 1.3. Recall that $\mathcal{L}y = y'''' + \gamma V y, V \in L^1_0(\mathbb{T}), \gamma \in \mathbb{R}$ and let $T^\gamma_m(\lambda) = T_m(\lambda, \gamma V), m = 1, 2, \rho^\gamma(\lambda) = \rho(\lambda, \gamma V), ...$ Due to (2.11) we have

$$T^\gamma_m = T^0_m + \gamma^2 T_{m,2} + \gamma^3 \widetilde{\eta}_m, \quad |\widetilde{\eta}_m(\lambda, \gamma)| \leqslant \frac{(m\|V\|)^3}{3!|z|^9} e^{m+\varkappa}, \quad \lambda \in \mathbb{C}, \tag{6.1}$$

$$T_{m,2}(\lambda) = \frac{1}{4}\int_0^m dt \int_0^t V(s)V(t)\varphi^0_3(m - t + s, \lambda)\varphi^0_3(t - s, \lambda)ds, \quad m = 1, 2, \tag{6.2}$$

where T^0_m, ρ^0 were given by (1.13), and $\varphi^0_3(t, \lambda)$ was given by (2.2) and $\widetilde{\eta}_m(\lambda, \gamma)$ is a real analytic function of $(\lambda, \gamma) \in \mathbb{C}^2$. Simple calculations imply

$$T^0_m(\lambda) = 1 + \frac{m^4}{4!}\lambda + O(\lambda^2), \quad \varphi^0_3(t, \lambda) = \frac{t^3}{6} + O(\lambda), \quad |\lambda| \to 0, \tag{6.3}$$

uniformly on $t \in [0, 2]$. Substituting this asymptotics into identity (6.2) we obtain

$$T_{m,2}(\lambda) = v_m + O(\lambda), \quad |\lambda| \to 0, \tag{6.4}$$

where v_m was given by (1.20). Using identity (1.4) we obtain

$$\rho^\gamma(\lambda) = \rho^0(\lambda) + \gamma^2 \widetilde{\rho}(\lambda, \gamma), \quad \rho^0(\lambda) = \frac{\lambda}{4} + O(\lambda^2), \quad \lambda \to 0, \tag{6.5}$$

$$\widetilde{\rho}(\lambda, \gamma) = \frac{T_{2,2}}{2} - 2T^0_1(\lambda)T_{1,2}(\lambda) + O(\gamma), \quad \gamma \to 0, \tag{6.6}$$

uniformly in any bounded domain in $\mathbb{C}$. The function $\rho^0(\lambda)$ has simple roots $\lambda = 0$ and $\rho^0(\lambda), \widetilde{\rho}(\lambda, \gamma)$ are analytic at the points $\lambda = 0, \gamma = 0$. Applying the Implicit Function Theorem to $\rho^\gamma(\lambda) = \rho^0(\lambda) + \gamma^2\widetilde{\rho}(\lambda, \gamma)$ and $\frac{\partial}{\partial\lambda}\rho^\gamma(\lambda)\Big|_{\lambda=\gamma=0} \neq 0$, we obtain a unique solution $r^-_0(\gamma), |\gamma| < \varepsilon, r^-_0(0) = 0$ of the equation $\rho^\gamma(\lambda) = 0, |\gamma| < \varepsilon$ for some $\varepsilon > 0$.

In order to prove asymptotics (1.18) we rewrite the equation $\rho^\gamma(\lambda) = 0$ in the form

$$\frac{\lambda}{4} + O(\lambda^2) = -\gamma^2\widetilde{\rho}(\lambda, \gamma), \quad \lambda = r^-_0(\gamma), \tag{6.7}$$

which yields $r^-_0(\gamma) = O(\gamma^2), \gamma \to 0$. Then using asymptotics (6.3), (6.4), (6.6), we obtain

$$\widetilde{\rho}(\lambda, \gamma) = \frac{v_2}{2} - 2v_1 + O(\gamma + \lambda). \tag{6.8}$$

Substituting the last asymptotics into (6.7), we have (1.18).

Identity (5.2) gives

$$D^\gamma_+(\lambda) = D^0_+(\lambda) + \gamma^2\widetilde{D}_+(\lambda, \gamma), \quad \widetilde{D}_+(\lambda, \gamma) = 2(2T^0_1 - 1)T_{1,2} - \frac{T_{2,2}}{2} + O(\gamma), \quad \gamma \to 0,$$

uniformly in any bounded domain in $\mathbb{C}$, and D_+^0 was given by (1.14), $D_+^0 = -\frac{\lambda}{4} + O(\lambda^2), |\lambda| \to 0$. The function $D_+^0(\lambda)$ has simple roots $\lambda = 0$ and $D_+^0(\lambda), \widetilde{D}_+(\lambda,\gamma)$ are analytic at the points $\lambda = 0, \gamma = 0$. Applying the Implicit Function Theorem to $D_+^\gamma(\lambda) = D_+^0(\lambda) + \gamma^2 g(\lambda,\gamma)$ and $\frac{\partial}{\partial\lambda} D_+^\gamma|_{\lambda=\gamma=0} \neq 0$, we obtain a unique solution $\lambda_0^+(\gamma), |\gamma| < \varepsilon, \lambda_0^+(0) = 0$ of the equation $D_+^\gamma(\lambda) = 0, |\gamma| < \varepsilon$ for some $\varepsilon > 0$.

We prove asymptotics (1.18). We write the equation $D_+^\gamma(\lambda) = 0$ in the form

$$-\frac{\lambda}{4} + O(\lambda^2) = -\gamma^2 \widetilde{D}_+(\lambda,\gamma), \quad \lambda = \lambda_0^+(\gamma), \tag{6.9}$$

which yields $\lambda_0^+(\gamma) = O(\gamma^2), \gamma \to 0$. Then using (6.3), (6.4) we obtain

$$\widetilde{D}_+(\lambda,\gamma) = 2v_1 - \frac{v_2}{2} + O(\gamma), \quad \gamma \to 0.$$

Substituting the last asymptotics into (6.9), we have (1.18).

We prove (1.19). Substituting asymptotics (1.18) into (6.1) and using (6.3), (6.4) we obtain

$$T_1^\gamma(\lambda_0^+) = 1 - \gamma^2 A + O(\gamma^3), \quad A = \frac{v_2}{12} - \frac{4v_1}{3}, \quad \gamma \to 0. \tag{6.10}$$

Using asymptotics (1.18) we have

$$\rho^\gamma(\lambda_0^+) = sy(\gamma), \quad y = (\rho^\gamma)'(r_0^-) + O(s), \quad s = \lambda_0^+ - r_0^- \to 0, \ \ as \ \ \gamma \to 0.$$

Substituting $\rho^\gamma(\lambda_0^+) = sy(\gamma)$ into the identity $D_+ = (T_1 - 1)^2 - \rho$ (see (5.2)), and using $D_+(\lambda_0^+) = 0$ we obtain

$$s = \lambda_0^+ - r_0^- = \frac{(T_1^\gamma(\lambda_0^+) - 1)^2}{y(\gamma)}. \tag{6.11}$$

Asymptotics (6.5), (6.6) and $(\rho^0)'(\lambda) = \frac{1}{4} + O(\lambda), |\lambda| \to 0$ give

$$y(\gamma) = (\rho^0)'(r_0^-) + O(\gamma^2) = \frac{1}{4} + O(\gamma^2), \tag{6.12}$$

where we have used asymptotics (1.18). Substituting (6.10), (6.12) into (6.11) we have (1.19).

Recall the identity $\Delta_m^\gamma = T_1^\gamma - (-1)^m \sqrt{\rho^\gamma}, m = 1, 2$. Then

$$\Delta_m^\gamma(\lambda) = T_1^\gamma(r_0^-) - (-1)^m \sqrt{\lambda - r_0^-}\sqrt{y(\gamma)} + O((\lambda - r_0^-)^{\frac{3}{2}}), \quad \lambda - r_0^- \to +0.$$

Hence the function Δ_1^γ is increasing in the interval $(r_0^-, r_0^- + \varepsilon)$ for some $\varepsilon > 0$ (see Fig.(**??**)) and the function Δ_2^γ is decreasing in this interval. Asymptotics (6.10) gives

$$\Delta_1^\gamma(r_0^-) = \Delta_2^\gamma(r_0^-) = T_1^\gamma(r_0^-) = 1 - \gamma^2 A + O(\gamma^3), \quad \gamma \to 0, \quad r_0^- = r_0^-(\gamma).$$

Assume that $A > 0$. Then there exists $\delta > 0$ such that $-1 < \Delta_1^\gamma(r_0^-) < 1$ for each $\gamma \in (0,\delta)$ and Δ_1^γ is increasing in the interval $(r_0^-, r_0^- + \varepsilon)$ with some $\varepsilon > 0$. Then by Theorem 1.1 iv, Δ_1^γ is increasing in the interval $(r_0^-, \lambda_0(\gamma))$, where $\Delta_1^\gamma(\lambda_0(\gamma)) = 1$. Hence $\lambda_0(\gamma) = \lambda_{2n}^\pm(\gamma)$

for some n. Note that $\lambda_0(0) = 0$, since $\Delta_1^0(\lambda) = \cosh z$. Then $\lambda_0(\gamma) = \lambda_0^+(\gamma) = \lambda_0^+$. Hence $-1 < \Delta_1^\gamma(\lambda) < 1, \lambda \in (r_0^-, \lambda_0^+)$ and $\Delta_1^\gamma(\lambda_0^+) = 1$. Moreover, substituting identities (6.1), (6.5) into the identities $\Delta_2^\gamma = T_1^\gamma - \sqrt{\rho^\gamma}$, we obtain $\Delta_2^\gamma = \cos z + o(\gamma), \gamma \to 0$. Then the function $\Delta_2^\gamma + 1, 0 \leqslant \gamma < \delta$ has not any zero in the interval (r_0^-, λ_0^+). Then $-1 < \Delta_2^\gamma(\lambda) < 1$ for each $\lambda \in (r_0^-, \lambda_0^+)$. Hence by Theorem 1.1 ii), the spectral interval (r_0^-, λ_0^+) has multiplicity 4.

Now we will show that $A > 0$ for all $V \in L_0^1(\mathbb{T}), V \neq 0$. Firstly we prove the following identity

$$A = \frac{1}{2 \cdot 288} \int_0^1 f(u) \int_u^1 V(t)V(t-u)dtdu, \quad f(u) = u^2(2u^4 - 6u^3 + 5u^2 - 1). \tag{6.13}$$

Let $A_0 = 2 \cdot 288A$. Identities (1.20), (1.19) give

$$A_0 = \frac{1}{6}\int_0^2 dt \int_0^t V(s)V(t)(2-t+s)^3(t-s)^3 ds - \frac{8}{3}h_1 = -\frac{8h_1}{3} + \frac{h_2}{3} + \frac{h_3}{6}, \tag{6.14}$$

where

$$h_1 = \int_0^1 dt \int_0^t V(s)V(t)(1-t+s)^3(t-s)^3 ds,$$

$$h_2 = \int_0^1 dt \int_0^t V(s)V(t)(2-t+s)^3(t-s)^3 ds, \quad h_3 = \int_0^1 dt \int_0^1 V(s)V(t)(1-t+s)^3(1+t-s)^3 ds.$$

Then

$$h_1 = I_3 - 3I_4 + 3I_5 - I_6, \quad h_2 = 8I_3 - 12I_4 + 6I_5 - I_6, \quad h_3 = -6I_2 + 6I_4 - 2I_6, \tag{6.15}$$

where

$$I_m = \int_0^1 dt \int_0^t V(s)V(t)(t-s)^m ds = \int_0^1 u^m du \int_u^1 V(t-u)V(t)dt, \ m \geqslant 0. \tag{6.16}$$

Substituting (6.15) into (6.14) and using (6.16) we obtain

$$A_0 = 5I_4 - 6I_5 + 2I_6 - I_2 = \int_0^1 f(s) \int_s^1 V(t)V(t-s)dtds, \tag{6.17}$$

where $f = u^2(2u^4 - 6u^3 + 5u^2 - 1)$, which yields (6.13).

Now we prove that $A > 0$. Using $f^{(j)}(0) = f^{(j)}(1), 0 \leqslant j \leqslant 4, f^{(5)}(1) = -f^{(5)}(0) = 6!$ we have

$$f(t) = \sum_n f_n e^{i2\pi nt}, \quad f_n = \frac{6!2}{(2\pi n)^6}, \quad n \neq 0, \quad f_0 = -\frac{1}{21}, \quad V(t) = \sum_{n\neq 0} \hat{V}_n e^{i2\pi nt}. \tag{6.18}$$

Substituting these identities into (6.17) we get

$$A_0 = \sum_{m,n} \hat{V}_n \hat{V}_m \int_0^1 f(s)ds \int_s^1 e^{i2\pi(n+m)t} e^{-i2\pi ns} dt = F_1 + F_2$$

where

$$F_2 = \sum_{m+n\neq 0} \frac{\hat{V}_n \hat{V}_m}{2\pi i(n+m)} \int_0^1 f(s) e^{-i2\pi ns}(1 - e^{i2\pi(n+m)s}) ds = \sum_{m+n\neq 0} \hat{V}_n \hat{V}_m \frac{f_n - f_m}{2\pi i(n+m)} = 0$$

and

$$F_1 = \sum_{-\infty}^{\infty} |\hat{V}_n|^2 \int_0^1 f(s)(1-s) e^{-i2\pi ns} ds.$$

Note that

$$\int_0^1 f(s)(1-s) e^{-i2\pi ns} ds = \sum_p \int_0^1 (1-s) f_p e^{i2\pi(p-n)s} ds = \sum_{p\neq n} \frac{-f_p}{i2\pi(p-n)} + \frac{f_n}{2}.$$

Then

$$F_1 = \sum_{n\neq 0} |\hat{V}_n|^2 \Big(\sum_{p\neq n} \frac{-f_p}{i2\pi(p-n)} + \frac{f_n}{2} \Big) = \sum_{n\neq 0} |\hat{V}_n|^2 \frac{f_n}{2} > 0$$

since $\hat{V}_0 = 0, F_1$ is real, $f_p > 0, p \neq 0.$ ■

7 Complex resonances.

Consider the operator $\mathcal{L}^\gamma = \frac{d^4}{dt^4} + \gamma\delta_{per}, \gamma \in \mathbb{C}$, where δ_{per}, $\delta_{per} = \sum \delta(t-n)$. Let $T_1^\gamma = T_1(\cdot, \gamma\delta_{per}), \rho^\gamma = \rho(\cdot, \gamma\delta_{per}), ...$ Recall that $\lambda = z^4$ and for the case $V = 0$ we have

$$T_1^0(\lambda) = c_+(z), \quad \rho^0(\lambda) = c_-^2(z), \quad c_\pm(z) = \frac{\cosh z \pm \cos z}{2}, \quad s_\pm(z) = \frac{\sinh z \pm \sin z}{2}. \tag{7.1}$$

Lemma 7.1. *For the operator $\mathcal{L}^\gamma = d^4/dt^4 + \gamma\delta_{per}$ the following identities are fulfilled:*

$$T_1^\gamma = T_1^0 - \gamma \frac{s_-}{4z^3}, \quad \rho^\gamma = \rho^0 - \gamma \frac{c_- s_+}{2z^3} + \gamma^2 \frac{s_-^2}{16 z^6}. \tag{7.2}$$

Proof. The solution $y(t)$ of the equation $y'''' + V^\gamma y = \lambda y$ and y', y'' are continuous and $y'''(n+0) - y'''(n-0) = -\gamma y(n)$, $n \in \mathbb{Z}$. Then the fundamental solutions $\varphi_j(t, \lambda), j = 0, 1, 2, 3,$ have the form

$$\varphi_j(t) = \varphi_j^0(t), \quad 0 \leqslant t < 1, \quad \varphi_j(t) = \varphi_j^0(t) - \gamma\varphi_3^0(t-1)\varphi_j^0(1), \quad 1 \leqslant t < 2,$$

$$\varphi_j(t) = \varphi_j^0(t) - \gamma\varphi_3^0(t-1)\varphi_j^0(1) - \gamma\varphi_3^0(t-2)\Big(\varphi_j^0(2) - \gamma\varphi_3^0(1)\varphi_j^0(1)\Big), \ 2 \leqslant t < 3,$$

here and below we write $\varphi_j(t) = \varphi_j(t, \lambda)$. Then

$$T_1^\gamma = \frac{1}{4} \sum_0^3 \varphi_j^{(j)}(1) = \frac{1}{4} \Big(\sum_0^3 (\varphi_j^0)^{(j)}(1) - \gamma\varphi_3^0(1) \Big) = T_1^0 - \frac{\gamma}{4}\varphi_3^0(1),$$

which yields the first identity in (7.2), and

$$T_2^\gamma = \frac{1}{4}\sum_0^3 \varphi_j^{(j)}(2) = T_2^0 - \frac{\gamma}{4}\sum_0^3 (\varphi_3^0)^{(j)}(1)\varphi_j^0(1) - \frac{\gamma}{4}\Big(\varphi_3^0(2) - \gamma(\varphi_3^0(1))^2\Big).$$

Identities (2.3) give $\sum_0^3 (\varphi_3^0)^{(j)}(1)\varphi_j^0(1) = \varphi_3^0(2)$ and we obtain

$$T_2^\gamma = T_2^0 - \frac{\gamma}{2}\varphi_3^0(2) + \frac{\gamma^2}{4}(\varphi_3^0(1))^2.$$

Then

$$\rho^\gamma = \frac{T_2^\gamma + 1}{2} - (T_1^\gamma)^2 = \frac{1}{2}\Big(T_2^0 - \frac{\gamma}{2}\varphi_3^0(2) + \frac{\gamma^2}{4}(\varphi_3^0(1))^2 + 1\Big) - \Big(T_1^0 - \frac{\gamma}{4}\varphi_3^0(1)\Big)^2$$
$$= \rho^0 - \frac{\gamma}{4}\varphi_3^0(2) + T_1^0\frac{\gamma}{2}\varphi_3^0(1) + \Big(\frac{\gamma}{4}\varphi_3^0(1)\Big)^2.$$

Using

$$2T_1^0\varphi_3^0(1) - \varphi_3^0(2) = \frac{1}{2z^3}\Big((\cosh z + \cos z)(\sinh z - \sin z) - (\sinh 2z - \sin 2z)\Big) = -\frac{2c_- s_+}{z^3},$$

we obtain the second identity in (7.2). ■

We shall show the existence of complex resonances. In this case γ is not small parameter. We rewrite ρ^γ, given by (7.2), in the form

$$\rho^\gamma(\lambda) = \Big(c_-(z) - \gamma\frac{s_+(z)}{4z^3}\Big)^2 - \frac{\gamma^2 \sinh z \sin z}{16z^6} = \frac{s_+^2(z)}{(4z^3)^2}(F_+(z) - \gamma)(F_-(z) - \gamma), \qquad (7.3)$$

$$F_\pm(z) = \frac{4z^3 c_-(z)}{s_+(z) \pm \sqrt{u}}, \quad u = \sinh z \sin z, \quad z \in E_n = (2\pi n, \pi(2n+1)) \qquad (7.4)$$

where $z = \lambda^{1/4}$ and $\sqrt{1} = 1$. The following properties of $F = F_+$ are fulfilled: for each $n \geqslant 1$ the functions F are analytic on the interval η_n and,

$$F'(z) \to -\infty \ \text{ as } \ z \to 2\pi n + 0, \quad \text{and} \ \ F'(z) \to \infty \ \text{ as } \ z \to (2n+1)\pi - 0.$$

Hence for each $n \geqslant 1$ there exist points $z_n \in E_n$, such that $F(z_n) = \min_{z\in E_n} F(z)$. Then the Taylor expansion of the function F at the point z_n is given by

$$F(z) = F(z_n) + \frac{(z-z_n)^2}{2}\Big(F''(z_n) + \widetilde{F}(z)\Big), \quad \widetilde{F}(z) = O(z - z_n) \ \text{ as } \ z - z_n \to 0. \qquad (7.5)$$

Moreover, for each fixed $z \in E_n$, we have

$$F(z) = 4z^3(1 + O(e^{-z/2})) = 4z^3(1 + O(e^{-\pi n})), \quad \text{as } \ n \to \infty. \qquad (7.6)$$

We prove that there exist the real and non-real branch points of the function $\Delta^\gamma(\lambda)$ for some γ. The corresponding behavior of the functions $\rho^\gamma(\lambda)$, $\Delta^\gamma(\lambda)$ is shown by Fig.2.

Now we prove Proposition 1.4. Let $\gamma_n = F(z_n)$.We prove that there exists $N > 0$ such that for each $n \geqslant N$ there exist $\varepsilon_n > 0$ and the functions $r_n^{\pm}(\gamma), \gamma_n - \varepsilon_n < \gamma < \gamma_n + \varepsilon_n$, such that $r_n^{\pm}(\gamma)$ are zeros of the function $\rho^{\gamma}(\lambda), r_n^{\pm}(0) = z_n^4$. Moreover, the following asymptotics are fulfilled:

$$r_n^{\pm}(\gamma) = z_n^4 \pm 4z_n^3 \Big(\frac{2\nu}{F''(z_n)}\Big)^{\frac{1}{2}} + O\Big(\nu^{\frac{3}{2}}\Big), \quad F''(z_n) > 0 \ \text{ as } \nu = \gamma - \gamma_n \to 0. \tag{7.7}$$

Proof of Proposition 1.4. Differentiation in identity (7.4) for $F = F_+$ yields

$$F' = gF, \quad g = \frac{3}{z} + \frac{s_+}{c_-} - h, \quad h = \frac{a'}{a}, \quad a = s_+ + \sqrt{u}. \tag{7.8}$$

Asymptotics (7.6) gives $F(z_n) \neq 0, n \geqslant N$. Then using identity $F'(z_n) = 0$ we obtain $g(z_n) = 0$, hence

$$h(z_n) = \frac{3}{z_n} + \frac{s_+(z_n)}{c_-(z_n)} = 1 + \frac{3}{z_n} + O(e^{-2\pi n}), \quad n \to +\infty. \tag{7.9}$$

We get

$$F'' = g'F + gF', \quad F''(z_n) = g'(z_n)F(z_n), \tag{7.10}$$

and

$$g' = g_0 - h', \quad g_0(z_n) = -\frac{3}{{z_n}^2} + \frac{c_+(z_n)}{c_-(z_n)} - \frac{s_+^2(z_n)}{c_-^2(z_n)} = O(n^{-2}), \quad n \to +\infty. \tag{7.11}$$

Consider h'. Differentiating identities (7.8) we obtain

$$h' = \frac{a''}{a} - h^2, \quad a' = c_+ + \frac{u'}{2\sqrt{u}}, \quad a'' = s_- + \frac{u''}{2\sqrt{u}} - \frac{1}{4\sqrt{u}}\Big(\frac{u'}{\sqrt{u}}\Big)^2. \tag{7.12}$$

Using identity (7.8) for a we have

$$\frac{a''(z_n)}{a(z_n)} = 1 + O(e^{-\pi n}), \quad n \to +\infty. \tag{7.13}$$

Substituting (7.9), (7.13) into (7.12) we have $h'(z_n) = -\frac{6}{z_n} + O(n^{-2})$. Then (7.11) gives $g'(z_n) = \frac{6}{z_n} + O(n^{-2})$. Thus (7.6), (7.10) imply

$$F''(z_n) = 24z_n^2(1 + O(n^{-1})), \quad n \to +\infty.$$

Thus, for each $r > 0$ there exists $N > 0$ such that $F''(z_n) \geqslant r$ for all $n \geqslant N$.

Let $n \geqslant N$. Substituting $F(z_n) = \gamma_n$ into (7.5) we have

$$F(z) = \gamma_n + \frac{(z - z_n)^2}{2}\Big(F''(z_n) + \widetilde{F}(z)\Big), \quad \widetilde{F}(z) = O(z - z_n), \quad z - z_n \to 0. \tag{7.14}$$

There exists $\delta > 0$ such that the function $\widetilde{F}(z)$ is analytic in the disk $\{|z - z_n| < \delta\}$ and $F''(z_n) + \widetilde{F}(z) > 0$ for $z - z_n < (-\delta, \delta)$. Then using (7.14) we rewrite the equation $F(z) - \gamma = 0$ in the form

$$\Phi_+(z,\gamma)\Phi_-(z,\gamma) = 0, \quad \Phi_\pm(z,\gamma) = \sqrt{\nu} \mp \frac{z - z_n}{\sqrt{2}}\sqrt{F''(z_n) + \widetilde{F}(z)}. \tag{7.15}$$

Using $(\Phi_\pm)'_z(z_n, \gamma_n) \neq 0$ and applying the Implicit Function Theorem we obtain that $\Phi_\pm(z,\gamma)$ has exactly one simple root $z_\pm(\gamma_n + \nu)$ in the disk $\{|\nu| < \varepsilon_n\}$ for some $\varepsilon_n > 0$ such that

$$z_\pm(\gamma) = z_n + \frac{\sqrt{2\nu}}{\sqrt{F''(z_n) + \widetilde{F}(z_\pm(\gamma))}}, \qquad z_\pm(\gamma_n) = z_n. \tag{7.16}$$

Thus, the function $F(z) - \gamma$, $|\nu| < \varepsilon_n$ has exactly two zeros $z_\pm(\gamma)$. Then (7.3) gives that the function $\rho^\gamma(\lambda)$, $|\gamma - \gamma_n| < \varepsilon_n$ has exactly two zeros $r_n^\pm(\gamma) = z_\pm^4(\gamma)$. Substituting asymptotics (7.5) for $\widetilde{F}$ into (7.16), we obtain (7.7). ■

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