

# REDUCED GUTZWILLER FORMULA WITH SYMMETRY: CASE OF A LIE GROUP

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ABSTRACT. We consider a classical Hamiltonian  $H$  on  $\mathbb{R}^{2d}$ , invariant by a Lie group of symmetry  $G$ , whose Weyl quantization  $\widehat{H}$  is a selfadjoint operator on  $L^2(\mathbb{R}^d)$ . If  $\chi$  is an irreducible character of  $G$ , we investigate the spectrum of its restriction  $\widehat{H}_\chi$  to the symmetry subspace  $L_\chi^2(\mathbb{R}^d)$  of  $L^2(\mathbb{R}^d)$  coming from the decomposition of Peter-Weyl. We give semi-classical Weyl asymptotics for the eigenvalues counting function of  $\widehat{H}_\chi$  in an interval of  $\mathbb{R}$ , and interpret it geometrically in terms of dynamics in the reduced space  $\mathbb{R}^{2d}/G$ . Besides, oscillations of the spectral density of  $\widehat{H}_\chi$  are described by a Gutzwiller trace formula involving periodic orbits of the reduced space, corresponding to quasi-periodic orbits of  $\mathbb{R}^{2d}$ .

RÉSUMÉ. On considère un hamiltonien classique  $H$  sur  $\mathbb{R}^{2d}$ , invariant par un groupe de Lie  $G$  de symétrie, et dont le quantifié de Weyl  $\widehat{H}$  est un opérateur autoadjoint sur  $L^2(\mathbb{R}^d)$ . Si  $\chi$  est un caractère irréductible de  $G$ , on étudie le spectre de sa restriction  $\widehat{H}_\chi$  au sous-espace de symétrie  $L_\chi^2(\mathbb{R}^d)$  de  $L^2(\mathbb{R}^d)$  provenant de la décomposition de Peter-Weyl. On donne une asymptotique de Weyl semi-classique pour la fonction de comptage des valeurs propres de  $\widehat{H}_\chi$  dans un intervalle  $\mathbb{R}$ , et on interprète géométriquement le premier terme dans l'espace réduit  $\mathbb{R}^{2d}/G$ . Par ailleurs, on décrit les oscillations de la densité spectrale de  $\widehat{H}_\chi$  en donnant une formule de trace à la Gutzwiller, faisant apparaître les orbites périodiques de l'espace réduit, qui correspondent à des orbites quasi-périodiques de  $\mathbb{R}^{2d}$ .

## 1. INTRODUCTION

The purpose of this work is to give semi-classical spectral asymptotics of a quantum Hamiltonian on  $\mathbb{R}^{2d}$  reduced by a compact Lie group of symmetry  $G$ . We will interpret coefficients in terms of reduced classical dynamics in  $\mathbb{R}^{2d}/G$ . This paper follows a preceding study on finite groups (see [6],[7]). For a more detailed introduction to the concepts, we refer the reader to this article.

Mathematically, first systematic quantum investigations with symmetry reduction were carried out on a Riemannian compact manifold  $M$  for the Laplacian, or for an elliptic differential operator, as it was done for a compact Lie group of symmetry simultaneously by Donnelly ([13]) and Brüning & Heintze ([4]) in 1978-79. They gave Weyl asymptotics of the eigenvalues counting function of the operator for high energy, and interpreted the results in terms of the reduced space  $M/G$ . The same study was done by Helffer and Robert in 1984-86 in  $\mathbb{R}^d$  for an elliptic pseudo-differential operator with a finite or compact Lie group of symmetry (see [20], [21]). The semi-classical version of this work was done by El Houakmi and Helffer in 1984-91, still for Weyl asymptotics (see [14], [15]). However the computation of the leading term wasn't totally achieved, which forms one of the goals of this article. Coming back to compact manifolds, at the end of the 80's, Guillemin and Uribe showed oscillations of the spectral density of a reduced elliptic pseudo-differential operator could be described by a trace formula involving periodic orbits of the reduced space (see [16], [17], [18]). Another aim of this paper is to give an analogue of this result in  $\mathbb{R}^d$  in the context of articles of Helffer and Robert previously quoted, using a different method. This investigation is also related to the work of Borthwick, Paul and

Uribe [3] (see also Charles [9]) with Toeplitz operator on Kähler manifolds: even if their point of view consists in quantizing directly the Hamiltonian in the reduced space, following the theory of geometric quantization and symplectic reduction of Kostant, Souriau, Guillemin et al, they give a reduced Gutzwiller formula in this context.

We briefly recall our setting:  $H : \mathbb{R}^{2d} \rightarrow \mathbb{R}$  is a smooth Hamiltonian.  $G$  is a compact Lie group of invertible linear applications of the configuration space  $\mathbb{R}^d$ . It acts symplectically on the phase space  $\mathbb{R}^d \times \mathbb{R}^d$  by  $M : G \rightarrow Sp(d, \mathbb{R})$  defined by:

$$(1.1) \quad M(g)(x, \xi) := (gx, {}^t g^{-1}\xi)$$

We assume that  $G$  is a symmetry for  $H$ , i.e.  $H$  is  $G$ -invariant:

$$(1.2) \quad H(M(g)z) = H(z), \quad \forall g \in G, \quad \forall z \in \mathbb{R}^{2d}.$$

The Hamiltonian system associated to  $H$  is:

$$(1.3) \quad \dot{z}_t = J\nabla H(z_t), \quad \text{where } J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}.$$

It has the property that its flow  $\Phi_t$  commutes with the symmetry  $M(g)$  for all  $g$  in  $G$ .

From the quantum point of view, under suitable assumptions (see (3.2)), the Weyl quantization of  $H$  is given by (if  $u \in \mathcal{S}(\mathbb{R}^d)$ ):

$$(1.4) \quad Op_h^w(H)u(x) = (2\pi h)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{\frac{i}{h}(x-y)\xi} H\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi, \quad \forall x \in \mathbb{R}^d.$$

In particular,  $Op_h^w(H)$  is essentially selfadjoint on  $\mathcal{S}(\mathbb{R}^d)$  and we denote by  $\hat{H}$  its selfadjoint extension with domain  $D(\hat{H})$ .  $G$  acts on the quantum space  $L^2(\mathbb{R}^d)$  through  $\tilde{M}$  defined for  $g \in G$  by :

$$(1.5) \quad \tilde{M}(g)(f)(x) := f(g^{-1}x), \quad \forall f \in L^2(\mathbb{R}^d), \quad \forall x \in \mathbb{R}^d.$$

If  $\chi$  is an irreducible character of  $G$ , we set  $d_\chi := \chi(Id)$ . Then, the symmetry subspace  $L_\chi^2(\mathbb{R}^d)$  associated to  $\chi$  is defined as the image of  $L^2(\mathbb{R}^d)$  by the projector:

$$(1.6) \quad P_\chi := d_\chi \int_G \overline{\chi(g)} \tilde{M}(g) dg,$$

where  $dg$  is the normalized Haar measure on  $G$ . Then  $L_\chi^2(\mathbb{R}^d)$  splits into a Hilbertian sum of  $L_\chi^2(\mathbb{R}^d)$ 's (Peter-Weyl decomposition), and the property (1.2) implies that each  $L_\chi^2(\mathbb{R}^d)$  is invariant by  $\hat{H}$ . The restriction  $\hat{H}_\chi$  of  $\hat{H}$  to  $L_\chi^2(\mathbb{R}^d)$ , will be called here the *reduced quantum Hamiltonian*.

Our goal is to investigate the spectrum of  $\hat{H}_\chi$  in a localized interval of energy as  $h$  goes to zero. Without symmetry, the celebrated 'Correspondence Principle' roughly says that semi-classical asymptotics link quantum objects defined through  $\hat{H}$  (as trace or eigenvalues counting function) with quantities of the classical Hamiltonian system (1.3) of  $H$ . In the framework of symmetries, since specialists of classical dynamics are used to investigate (1.3) in the quotient  $\mathbb{R}^{2d}/G$ , we expect that semi-classical asymptotics would link the spectrum of  $\hat{H}_\chi$  with quantities of the Hamiltonian system in  $\mathbb{R}^{2d}/G$ , or more precisely in  $\Omega_0/G$ , where  $\Omega_0 \subset \mathbb{R}^{2d}$  is the zero level of the momentum map of  $G$  (see section 2 for details).

It is easy to check that  $\Omega_0$  is invariant by the action of  $G$ . We denote the *reduced space* by:

$$(1.7) \quad \Omega_{red} := \Omega_0/G.$$

Let  $\pi : \Omega_0 \rightarrow \Omega_{red}$  be the canonical projection on the quotient. Thanks to (1.2), we can clearly define the reduced classical Hamiltonian  $\tilde{H} : \Omega_{red} \rightarrow \mathbb{R}$  by:

$$(1.8) \quad \tilde{H}(\pi(z)) := H(z), \quad \forall z \in \Omega_0.$$

The topological flow  $\tilde{\Phi}_t : \Omega_{red} \rightarrow \Omega_{red}$  is defined for all  $t$  such that the flow  $\Phi_t$  of  $H$  exists, by:

$$(1.9) \quad \tilde{\Phi}_t(\pi(z)) := \pi(\Phi_t(z)), \quad \forall z \in \Omega_0.$$

Let  $U$  be an open set of  $\mathbb{R}^{2d}$  invariant by the action of  $G$ . Under suitable assumptions (mainly the fact that stabilizers are conjugate on  $\Omega_0 \cap U$  –see Definition 2.1–), then  $\Omega_0 \cap U$  is a smooth submanifold of  $\mathbb{R}^{2d}$ ,  $(\Omega_0 \cap U)/G \subset \Omega_{red}$  inherits a structure of smooth symplectic manifold from  $\mathbb{R}^{2d}$ , such that the restriction of  $\pi$  and  $\tilde{H}$  would be smooth, and such that  $\tilde{\Phi}_t$  would be the flow of the Hamiltonian  $\tilde{H}$ . The Riemannian structure of  $\Omega_0$  also descends to the quotient, and we get a notion of volume on  $\Omega_{red}$ . Note that, since all stabilizers are conjugate on  $\Omega_0 \cap U$ , all  $G$ -orbits of points of  $\Omega_0 \cap U$  have the same dimension as submanifolds of  $\mathbb{R}^{2d}$ . A first result is the following:

**Theorem 1.1.** *Let  $\varepsilon > 0$ ,  $E_1 < E_2$  in  $\mathbb{R}$  and set  $U := H^{-1}(]E_1 - \varepsilon, E_2 + \varepsilon[)$ . Assume that conditions of symplectic reduction (definition 2.1) are fulfilled on  $\Omega_0 \cap U$ . Suppose that  $H^{-1}(]E_1 - \varepsilon, E_2 + \varepsilon[)$  is compact and that  $E_1$  and  $E_2$  are non critical values of  $\tilde{H}$ . Then, for small  $h$ 's, the spectrum of  $\hat{H}_\chi$  is discrete in  $I := [E_1, E_2]$ , and, if  $N_{I,\chi}(h)$  denotes the number of eigenvalues of  $\hat{H}_\chi$  in  $I$  (with multiplicity), then we have:*

$$(1.10) \quad N_{I,\chi}(h) = (2\pi h)^{k_0-d} d_\chi \text{Vol}_{red}(\tilde{H}^{-1}(I)) \left[ \rho_{\chi|_{H_0}} : \mathbb{1} \right] + O(h^{k_0-d+1}).$$

where  $k_0$  is the common dimension of  $G$ -orbits on  $\Omega_0 \cap U$ , and  $\text{Vol}_{red}$  is the Riemannian volume on  $\Omega_{red}$ . The algebraic quantity  $\left[ \rho_{\chi|_{H_0}} : \mathbb{1} \right]$  is an integer described in Theorem 3.1.

This result generalizes the ones of Donnelly and Brüning & Heintze to the case of  $\mathbb{R}^d$  in a quantum semi-classical context. It was already conjectured for high energies in [21], where authors gave theoretical asymptotics using BKW methods, as in [15] for a semi-classical version.

Our second main result is devoted to the oscillations of the spectral density of states of  $\hat{H}_\chi$  in a neighbourhood of an energy  $E \in \mathbb{R}$ . We get a trace formula similar to the one of Guillemin and Uribe ([18]), involving a sum over periodic orbits with energy  $E$  of the Hamiltonian system of  $\tilde{H}$  in  $\Omega_{red}$ . We set:

$$(1.11) \quad \widetilde{\Sigma}_E := \{\tilde{H} = E\} \subset \Omega_{red} \text{ and } \mathcal{L}_{red}(E) := \{t \in \mathbb{R} : \exists x \in \widetilde{\Sigma}_E : \tilde{\Phi}_t(x) = x.\}$$

We suppose that:

- There exists  $\delta E > 0$  such that  $H^{-1}(]E - \delta E, E + \delta E[)$  is compact.
- $f : \mathbb{R} \rightarrow \mathbb{R}$  is such that its Fourier transform  $\hat{f}$  is smooth and compactly supported.
- $\psi : \mathbb{R} \rightarrow \mathbb{R}$  is smooth and compactly supported in  $]E - \delta E, E + \delta E[$ .

Then, for small  $h$ 's,  $\psi(\hat{H}_\chi)$  is trace class and we denote by  $\mathfrak{G}_\chi(h)$  the well defined following trace:

$$(1.12) \quad \mathfrak{G}_\chi(h) := \text{Tr} \left( \psi(\hat{H}_\chi) f \left( \frac{E - \hat{H}_\chi}{h} \right) \right).$$

If  $t_0 \neq 0$ , let  $\mathcal{P}_{red}(E, t_0)$  be the set of periodic orbits in  $\widetilde{\Sigma}_E$  admitting  $t_0$  as a period.

**Theorem 1.2.** *Set  $U := H^{-1}(]E - \delta E, E + \delta E[)$ . Suppose that hypotheses of symplectic reduction are fulfilled on  $\Omega_0 \cap U$ . Moreover, suppose that periodic orbits of  $\widetilde{\Sigma}_E \subset \Omega_{red}$  having a period in  $\text{Supp} \hat{f}$  are non degenerate (in the sense of [10]) for these periods. Suppose also that  $0 \notin \text{Supp}(\hat{f})$ . Then  $\mathfrak{G}_\chi(h)$  has a complete asymptotic expansion in powers of  $h$  as  $h \rightarrow 0^+$  (modulo oscillating terms of type  $e^{\frac{i\pi}{h}}$ ), whose coefficients are distributions in  $\hat{f}$  with support in  $\mathcal{L}_{red}(E) \cap \text{Supp} \hat{f}$ . Moreover, the first term is given by:  $\mathfrak{G}_\chi(h) =$*

$$(1.13) \quad \psi(E) d_\chi \sum_{t_0 \in \mathcal{L}_{red}(E) \cap \text{Supp} \hat{f}} \hat{f}(t_0) \sum_{\bar{\gamma} \in \mathcal{P}_{red}(E, t_0)} e^{\frac{i}{h} S_{\bar{\gamma}}(t_0)} \frac{1}{2\pi} \int_{\Lambda_{\bar{\gamma}, t_0}} \overline{\chi(g)} d(t_0, z, g) d\sigma_{\Lambda_{\bar{\gamma}}}(z, g) + O(h).$$

where  $S_{\bar{\gamma}}(t_0) := \int_0^{t_0} p_t(z)\dot{q}_t(z)dt$  (with  $\Phi_t(z) = (q_t(z), p_t(z)) \in \mathbb{R}^d \times \mathbb{R}^d$ ) doesn't depend on  $z$  such that  $\pi(z) \in \bar{\gamma}$ , and:

$$(1.14) \quad \Lambda_{\bar{\gamma}, t_0} := \{(z, g) \in (\Omega_0 \cap \Sigma_E) \times G : M(g)\Phi_{t_0}(z) = z \text{ and } \pi(z) \in \bar{\gamma}\} \subset \mathbb{R}^{2d} \times G,$$

The density  $d(t_0, z, g)$  doesn't depend on  $h$  and  $\chi$  and is detailed in Theorem 4.6 using quantities describing the classical dynamical system (1.3).

Note that, even if periodic orbits are non degenerate in the reduced space  $\Omega_{red}$ , the ones of  $\Omega_0$  are generally degenerate in  $\mathbb{R}^{2d}$ . Indeed, elements of  $G$  map a periodic orbit of  $\Omega_0$  into another one of same period, which creates tubes of periodic orbits with same period and doesn't match with a non degenerate situation.

The author expects that one can calculate the density  $d(t_0, z, g)$  in terms of primitive period, Maslov index, and of the energy restricted Poincaré map of the periodic orbit  $\bar{\gamma}$ . This seems to be a non-trivial calculus we have been able to complete only for finite groups for the moment (see [7]). See also the work of the physicist S.C. Creagh in [12]. If one omits the assumption of non-degeneracy, under hypothesis of 'G-clean flow' (see Definition 4.4), we still get an asymptotic expansion, which depends on the connected components of the set:

$$(1.15) \quad \mathcal{C}_E = \{(t, z, g) \in \text{Supp}(\hat{f}) \times \mathbb{R}^{2d} \times G : z \in (\Omega_0 \cap \Sigma_E), M(g)\Phi_t(z) = z\}.$$

As in the article on finite groups [7], we will use the work of Combescure and Robert on coherent states instead of a traditional WKB method. The structure of this paper is the following: In section 2, we precise our setting to get a nice smooth structure on the reduced space  $\Omega_{red}$  and give examples of classical and quantum reduced Hamiltonians. Section 3 is dedicated to the so called 'weak asymptotics', i.e. the asymptotic expansion of  $\text{Tr}(f(\hat{H}_\chi))$  when  $f(\hat{H}_\chi)$  is trace class. This will help us to compute geometrically the leading term of (1.10) for Weyl asymptotics. Then we adapt the method of [10] using coherent states, which leads us to an application of the generalised stationary phase theorem in section 4. We find optimal conditions (called G-clean flow conditions) to apply this theorem to our case and give theoretical asymptotics. In section 5, we describe the particular case where  $\text{Supp}(\hat{f})$  is located near zero, and get Theorem 1.1. Section 6 is dedicated to the case where we suppose that periodic orbits are non degenerate in the reduced space, and leads to Theorem 1.2.

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## 2. SYMPLECTIC AND QUANTUM REDUCTION

**2.1. Symplectic reduction.** Let  $H : \mathbb{R}^{2d} \rightarrow \mathbb{R}$  be a smooth Hamiltonian invariant by a compact Lie group  $G$  of dimension  $p$  as in section 1. The theorem of E.Noether says that, if  $G$  is not finite, then  $G$  provides the system (1.3) with integrals. In our case, these integrals are easy to compute. If  $A \in \mathfrak{G}$ , the Lie algebra of  $G$ , we define  $M(A)$  as:

$$(2.1) \quad M(A) := \begin{pmatrix} A & 0 \\ 0 & -{}^t A \end{pmatrix}.$$

Then, if  $F_A : \mathbb{R}^{2d} \rightarrow \mathbb{R}$  is defined by  $F_A(z) := \frac{1}{2} \langle JM(A)z, z \rangle$ , then  $F_A$  is a first integral of  $H$ . Indeed, one can differentiate at  $t = 0$  the identity:

$$H(e^{tM(A)}z) = H(z),$$

to get  $\{H, F_A\} = 0$ , where  $\{, \}$  denotes the Poisson bracket ( $JM(A)$  is symmetric since for  $g \in G$ ,  $M(g)$  is symplectic). Now, if  $A_1, \dots, A_p$  is a basis of  $\mathfrak{G}$ , then we define the *momentum*

map  $\mathbb{F} : \mathbb{R}^{2d} \rightarrow \mathbb{R}^p$  by  $\mathbb{F} := (F_{A_1}, \dots, F_{A_p})$ . The zero level of  $\mathbb{F}$  is:

$$(2.2) \quad \Omega_0 := \mathbb{F}^{-1}(\{0\}) = \bigcap_{A \in \mathcal{G}} F_A^{-1}(\{0\}).$$

One can remark that  $\Omega_0$  is homogeneous and that  $0 \in \Omega_0$ . Furthermore, since  $M$  is symplectic, it is easy to check that  $\Omega_0$  is invariant by the action of  $G$ . Hence we can define  $\Omega_{red}$  as in (1.7), and  $\pi$  as the projection on this quotient. We want to get, roughly speaking, a smooth structure on  $\Omega_{red}$ . A natural hypothesis would be to suppose that  $\Omega_0$  is a manifold and that all stabilizers are conjugate on  $\Omega_0$  (see for example [22], Theorem 4.18 p.196). Nevertheless, as we already said, zero is in  $\Omega_0$ , and its stabilizer is  $G$  itself since the action is linear. One can also make this hypothesis only on  $\Omega_0 \setminus \{0\}$ . However, this would exclude the case of a cylindrical symmetry as we will see below. A less restrictive hypothesis consists in demanding a smooth structure only on a part of  $\Omega_{red}$ , which will be enough for our quantum application.

**Definition 2.1.** *If  $U$  is an open set of  $\mathbb{R}^{2d}$  invariant by the action of  $G$ , we say that ‘hypotheses of reduction are satisfied on  $U \cap \Omega_0$ ’ if  $U \cap \Omega_0 \neq \emptyset$  and if following assumptions are fulfilled:*

- *Stabilizers of points of  $\Omega_0 \cap U$  are all conjugate subgroups of  $G$ .*
- *$\forall z \in \Omega_0 \cap U, \dim(\Omega_0 \cap U) = 2d - \dim(G(z))$ .*

We then have the following theorem generalizing the case of a free action originally given by J.Marsden and A.Weinstein (one can find a proof of this theorem in the book [23], Theorem 8.1.1 p.202):

**Theorem 2.2.** *(Symplectic reduction)*

*If  $U$  is an open set of  $\mathbb{R}^{2d}$  such that hypotheses of reduction are satisfied on  $U \cap \Omega_0$ , then  $U \cap \Omega_0$  is a smooth submanifold of  $\mathbb{R}^{2d}$ , and there exists a unique structure of smooth manifold on  $(\Omega_0 \cap U)/G \subset \Omega_{red}$  such that the restriction of  $\pi$  to  $\Omega_0 \cap U$  is a smooth submersion. Moreover, there exists a unique symplectic form  $w_{red}$  on  $(\Omega_0 \cap U)/G$  such that  $\pi^*w_{red}$  is the restriction of  $\langle J, \cdot \rangle_{\mathbb{R}^{2d}}$  to  $\Omega_0 \cap U$ . Finally, the restriction  $\tilde{H} : (\Omega_0 \cap U)/G \rightarrow \mathbb{R}$  is smooth,  $\pi$  maps the integral curves of (1.3) lying in  $\Omega_0 \cap U$  on those of the Hamiltonian system induced by  $\tilde{H}$  on  $(\Omega_{red}, w_{red})$ , and  $\tilde{\Phi}_t$  is the flow of  $\tilde{H}$ .*

Under hypotheses of definition 2.1, we will denote by:

$$(2.3) \quad H_0 \subset G \text{ the stabilizer of one point } z_0 \text{ in } \Omega_0 \cap U, \text{ and } k_0 := \dim(G(z_0)),$$

$k_0$  being then the common dimension of  $G$ -orbits in  $\Omega_0 \cap U$ . Thus we have  $\dim((\Omega_0 \cap U)/G) = 2(d - k_0)$ . Note that  $(\Omega_0 \cap U)/G$  inherits of a Riemannian structure from the submersion  $\pi : \Omega_0 \cap U \rightarrow (\Omega_0 \cap U)/G$ , which in particular gives a measure on this set. Moreover, by a dimensional argument, we have  $T_z \Omega_0 = (J\mathcal{G}z)^\perp$ .

*Remark:* most of time we will take  $U := H^{-1}(I)$ , where  $H$  is  $G$ -invariant and  $I$  is an open interval of  $\mathbb{R}$ . Thus, hypotheses of reduction won’t be fulfilled if  $0 \in H^{-1}(I)$ . This would require an additional treatment as it was done in [15].

Now, we give examples of classical reduction:

–Spherical symmetry:  $G = SO(d)$ ,  $p = \frac{d(d-1)}{2}$ . It is easily seen that we have the following results:

$$\Omega_0 = \{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d : \text{vectors } x \text{ and } \xi \text{ are linearly dependant}\}.$$

Stabilizers are conjugate in  $\Omega_0 \setminus \{0\}$ , which is a submanifold of dimension  $d + 1$  of  $\mathbb{R}^{2d}$ , and thus we have  $\dim(\Omega_0 \setminus \{0\}) = 2d - \dim(G(z))$  for all  $z$  in  $\Omega_0 \setminus \{0\}$ . Hence, for an invariant open set  $U$  in  $\mathbb{R}^d \times \mathbb{R}^d$  such that  $\Omega_0 \cap U \neq \emptyset$ , hypotheses of reduction are fulfilled on  $\Omega_0 \cap U$  if and only if  $0 \notin U$ .

–Cylindrical symmetry:  $d = 3$  and  $G$  is the group of rotations of  $\mathbb{R}^3$  around the vertical axis. It is also easy to see that:

$$\Omega_0 = \{(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3 : (x_1, x_2) \text{ and } (\xi_1, \xi_2) \text{ are colinear}\}.$$

$\Omega_0$  is the disjoint union of two sets  $\Omega_1$  and  $\Omega_2$  with:

$$\begin{cases} \Omega_1 := \{(0, 0, x_3; 0, 0, \xi_3) : x_3 \in \mathbb{R}, \xi_3 \in \mathbb{R}\}. \\ \Omega_2 := \{(x', x_3; \xi', \xi_3) : x' \text{ and } \xi' \text{ are colinear in } \mathbb{R}^2, \text{ and } (x', \xi') \neq 0\}. \end{cases}$$

On  $\Omega_1$ , stabilizers are conjugate to  $G$  itself, whereas on  $\Omega_2$  they are equal to  $\{Id_{\mathbb{R}^3}\}$ . Moreover,  $\Omega_1$  is a plane and  $\dim(\Omega_2) = 5 = 2d - G(z)$  if  $z \in \Omega_2$ . Thus for an invariant open set  $U$  in  $\mathbb{R}^3 \times \mathbb{R}^3$  such that  $\Omega_0 \cap U \neq \emptyset$ , hypotheses of reduction are fulfilled on  $\Omega_0 \cap U$  if and only if  $U \cap \Omega_1 = \emptyset$ .

**2.2. Quantum reduction.** For general background on the decomposition of Peter-Weyl, reduced Hamiltonians and interpretation of symmetry, we refer the reader to [7], [8], [21], and [28], [24]). We just recall that if  $\widehat{G}$  denotes the set of irreducible characters of  $G$ , then  $G$  is countable (since  $G$  is compact), and with the definition of  $L_\chi^2(\mathbb{R}^d)$  given in section 1, we have the Hilbertian decomposition:

$$(2.4) \quad L^2(\mathbb{R}^d) = \bigoplus_{\chi \in \widehat{G}}^\perp L_\chi^2(\mathbb{R}^d),$$

which comes from the identity:

$$(2.5) \quad \tilde{M}(g)^{-1} Op_h^w(H) \tilde{M}(g) = Op_h^w(H \circ M(g)), \quad \forall g \in G.$$

Basic properties of the  $L_\chi^2(\mathbb{R}^d)$ 's are the same as in the case of finite groups, excepted for the spectrum inclusion  $\bigcup_{\chi \in \widehat{G}} \sigma(\widehat{H}_\chi) \subset \sigma(\widehat{H})$ , without equality in general (see [8] p.11-12).

Then we give examples of quantum reduction: for a character  $\chi$  of degree 1, we have a simple description of  $L_\chi^2(\mathbb{R}^d)$  (see [21] or [7]):

$$L_\chi^2(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \tilde{M}(g)f = \chi(g)f\}.$$

This is in particular the case with  $\chi = \chi_0$ , the character of the trivial representation of  $G$ , i.e. the representation of degree 1, constant equal to identity. If we endow  $\mathbb{R}^d/G$  with the image measure of the Lebesgue measure on  $\mathbb{R}^d$  by the canonical projection  $\pi$  on the quotient, we note that the map  $u \mapsto u \circ \pi$  identify  $L_{\chi_0}^2(\mathbb{R}^d)$  with  $L^2(\mathbb{R}^d/G)$ . When the group is abelian, characters are of degree 1 (see [27]). This is the case for  $G = SO(2)$  with  $d = 2$ . One can show that  $\widehat{G}$  is indexed by  $\mathbb{Z}$ , and if  $R_\theta$  is the rotation of angle  $\theta$  in the anticlockwise sense, then  $\widehat{G} = (\chi_n)_{n \in \mathbb{Z}}$  with  $\chi_n(R_\theta) := e^{in\theta}$  and:

$$L_{\chi_n}^2(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) : \tilde{f}(r, \theta) = e^{-in\theta} g(r), g \in L^2(\mathbb{R}_+, r dr)\},$$

where  $\tilde{f}$  is the expression of  $f$  in polar coordinates. Using the expression of the Laplacian in spherical coordinates, if  $H(x, \xi) := |\xi|^2 + V(x)$  where  $V(x) = V_0(|x|)$  is radial in  $\mathbb{R}^2$ , then (with  $h = 1$ ) we get that  $\widehat{H}_{\chi_n}$  is unitary equivalent to the following operator on  $L^2(\mathbb{R}_+, r dr)$ :

$$-\partial_r^2 - \frac{1}{r} \partial_r + \frac{n^2}{r^2} + V_0(r).$$

An example with characters of arbitrary high degrees is given by  $G = SO(3)$ . Conjugation classes are given by rotations with same angle (non oriented). Let  $\tilde{R}_\theta$  denotes the rotation with angle  $\theta$  around the vertical axis, then  $\widehat{G} = (\chi_n)_{n \in \mathbb{N}}$  with:

$$\chi_n(\tilde{R}_\theta) := \sum_{k=-n}^n e^{ik\theta} = \sin((2n+1)\theta/2) / \sin(\theta/2), \quad \theta \in [0, 2\pi].$$

Hence,  $d_{\chi_n} = 2n + 1$ . Moreover, we can describe the symmetry subspaces using spherical harmonics  $(Y_{n,k})$ , with  $n \in \mathbb{N}$  and  $k \in \{-n, \dots, n\}$  ( $n$  is the quantum azimuthal number), which

are the eigenfunctions of the Laplacian in spherical coordinates ( $-\Delta_{\text{sph}} Y_{n,k} = n(n+1)Y_{n,k}$ ):

$$L^2_{\chi_n}(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3) : \tilde{f}(r, \theta, \varphi) = g(r) \sum_{k=-n}^n \lambda_k Y_{n,k}(\theta, \varphi), \quad \lambda_k \in \mathbb{C}, g \in L^2(\mathbb{R}_+, r^2 dr)\}.$$

where  $\tilde{f}$  is the expression of  $f$  in spherical coordinates (see [8] p.7-8 and [24] p.115). If  $H(x, \xi) := |\xi|^2 + V(x)$  where  $V(x) = V_0(|x|)$  is radial in  $\mathbb{R}^3$ , then (with  $h = 1$ ) we get that  $\widehat{H}_{\chi_n}$  is unitary equivalent to the following operator on  $L^2(\mathbb{R}_+, r^2 dr)$ :

$$-\partial_r^2 - \frac{2}{r} \partial_r + \frac{n(n+1)}{r^2} + V_0(r).$$

### 3. WEAK ASYMPTOTICS

The following result is interesting in itself and is also a usual way to compute the first term of the asymptotic expansion of the eigenvalues counting function of  $\widehat{H}_\chi$  (Theorem 1.1).

**Theorem 3.1.** *Let  $G$  be a compact Lie group of  $Gl(d, \mathbb{R})$  and  $H : \mathbb{R}^{2d} \rightarrow \mathbb{R}$  be a smooth Hamiltonian  $G$ -invariant satisfying (3.2). Let  $E_1 < E_2$  be such that  $H^{-1}(]E_1 - \varepsilon, E_2 + \varepsilon])$  is compact (where  $\varepsilon > 0$ ). If  $U := H^{-1}(]E_1 - \varepsilon, E_2 + \varepsilon])$  is such that hypotheses of reduction are satisfied on  $\Omega_0 \cap U$ , if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is smooth, compactly supported in  $]E_1, E_2]$ , if  $\chi \in \widehat{G}$ , then, for small  $h$ 's,  $f(\widehat{H}_\chi)$  is trace class and  $\text{Tr}(f(\widehat{H}_\chi))$  has a complete asymptotic expansion in powers of  $h$  as  $h \rightarrow 0^+$ , with first term:*

$$(3.1) \quad \text{Tr}(f(\widehat{H}_\chi)) = (2\pi h)^{k_0-d} d_\chi \int_{\Omega_{red}} f(\tilde{H}(x)) d\sigma_{red}(x) \cdot [\rho_{\chi|_{H_0}} : \mathbb{1}] + O(h^{k_0-d+1}).$$

Here  $d\sigma_{red}$  is the measure corresponding to the Riemannian structure on  $(\Omega_0 \cap U)/G$ ,  $\tilde{H}$  is given by (1.8),  $k_0$  is the common dimension of  $G$ -orbits on  $\Omega_0 \cap U$ , and  $[\rho_{\chi|_{H_0}} : \mathbb{1}]$  is an integer. Namely, if  $H_0$  is any stabilizer of  $\Omega_0 \cap U$  and  $\rho_\chi$  is any representation with character  $\chi$ , then it is the number of times that the trivial representation  $\mathbb{1}$  is contained in the decomposition into irreducible representations of  $\rho_\chi$  restricted to  $H_0$ .

Next sections are dedicated to the proof of this theorem.

**3.1. First reductions.** Suppose that  $H$  satisfies the following assumptions (where  $C, C_\alpha > 0$ ),

$$(3.2) \quad \begin{cases} \langle H(z) \rangle \leq C \langle H(z') \rangle + \langle z - z' \rangle^m, & \forall z, z' \in \mathbb{R}^{2d}. \\ |\partial_z^\alpha H(z)| \leq C_\alpha \langle H(z) \rangle, & \forall z \in \mathbb{R}^{2d}, \forall \alpha \in \mathbb{N}^{2d}. \\ H \text{ has a lower bound on } \mathbb{R}^{2d}. \end{cases}$$

where  $\langle z \rangle := (1 + |z|^2)^{\frac{1}{2}}$ . Then, since  $\text{Supp}(f) \subset ]E_1, E_2]$ , we can write for  $N_0 \in \mathbb{N}$  (see [19]):

$$(3.3) \quad f(\widehat{H}) = \sum_{j=0}^{N_0} h^j O p_h^w(a_j) + h^{N_0+1} R_{N_0+1}(h).$$

where  $\text{Supp}(a_j) \subset H^{-1}(]E_1 - \varepsilon, E_2 + \varepsilon])$ ,  $a_0(z) = f(H(z))$ , with  $\sup_{0 < h \leq 1} \|R_{N_0+1}(h)\|_{\text{Tr}} \leq C h^{-d}$ .

We easily get that all  $a_j$  are  $G$ -invariant. Thus, we are led to show that, if  $a$  is a smooth  $G$ -invariant function compactly supported in  $U \subset \mathbb{R}^{2d}$ , and  $\widehat{A} := O p_h^w(a)$ , then  $\text{Tr}(\widehat{A}_\chi)$  has a complete asymptotic expansion in powers of  $h$  when  $h \rightarrow 0^+$ , and we have:

$$(3.4) \quad \text{Tr}(\widehat{A}_\chi) = (2\pi h)^{k_0-d} d_\chi \int_{\Omega_{red}} \tilde{a}(x) d\sigma_{red}(x) \cdot [\rho_{\chi|_{H_0}} : \mathbb{1}] + O(h^{k_0-d+1}).$$

where  $\tilde{a}(\pi(z)) := a(z)$ . By the trace formula using coherent states (see [7], [10]), we have:

$$\text{Tr}(\widehat{A}_\chi) = \text{Tr}(\widehat{A} P_\chi) = d_\chi (2\pi h)^{-d} \int_G \int_{\mathbb{R}^{2d}} \overline{\chi(g)} \langle \widehat{A} \varphi_\alpha, \tilde{M}(g)^{-1} \varphi_\alpha \rangle d\alpha dg.$$

As in [10], we can show that there exists a compact set  $K$  in  $\mathbb{R}^{2d}$  such that:

$$\int_{\mathbb{R}^{2d} \setminus K} |\langle \widehat{A} \varphi_\alpha; \tilde{M}(g)^{-1} \varphi_\alpha \rangle_{L^2(\mathbb{R}^d)}| d\alpha = O(h^\infty),$$

uniformly in  $g \in G$ . In view of Lemma 3.1. of [10], if  $N \in \mathbb{N}^*$ , then there exists  $C_{d,N} > 0$  with:

$$\left\| Op_h^w(a) \varphi_\alpha - \sum_{k=0}^N h^{\frac{k}{2}} \sum_{\gamma \in \mathbb{N}^{2d}, |\gamma|=k} \frac{\partial^\gamma a(\alpha)}{\gamma!} \Psi_{\gamma,\alpha} \right\|_{L^2(\mathbb{R}^d)} \leq C_{d,N} h^{\frac{N+1}{2}}.$$

where  $\Psi_{\gamma,\alpha} := \mathcal{T}_h(\alpha) \Lambda_h Op_1^w(z^\gamma) \tilde{\psi}_0$ . For notations on coherent states, we refer to [10] and [7]. We can suppose that  $\text{Supp}(a) \subset K$  and if  $\chi_1$  is a smooth function compactly supported in  $\mathbb{R}^{2d}$  with  $\chi_1 = 1$  on  $K$ , writing  $1 = \chi_1 + (1 - \chi_1)$ , we get:

$$\text{Tr}(\widehat{A}_\chi) = d_\chi (2\pi h)^{-d} \sum_{k=0}^N h^{\frac{k}{2}} \sum_{\gamma \in \mathbb{N}^{2d}, |\gamma|=k} \int_G \int_{\mathbb{R}^{2d}} \overline{\chi(g)} \frac{\partial^\gamma a(\alpha)}{\gamma!} m_\gamma(\alpha, g) d\alpha dg + O(h^{\frac{N+1}{2}-d}),$$

where

$$m_\gamma(\alpha, g) := \langle \mathcal{T}_h(\alpha) \Lambda_h Op_1^w(z^\gamma) \tilde{\psi}_0; \tilde{M}(g)^{-1} \mathcal{T}_h(\alpha) \Lambda_h \tilde{\psi}_0 \rangle.$$

Thanks to the metaplectic property of  $\tilde{M}$  – see (2.5) –, since  $\mathcal{T}_h(\alpha) = Op_h^w(\exp(\frac{i}{h}(px - q\xi)))$  (if  $\alpha =: (q, p)$ ), we can write:

$$\tilde{M}(g)^{-1} \mathcal{T}_h(\alpha) = \mathcal{T}_h(M(g^{-1})\alpha) \tilde{M}(g)^{-1}.$$

Let  $Q_\gamma$  be the polynomial such that  $Op_1^w(z^\gamma) \tilde{\psi}_0 = Q_\gamma \tilde{\psi}_0$ . By using formulas on coherent states of [7] or [8], we get:

$$(3.5) \quad m_\gamma(\alpha, g) = e^{-\frac{i}{2h} \langle JM(g^{-1})\alpha, \alpha \rangle} \langle \mathcal{T}_1 \left( \frac{\alpha - M(g^{-1})\alpha}{\sqrt{h}} \right) Q_\gamma \tilde{\psi}_0; \tilde{M}(g)^{-1} \tilde{\psi}_0 \rangle.$$

Here, we can use the same trick as in [7] which allows us to suppose that  $G$  is composed of *isometries* (we recall that, since  $G$  is compact, by an averaging argument, it is conjugate to a subgroup of the orthogonal group, and one is led to consider isometries by making a change of Hamiltonian). By a clear change of variables in the integral given by the scalar product in (3.5), we get:

$$m_\gamma(\alpha, g) = (\pi h)^{-\frac{d}{2}} e^{-\frac{i}{2h} \langle JM(g^{-1})\alpha, \alpha \rangle} e^{\frac{i}{2h} \langle (I-t)g)p, (I-g^{-1})q \rangle} e^{-\frac{1}{2h} |(I-g^{-1})q|^2} \\ \times \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{h}} e^{\frac{i}{h} \langle (I-g^{-1})q + (I-t)g)p; y \rangle} Q_\gamma \left( \frac{y}{\sqrt{h}} \right) dy.$$

Note that, since  $Q_\gamma$  has the same parity as  $|\gamma| = k$ , only *entire* powers of  $h$  will arise in the asymptotics. Then, after making the change of variable  $y' = \frac{y}{\sqrt{h}}$ , we write:

$$Q_\gamma(x) = \sum_{|\mu| \leq |\gamma|} \kappa_{\mu,\gamma} x^\mu.$$

We set  $\beta_0 := i[i(I-g^{-1})q + (I-t)g)p]$  and use the calculus of the Gaussian given by Lemma 3.2 of [7] to get:

$$m_\gamma(\alpha, g) = e^{\frac{i}{h} \Phi(\alpha, g)} \sum_{|\mu| \leq |\gamma|} \kappa_{\mu,\gamma} \sum_{\eta \leq \mu} h^{-\frac{\eta}{2}} \left( \frac{1}{2} \beta_0 \right)^\eta P_\eta(2I_d),$$

where  $P_\eta$  is a polynomial independant of  $h$ ,  $\alpha$  and  $g$ , with  $P_0(2I_d) = 1$ , and:

$$(3.6) \quad \Phi(\alpha, g) := \langle B\alpha, \alpha \rangle \quad \text{with} \quad B = \frac{1}{4} J(M(g) - M(g^{-1})) + \frac{i}{4} (I - M(g))(I - M(g^{-1})).$$

Finally, we have:

$$(3.7) \quad \text{Tr}(\widehat{A}_\chi) = (2\pi h)^{-d} d_\chi \sum_{k=0}^N h^{\frac{k}{2}} \sum_{\gamma \in \mathbb{N}^{2d}, |\gamma|=k} \sum_{|\mu| \leq |\gamma|} \sum_{\eta \leq \mu} \frac{\kappa_{\mu,\gamma}}{\gamma!} P_\eta(2I_d) h^{-\frac{|\eta|}{2}} I_{\gamma,\eta}(h) + O(h^{\frac{N+1}{2}-d}).$$

with :

$$(3.8) \quad I_{\gamma, \eta}(h) := \int_G \int_{\mathbb{R}^{2d}} e^{\frac{i}{h}\Phi(\alpha, g)} \overline{\chi(g)} \partial^\gamma a(\alpha) F(\alpha, g)^\eta d\alpha dg$$

$$(3.9) \quad F(\alpha, g) := \frac{i}{2} [i(I - g^{-1})q + (I - g)p].$$

Hence, we are led to a stationary phase problem, to find an asymptotic expansion of  $I_{\gamma, \eta}(h)$ . Note that we will see that  $F(\alpha, g) = 0$  on the critical set of the phase. Thus, the asymptotics of  $I_{\gamma, \eta}(h)$  will start with a shift of  $h^{\frac{|\eta|}{2}}$ , which will compensate the term  $h^{-\frac{|\eta|}{2}}$  in (3.7).

**3.2. Phase analysis.** We want to apply the generalised stationary phase theorem in the form of [10]. First we note that, if  $(z, g) \in \mathbb{R}^{2d} \times G$ , then  $\Im\Phi(z, g) = \frac{1}{4}|(I - M(g^{-1}))z|^2 \geq 0$ . In the rest of the article, we often won't make the difference between  $A \in \mathfrak{G}$  and  $M(A)$  (or between  $g \in G$  and  $M(g)$ ) in order to lighten notations.

– *The critical set:* it is easily seen that

$$\Im\Phi(z, g) = 0 \iff \partial_z \Phi(z, g) = 0 \iff M(g)z = z.$$

Moreover, if  $A \in \mathfrak{G}$ , and  $g \in G$ , then:

$$4\partial_g \Phi(z, g)(Ag) = \langle [J(Ag + g^{-1}A) - iAg(I - g^{-1}) + i(I - g)g^{-1}A]z, z \rangle.$$

If, besides,  $\Im\Phi(z, g) = 0$ , then:  $\partial_g \Phi(z, g)(Ag) = 0 \iff \langle JM(A)z; z \rangle = 0$ . Thus we have proved that the critical set  $\Gamma_0 := \{\Im\Phi = 0\} \cap \{\nabla\Phi = 0\} \cap \{U \times G\}^1$  satisfies:

$$(3.10) \quad \Gamma_0 = \{(z, g) \in \Omega_0 \times G : M(g)z = z \text{ and } z \in U\}.$$

**Lemma 3.2.**  $\Gamma_0$  is a smooth submanifold of  $\mathbb{R}^{2d} \times G$  of dimension  $\dim \Gamma_0 = 2d - 2k_0 + p$ , where  $p$  is the dimension of  $G$ . Moreover, if  $(z, g) \in \Gamma_0$ , then:

$$(3.11) \quad T_{(z, g)}\Gamma_0 = \{(\alpha, Ag) : \alpha \in T_z\Omega_0, A \in \mathfrak{G} \text{ and } (M(g) - I)\alpha + Az = 0\}.$$

Proof: we will make deep use of the reduction theorem 2.2. Set:

$$R_0 := \{(z, M(g)z) : z \in \Omega_0 \cap U, g \in G\}.$$

We note that  $R_0 = (\pi \times \pi)^{-1}(\text{diag}[(\Omega_0 \cap U)/G])$ . By Theorem 2.2,  $\pi \times \pi$  is a submersion on  $(\Omega_0 \cap U)^2$ , thus  $R_0$  is a submanifold of dimension  $\dim R_0 = 2d$ , and if  $M(g)z = z \in \Omega_0 \cap U$ , then we have:

$$(3.12) \quad T_{(z, z)}R_0 = \{(\alpha, \beta) \in T_z\Omega_0 \times T_z\Omega_0 : d_z\pi(\alpha) = d_z\pi(\beta)\}.$$

From the fact that  $\pi$  is a submersion, we also deduce that  $\forall z \in (\Omega_0 \cap U)$ ,  $\ker d_z\pi = \mathfrak{G}z$ . Moreover, one can differentiate the identity  $\pi(M(g)x) = \pi(x)$  with respect to  $x \in \Omega_0$  to get that, if  $M(g)z = z$ , then  $\forall \alpha \in T_z\Omega_0$ ,  $(M(g) - I)\alpha \in \mathfrak{G}z$ . Thus, by (3.12) we get:

$$(3.13) \quad T_{(z, z)}R_0 = \{(\alpha, M(g)\alpha + Az) : \alpha \in T_z\Omega_0, A \in \mathfrak{G}\}.$$

Let  $\varphi_0 : \begin{cases} \Omega_0 \times G \rightarrow R_0 \\ (z, g) \mapsto (z, M(g)z) \end{cases}$ . We have:  $\text{diag}(\Omega_0 \cap U) \subset R_0$ , and  $\Gamma_0 = \varphi_0^{-1}(\text{diag}(\Omega_0 \cap U))$ .

Let  $(z, g) \in \Gamma_0$ . If  $\alpha \in T_z\Omega_0$  and  $A \in \mathfrak{G}$ , then we have:

$$d_{(z, g)}\varphi_0(\alpha, Ag) = (\alpha, M(g)\alpha + Az).$$

From (3.13), we deduce that  $\text{Im}(d_{(z, g)}\varphi_0) = T_{(z, g)z}R_0$ . Thus  $\varphi_0$  is a submersion, which ends the proof of the lemma.  $\square$

<sup>1</sup>By the non stationary phase theorem we can restrict ourselves to  $(z, g) \in \mathbb{R}^{2d} \times G$  such that  $z \in U$ , since we have from (3.3):  $\text{supp}(a) \subset U = H^{-1}(|E_1 - \varepsilon, E_2 + \varepsilon|)$ .

– *Calculus of the kernel of the Hessian of  $\Phi$ :*

Let  $(z_0, g_0) \in \Gamma_0$ . We define the following local chart of  $\mathbb{R}^{2d} \times G$  at  $(z_0, g_0)$ :

$$(3.14) \quad \varphi(z, s) := (z, \exp(\sum_{i=1}^p s_i A_i) g_0).$$

where  $(A_1, \dots, A_p)$  is a basis of  $\mathfrak{G}$  that we should choose later. We denote by:

$$(3.15) \quad \text{Hess } \Phi(z_0, g_0) := \left( \frac{\partial^2 (\Phi \circ \varphi)}{\partial x_i \partial x_j} (z_0, 0) \right)_{1 \leq i, j \leq 2d+p}$$

the Hessian matrix in the canonical basis of  $\mathbb{R}^{2d} \times \mathbb{R}^p$  of  $\Phi \circ \varphi$ . We clearly have:

$$\Phi''(z_0, g_0)|_{\mathcal{N}_{(z_0, g_0)} \Gamma_0} \text{ is non degenerate} \iff d_{(z_0, 0)} \varphi(\ker_{\mathbb{R}} \text{Hess } \Phi(z_0, g_0)) \subset T_{(z_0, g_0)} \Gamma_0,$$

where  $\ker_{\mathbb{R}} \text{Hess } \Phi(z_0, g_0) := \ker(\Re(\text{Hess } \Phi(z_0, g_0))) \cap \ker(\Im(\text{Hess } \Phi(z_0, g_0)))$ . To compute the matrix  $\text{Hess } \Phi(z_0, g_0)$ , we recall that:

$$(3.16) \quad \frac{\partial^2}{\partial s_i \partial s_j} (e^{\sum_{r=1}^p s_r A_r})|_{s=0} = \frac{1}{2} (A_i A_j + A_j A_i).$$

After computation, if we denote  $M(g)$  by  $g$ , if ‘ $i$ ’ is associated to a line and ‘ $j$ ’ to a column,  $i, j = 1, \dots, p$ , we get:

$$\text{Hess } \Phi(z, g) = \left( \begin{array}{c|c} \frac{1}{2} J(g - g^{-1}) + \frac{i}{2} (I - g)(I - g^{-1}) & \frac{1}{2} [J(I + g^{-1}) + i(g^{-1} - I)] A_j z \\ \hline \frac{1}{2} [(J(I + g^{-1}) + i(g^{-1} - I)) A_i z] & \frac{i}{2} \langle A_i z, A_j z \rangle \end{array} \right).$$

To compute the kernel of the Hessian of  $\Phi$ , we need to use the following formula:

$$(3.17) \quad \forall A \in \mathfrak{G}, \forall B \in \mathfrak{G}, \forall z \in \Omega_0, \quad \langle Az, JBz \rangle = 0.$$

Formula (3.17) comes from the fact that  $\Omega_0$  is invariant by  $G$  and is obtained by differentiating at  $t = 0$  the identity  $F_A(e^{tB} z) = 0$ . We set  $x := (g - I)\alpha + Az$ . Then, we get:

$$(3.18) \quad (\alpha, s) \in \ker_{\mathbb{R}} \text{Hess } \Phi(z, g) \iff \begin{cases} (I + g^{-1})x = 0 \\ \langle (I + g^{-1})JA_j z; \alpha \rangle = 0 \quad \forall j = 1, \dots, p \\ (g^{-1} - I)x = 0 \\ \langle A_j z; x \rangle = 0 \quad \forall j = 1, \dots, p \end{cases}$$

which is equivalent to  $x = 0$  and  $(I + g)\alpha \in (J\mathfrak{G}z)^\perp$ . In addition,  $x = 0 \iff (g - I)\alpha = -Az$ , and, in view of (3.17),  $Az \perp J\mathfrak{G}z$ . Thus (3.18)  $\iff x = 0$  and  $\alpha \in (J\mathfrak{G}z)^\perp$ . Therefore:

$$(3.19) \quad d_{(z, 0)} \varphi[\ker_{\mathbb{R}} \text{Hess } \Phi(z, g)] = \{(\alpha, Ag) \in \mathbb{R}^{2d} \times \mathfrak{G} : (M(g) - I)\alpha + Az = 0 \text{ et } \alpha \in (J\mathfrak{G}z)^\perp\}.$$

According to Lemma 3.2, we have  $d_{(z, 0)} \varphi[\ker_{\mathbb{R}} \text{Hess } \Phi(z, g)] = T_{(z, g)} \Gamma_0$ . Hence, we have shown that there is a theoretical asymptotic expansion of  $\text{Tr}(f(\widehat{H}_\chi))$ . We have now to compute the first term and interpret it geometrically.

**3.3. Computation of the leading term.** Since  $Q_0 = 1$  and  $P_0 = 1$ , we get from (3.7), (3.8):

$$\text{Tr}(\widehat{A}_\chi) \underset{h \rightarrow 0^+}{\sim} (2\pi h)^{-d} d_\chi \int_G \int_{\mathbb{R}^{2d}} e^{\frac{i}{h} \Phi(z, g)} \overline{\chi(g)} a(z) dz dg.$$

By the generalized stationary phase theorem, we obtain from Lemma 3.2:

$$(3.20) \quad \text{Tr}(\widehat{A}_\chi) = (2\pi h)^{k_0 - d} d_\chi \int_{\Gamma_0} \frac{\overline{\chi(g)} a(z)}{\det_+^{\frac{1}{2}} \left( \frac{\Phi''(z, g)}{i} \Big|_{\mathcal{N}_{(z, g)} \Gamma_0} \right)} d\sigma_{\Gamma_0}(z, g) + O(h^{k_0 - d + 1}).$$

where  $d\sigma_{\Gamma_0}$  denotes the Riemannian measure on  $\Gamma_0$ . We have to compute the determinant of this expression, and next we will give an integration formula to get from  $\Gamma_0$  to  $\Omega_{red}$ .

– *Computation of the determinant of the transversal Hessian:*

Fix  $z$  in  $\mathbb{R}^{2d}$  and  $g$  in  $G$  such that  $M(g)z = z$ . We denote by  $Stab(z) \subset G$  the stabilizer of  $z$ .

We endow the space of  $d \times d$  matrices with the Riemannian structure coming from the following scalar product:

$$(3.21) \quad \langle\langle A, B \rangle\rangle := \text{Tr}({}^t AB), \quad \text{for all matrices } A \text{ and } B.$$

We choose a basis  $A_1, \dots, A_p$  of  $\mathcal{G}$  such that:

$$(3.22) \quad A_1, \dots, A_{k_0} \text{ is an orthonormal basis of } (T_{Id} \text{Stab}(z))^\perp \text{ for } \langle\langle ; \rangle\rangle.$$

$$(3.23) \quad A_{k_0+1}, \dots, A_p \text{ is an orthonormal basis of } T_{Id} \text{Stab}(z) \text{ for } \langle\langle ; \rangle\rangle.$$

By definition, we have:

$$\det \left( \Phi''(z, g)|_{\mathcal{N}_{(z, g)}\Gamma_0} \right) = \det \left( (\Phi''(z, g)(\mu_i, \mu_j))_{1 \leq i, j \leq 2k_0} \right)$$

where  $(\mu_1, \dots, \mu_{2k_0})$  is an orthonormal basis of  $\mathcal{N}_{(z, g)}\Gamma_0$ . We have:

$$\det \left( \Phi''(z, g)|_{\mathcal{N}_{(z, g)}\Gamma_0} \right) = \det \left( (\langle \text{Hess } \Phi(z, g)(d_{(z, 0)}\varphi)^{-1}(\mu_i), (d_{(z, 0)}\varphi)^{-1}(\mu_j) \rangle)_{1 \leq i, j \leq 2k_0} \right).$$

Note that the differential of the chart  $d_{(z, 0)}\varphi$  is an isometry since  $A_1, \dots, A_p$  is an orthonormal basis of  $\mathcal{G}$ . Thus, if  $\varepsilon_i := (d_{(z, 0)}\varphi)^{-1}(\mu_i)$ , then  $(\varepsilon_1, \dots, \varepsilon_{2k_0})$  is an orthonormal basis of  $(d_{(z, 0)}\varphi)^{-1}(\mathcal{N}_{(z, g)}\Gamma_0) = \mathcal{F}^\perp$ , where:

$$\mathcal{F} := \{(\alpha, s) \in \mathbb{R}^{2d} \times \mathbb{R}^p : (M(g) - Id)\alpha + \sum_{i=1}^p s_i A_i z = 0, \alpha \in T_z \Omega_0\}.$$

By definition of  $\Phi$ , we note that  $\text{Hess } \Phi(z, g)(\mathcal{F}^\perp) \subset \mathcal{F}^\perp + i\mathcal{F}^\perp$ . Hence, we have:

$$\det \left( \frac{\Phi''(z, g)|_{\mathcal{N}_{(z, g)}\Gamma_0}}{i} \right) = \det \left( \frac{\mathcal{A}|_{\mathcal{F}^\perp}}{i} \right),$$

where  $\mathcal{A}|_{\mathcal{F}^\perp}$  denotes the matrix of the restriction of  $\text{Hess } \Phi(z, g)$  to  $\mathcal{F}^\perp$  in any basis of  $\mathcal{F}^\perp$ . We point out the fact that, differentiating the equality:  $e^{tA}z = z$  at  $t = 0$ , we have, for  $A \in \mathcal{G}$ :

$$(3.24) \quad Az = 0 \iff A \in T_{Id} \text{Stab}(z).$$

As a corollary, we note that  $(A_1 z, \dots, A_{k_0} z)$  is a basis of  $\mathcal{G}z$ . The following lemma shows off a basis of  $\mathcal{F}^\perp$ :

**Lemma 3.3.** *Let  $(B_1 z, \dots, B_{k_0} z)$  be a basis of  $\mathcal{G}z$ . We set in  $\mathbb{R}^{2d} \times \mathbb{R}^p$ , for  $j = 1, \dots, k_0$  :*

$$\varepsilon_j := (JB_j z, 0), \quad \varepsilon'_j := ((M(g^{-1}) - I)B_j z, \langle A_j z, B_j z \rangle, 0) \quad (j = 1, \dots, k_0).$$

*Then  $\mathcal{B} := (\varepsilon_1, \dots, \varepsilon_{k_0}, \varepsilon'_1, \dots, \varepsilon'_{k_0})$  is a basis of  $\mathcal{F}^\perp$ .*

The proof is straightforward using (3.17), (3.22), (3.23), (3.24) and the fact that  $T_z \Omega_0 = (J\mathcal{G}z)^\perp$ . Calculating  $\mathcal{A}|_{\mathcal{F}^\perp}$  in this basis, a tedious but basic computation leads to:

$$\det \left( \frac{\mathcal{A}|_{\mathcal{F}^\perp}}{i} \right) = \det \left( \frac{\frac{1}{2}(I - g_0)(I - g_0^{-1}) \mid \frac{1}{2i}(I + g_0^{-1})(g_0 - I)(g_0^{-1} - I) + \frac{1}{2i}(I + g_0^{-1})M^t M}{\frac{1}{2}(I + g_0) \mid \frac{1}{2}(g_0 - I)(g_0^{-1} - I) + \frac{1}{2}M^t M} \right),$$

where  $M$  is the  $k_0 \times k_0$  matrix with general term  $\langle B_i z, A_j z \rangle$  ( $i, j = 1, \dots, k_0$ ), and  $g_0$  is the matrix of the restriction of  $M(g)$  to  $\mathcal{G}z$  in the basis  $(B_1 z, \dots, B_{k_0} z)$ . Then, by the line operation  $L_1 \leftarrow L_1 - \frac{1}{2}(I + g_0^{-1})L_2$ , we get:

$$\det \left( \frac{\mathcal{A}|_{\mathcal{F}^\perp}}{i} \right) = \det[(I - g_0)(I - g_0^{-1}) + M^t M].$$

If one denotes by  $f : \mathcal{G}z \rightarrow \mathcal{G}z$  the linear application defined by:

$$(3.25) \quad f(x) := \sum_{r=1}^{k_0} \langle A_r z, x \rangle A_r z, \quad \forall x \in \mathcal{G}z,$$

then we remark that the matrix of  $f$  in the basis  $(B_1 z, \dots, B_{k_0} z)$  is equal to  ${}^t M M$ . Therefore, we have:

$$(3.26) \quad \det \left( \frac{\Phi''(z, g)}{i} \Big|_{\mathcal{N}_{(z, g)\Gamma_0}} \right) = \det[(I - M(g))(I - M(g)^{-1})|_{\mathcal{G}z} + f].$$

– *Integration formula on  $\Gamma_0$* : we have to deduce an integral over  $\Omega_{red}$  from an integral over  $\Gamma_0$ . A first step consists in passing from  $\Gamma_0$  to  $\Omega_0$ . We will use the following integration lemma using submersions (a proof can be found in [8] or more generally in [5]):

**Lemma 3.4.** *Let  $M$  and  $N$  be two Riemannian manifolds, and  $F : M \rightarrow N$  a smooth submersion. Let  $\varphi$  be smooth with compact support in  $M$ . Then we have:*

$$\int_M \varphi(x) d\sigma_M(x) = \int_N \left[ \int_{\Sigma_n} \varphi(y) \frac{d\sigma_{\Sigma_n}(y)}{|\det(d_y F^t(d_y F))|^{\frac{1}{2}}} \right] d\sigma_N(n).$$

where  $\Sigma_n := F^{-1}(\{n\})$ , and  $d\sigma_{\Sigma_n}$  is the Riemannian measure induced by the one of  $M$  on  $\Sigma_n$ .

Let  $\pi_1 : \Gamma_0 \rightarrow \Omega_0 \cap U$  be defined by:  $\pi_1(z, g) = z$ . We recall that (see (3.12)) if  $M(g)z = z$ , then  $\forall \alpha \in T_z \Omega_0$ ,  $(M(g) - I)\alpha \in \mathcal{G}z$ . From this fact, we deduce that  $\pi_1$  is a submersion. Thus, we can apply Lemma 3.4, noting that  $\Gamma_0$  is endowed with the Riemannian structure coming from the scalar product (3.21) and  $G$  with the one of  $d\tilde{g}$  Riemannian measure coming from the same scalar product ( $d\tilde{g} = Vol(G)dg$ )<sup>2</sup>. If we denote  $Stab(z)$  by  $H_z$ , and if  $\varphi$  is smooth, compactly supported in  $\Gamma_0$ , then we have:

$$(3.27) \quad \int_{\Gamma_0} \varphi(z, g) d\sigma_{\Gamma_0}(z, g) = \frac{1}{Vol(G)} \int_{\Omega_0} \left[ \int_{H_z} \varphi(z, y) \frac{d\tilde{\sigma}_{H_z}(y)}{|\det(d_{(z, y)} \pi_1 \cdot {}^t(d_{(z, y)} \pi_1))|^{\frac{1}{2}}} \right] d\sigma_{\Omega_0}(z),$$

where  $d\tilde{\sigma}_{H_z}$  also denotes the Riemannian measure from (3.21) on  $H_z$ . We recall that  $H_0$  is the stabilizer of a fixed element  $z_0$  of  $\Omega_0 \cap U$ . If  $u \in \mathcal{C}^\infty(H_z)$ , and if  $g_z \in G$  is such that  $g_z H_0 g_z^{-1} = H_z$ , then it is easy to check that:

$$\int_{H_z} u(h) d\sigma_{H_z}(h) = \int_{H_0} u(g_z h g_z^{-1}) d\sigma_{H_0}(h),$$

where  $d\sigma_{H_z}$  and  $d\sigma_{H_0}$  are the normalized Haar measures on  $H_z$  and  $H_0$ . Therefore, using the fact that  $Vol(H_z) = Vol(H_0)$  for (3.21) (remember that  $G$  is made of isometries), we have, in view of (3.27):

$$(3.28) \quad \int_{\Gamma_0} \varphi(z, g) d\sigma_{\Gamma_0}(z, g) = \int_{\Omega_0} \frac{Vol(H_0)}{Vol(G)} \left[ \int_{H_0} \varphi(z, g_z h g_z^{-1}) \frac{d\sigma_{H_0}(h)}{|\det(d_{(z, g_z h g_z^{-1})} \pi_1 \cdot {}^t(d_{(z, g_z h g_z^{-1})} \pi_1))|^{\frac{1}{2}}} \right] d\sigma_{\Omega_0}(z).$$

**Lemma 3.5.** *If  $z \in \Omega_0 \cap U$  and  $g \in G$  are such that  $M(g)z = z$ , then:*

$$\det [d_{(z, g)} \pi_1 \cdot {}^t(d_{(z, g)} \pi_1)]^{-1} = \det^{-1}(f) \det[(I - g)(I - g^{-1})|_{\mathcal{G}z} + f].$$

where  $f$  is given by (3.25).

*Proof*: first we remark that, if  $g \in G$ , then  $T_z \Omega_0 = (J\mathcal{G}z)^\perp$  is invariant by  $M(g)$ . Moreover, we note that, in view of (3.17),  $\mathcal{G}z \subset T_z \Omega_0$ . Let us denote by  $\mathcal{F}_0$  the orthogonal of  $\mathcal{G}z$  in  $T_z \Omega_0$ . We note that  $\mathcal{F}_0$  is also invariant by  $M(g)$ . Thus, if  $\alpha \in \mathcal{F}_0$ , then  $(M(g) - I)\alpha \in \mathcal{F}_0$ . But remembering that  $(M(g) - I)\alpha \in \mathcal{G}z$ , we get that  $M(g) = Id$  on  $\mathcal{F}_0$ .

Besides, if  $\alpha \in T_z \Omega_0$ , by definition of a transpose map,  ${}^t d_{(z, g)} \pi_1(\alpha)$  is the unique element  $(\alpha_0, A_0 g)$  of  $T_{(z, g)} \Gamma_0$  satisfying the fact that, for all  $(\beta, Ag) \in T_{(z, g)} \Gamma_0$  we have:

$$(3.29) \quad \langle \beta, \alpha \rangle_{\mathbb{R}^{2d}} = \langle \beta, \alpha_0 \rangle_{\mathbb{R}^{2d}} + \langle \langle A, A_0 \rangle \rangle = 0.$$

<sup>2</sup> $Vol(G)$  denotes the volume of  $G$  with respect to the Riemannian structure of (3.21).

Moreover, if  $\alpha \in \mathcal{F}$ , since  $M(g)\alpha = \alpha$ , we have  $(\alpha, 0) \in T_{(z,g)}\Gamma_0$ . Therefore, we have  ${}^t d_{(z,g)}\pi_1(\alpha) = (\alpha, 0)$ , and thus  $d_{(z,g)}\pi_1 \cdot {}^t d_{(z,g)}\pi_1 = Id$  on  $\mathcal{F}_0$ . Hence, we have:

$$\det(d_{(z,g)}\pi_1 \cdot {}^t d_{(z,g)}\pi_1) = \det(d_{(z,g)}\pi_1 \cdot {}^t d_{(z,g)}\pi_1|_{\mathcal{G}z}).$$

Let us show that  $d_{(z,g)}\pi_1 \cdot {}^t d_{(z,g)}\pi_1|_{\mathcal{G}z} = (f + (I - M(g))(I - M(g^{-1})))^{-1} \circ f$ . If  $\alpha \in \mathcal{G}z$ , if we set  $(\alpha_0, A_0g) := {}^t d_{(z,g)}\pi_1(\alpha)$ , then we have to show that:

$$f(\alpha) = (f + (I - M(g^{-1}))(I - M(g)))\alpha_0,$$

that is to say that (since  $(A_1z, \dots, A_{k_0}z)$  is a basis of  $\mathcal{G}z$ , for  $i = 1, \dots, k_0$ , we have:

$$(3.30) \quad \langle A_i z, f(\alpha - \alpha_0) \rangle = \langle A_i z, (I - M(g^{-1}))(I - M(g))\alpha_0 \rangle.$$

The l.h.s. of (3.30) is equal to  $\sum_{j=1}^{k_0} \langle A_i z, A_j z \rangle \langle \alpha - \alpha_0, A_j z \rangle$ . Moreover, note that we have:  $(A_j z, [A_j - gA_j g^{-1}]g) \in T_{(z,g)}\Gamma_0$ . Thus, by (3.29), we have:

$$(3.31) \quad \langle A_j z, \alpha - \alpha_0 \rangle = \langle (A_j - gA_j g^{-1}), A_0 \rangle.$$

Besides,

$$\langle A_i z, (I - g^{-1})(I - g)\alpha_0 \rangle = \langle A_i z, (I - g^{-1})A_0 z \rangle = \langle A_i z, (A_0 - g^{-1}A_0g)z \rangle.$$

Then, decomposing  $A_0 - g^{-1}A_0g$  in the orthonormal basis  $(A_1, \dots, A_{k_0})$  of  $[T_{Id}Stab(z)]^\perp$ , we get:

$$\langle A_i z, (I - g^{-1})(I - g)\alpha_0 \rangle = \sum_{j=1}^{k_0} \langle A_i z, A_j z \rangle \langle A_0 - g^{-1}A_0g, A_j \rangle.$$

From this last equality and (3.31), we get (3.30), which ends the proof of Lemma 3.5.  $\square$

We apply this lemma and (3.28) to  $\varphi(z, g) := \overline{\chi(g)}a(z)\det^{-\frac{1}{2}}\left(\frac{\Phi''(z,g)}{i}|_{\mathcal{N}(z,g)\Gamma_0}\right)$ , to get:

$$(3.32) \quad \int_{\Gamma_0} \overline{\chi(g)}a(z)\det^{-\frac{1}{2}}\left(\frac{\Phi''(z,g)}{i}|_{\mathcal{N}(z,g)\Gamma_0}\right) d\sigma_{\Gamma_0}(z, g) = \frac{Vol(H_0)}{Vol(G)} \int_{\Omega_0} \frac{a(z)}{\det^{\frac{1}{2}}(f)} d\sigma_{\Omega_0}(z) \cdot \int_{H_0} \overline{\chi(h)} d\sigma_{H_0}(h).$$

Note that, according to [27] or [28], we have:

$$(3.33) \quad \int_{H_0} \overline{\chi(h)} d\sigma_{H_0}(h) = [\rho_{\chi|_{H_0}} : \mathbb{1}].$$

Finally, we are greatly indebted to Gilles Carron for giving us the proof of the following lemma:

**Lemma 3.6.**  $\det^{\frac{1}{2}}(f) = Vol(G(z)) \frac{Vol(H_0)}{Vol(G)}$ , where  $Vol(G)$  and  $Vol(H_0)$  are calculated as Riemannian volumes for the scalar product (3.21).

*Proof:* let  $(\varepsilon_1, \dots, \varepsilon_{k_0})$  be an orthonormal basis of  $\mathcal{G}z$ . We have:

$$(3.34) \quad \det(f) = \det[(\langle A_i z, \varepsilon_j \rangle)_{(1 \leq i, j \leq k_0)}]^2.$$

The idea of the proof is to use the link between  $G$  and  $G(z)$  through  $G/H$ , using the fact that there is a unique  $G$ -invariant volume form<sup>3</sup> on  $G/H$  (up to a constant). We denote here by  $e$  the neutral element of  $G$  ( $e = Id$ ),  $H := Stab(z)$  and  $\mathcal{H} := T_e H$ . Let  $\pi : G \rightarrow G/H$  be the canonical projection on the quotient. Via  $\pi$ ,  $G/H$  inherits a Riemannian structure from  $G$ . Namely, if  $g$  is any element of  $G$ , let  $\psi_g$  denote the reciprocal application of the restriction of  $d_g \pi$  to  $(\ker d_g \pi)^\perp = (g\mathcal{H})^\perp$ . Then the metric  $\delta$  on  $G/H$  is defined by:

$$\forall g \in G, \forall u, v \in T_{\pi(g)}(G/H), \quad \delta_{\pi(g)}(u, v) = \langle \psi_g(u), \psi_g(v) \rangle.$$

<sup>3</sup>A volume form on  $G/H$  is said to be  $G$ -invariant if we have for all  $g$  in  $G$ :  $L_g^* \alpha = \alpha$ , where  $L_g : G/H \rightarrow G/H$  is defined by  $L_g(g_0H) = (gg_0)H$  for  $g_0 \in G$ .

Let  $\omega$  be the associated Riemannian volume form on  $G/H$ . We claim that:

$$(3.35) \quad \begin{cases} \omega \text{ is } G\text{-invariant.} \\ \omega(eH)(\psi_e^{-1}(A_1), \dots, \psi_e^{-1}(A_{k_0})) = 1. \\ \int_{G/H} \omega = \frac{Vol G}{Vol H}. \end{cases}$$

(use Lemma 3.4 and the fact that  $\Psi_e$  is an isometry).

Another volum form on  $G/H$  is given by  $\Phi^* \mu$ , where  $\Phi : G/H \rightarrow G(z)$  is defined by  $\Phi(gH) = gz$ , for any  $g$  in  $G$ , and  $\mu$  denotes the volume form associated to the euclidian volume on  $G(z)$ . We claim that:

$$(3.36) \quad \begin{cases} \Phi^* \mu \text{ is } G\text{-invariant.} \\ \Phi^* \mu(eH)(\psi_e^{-1}(A_1), \dots, \psi_e^{-1}(A_{k_0})) = 1. \\ \int_{G/H} \Phi^* \mu = Vol G(z). \end{cases}$$

Indeed, for  $g \in G$ ,  $\Phi \circ L_g = g \cdot \Phi$ , and  $\mu$  is  $G$ -invariant. Moreover,  $\psi_e^{-1}(A_i) = d_e \pi(A_i)$ , and since  $\Phi \circ \pi(g) = gz$ , we have  $d_{eH} \Phi \circ d_e \pi(A_i) = A_i z$  for  $i = 1, \dots, k_0$ .

Now, if  $\Omega^{k_0}(G/H)^G$  denotes the set of all  $G$ -invariant volume forms of  $G/H$ , we note that the application  $\alpha \mapsto \alpha(eH)$  from  $\Omega^{k_0}(G/H)^G$  to  $\Lambda^{k_0}(T_{eH}(G/H))$  is an isomorphism. In particular  $\dim(\Omega^{k_0}(G/H)^G) = 1$ , and thus we deduce that there exists  $\lambda$  in  $\mathbb{R}$  such that:

$$\Phi^* \mu = \lambda \omega.$$

By (3.35) and (3.36), we obtain that  $\lambda = Vol(G(z)) \frac{Vol(H)}{Vol(G)}$ , and, besides, that  $\lambda = \det(\varepsilon_i, A_j z) = (\det(f))^{\frac{1}{2}}$ .  $\square$

Thus, in view of (3.20):

$$\text{Tr}(\widehat{A}_\chi) = (2\pi h)^{k_0-d} d_\chi \int_{\Omega_0} a(z) \frac{d\sigma_{\Omega_0}(z)}{Vol(G(z))} [\rho_{\chi|_{H_0}} : \mathbb{1}] + O(h^{k_0-d+1}).$$

where  $dh$  is the Haar measure on  $H_0$ . The proof is clear if we remark that: if  $v \in \mathcal{C}_0^\infty((\Omega_0 \cap U)/G)$ , we have, using Lemma 3.4:

$$(3.37) \quad \int_{\Omega_{red}} v(x) d\sigma_{red}(x) = \int_{\Omega_0} v(\pi(z)) \frac{d\sigma_{\Omega_0}(z)}{Vol(G(z))}.$$

#### 4. REDUCED GUTZWILLER FORMULA: $G$ -CLEAN FLOW CONDITIONS

We now focus on the asymptotics of the reduced spectral density (1.12) (see Introduction for notations). The case where  $\hat{f}$  is supported near zero will lead to Theorem 1.1, and the case where  $0 \notin \text{Supp} \hat{f}$  to Theorem 1.2. By formula (2.3) of [7] and linearity of the trace, we have:

$$\mathfrak{S}_\chi(h) := d_\chi \int_G \overline{\chi(g)} \text{Tr} \left( \psi(\widehat{H}) f \left( \frac{E - \widehat{H}}{h} \right) \tilde{M}(g) \right) dg.$$

Note that the proof of [7], in particular section 3 ('Reduction of the proof by coherent states') gives us an asymptotic expansion at fixed  $g$  of the quantity  $I_g(h) := \text{Tr} \left( \psi(\widehat{H}) f \left( \frac{E - \widehat{H}}{h} \right) \tilde{M}(g) \right)$  for which the rest is uniform with respect to  $g \in G$ , since  $G$  is compact. Namely, we made hypothesis (3.2) on  $H$  in order to have a pleasant description of term  $\psi(\widehat{H})$  by functional calculus. Under this assumption, using the theorem of propagation of coherent states given by Combes and Robert in [11] and [26], we found that  $I_g(h)$  has an asymptotic expansion in powers of  $h$  with coefficients depending on  $h$  and of the form:

$$\int_{\mathbb{R}_t} \int_{\mathbb{R}_z^{2d}} \exp \left( \frac{i}{h} \varphi_E(t, z, g) \right) \hat{f}(t) a_g(z) dz dt.$$

where  $a_g : \mathbb{R}^{2d} \rightarrow \mathbb{C}$  is supported in  $H^{-1}([E - \delta E, E + \delta E])$ , and  $\varphi_E = \varphi_1 + i\varphi_2$  with:

$$(4.1) \quad \begin{cases} \varphi_1(t, z, g) := (E - H(z))t + \frac{1}{2} \langle M(g)^{-1}z, Jz \rangle - \frac{1}{2} \int_0^t (z_s - M(g^{-1})z) Jz_s ds \\ \varphi_2(t, z, g) := \frac{i}{4} \langle (I - \widehat{W}_t)(M(g)z_t - z); (M(g)z_t - z) \rangle. \end{cases}$$

where  $z_t := \Phi_t(z)$ ,  $\Phi_t$  being the flow of (1.3). We set  $F_z(t) := \partial_z \Phi_t(z) \in Sp(d, \mathbb{R})$  and

$$(4.2) \quad F_z(t) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{where } A, B, C, D \text{ are } d \times d \text{ matrices.}$$

Then  $\widehat{W}_t := \begin{pmatrix} W_t & -iW_t \\ -iW_t & -W_t \end{pmatrix}$  with  $\frac{1}{2}(I + W_t) := (I - i^t g^{-1} M_0 g^{-1})^{-1}$ , where we have set  $M_0 := (C + iD)(A + iB)^{-1}$ . Moreover, we have from [7] that  $\|W_t\|_{\mathcal{L}(\mathbb{C}^d)} < 1$ . Therefore,  $\mathcal{G}_\chi(h)$  has an asymptotic expansion in powers of  $h$  with coefficients depending on  $h$  and of the form:

$$(4.3) \quad J(h) = \int_G \int_{\mathbb{R}_t} \int_{\mathbb{R}^{2d}} \exp\left(\frac{i}{h} \varphi_E(t, z, g)\right) \hat{f}(t) a_g(z) dz dt dg.$$

where  $a_g : \mathbb{R}^{2d} \rightarrow \mathbb{C}$  is supported in  $H^{-1}([E - \delta E, E + \delta E])$ . In particular, in view of [7], we get:

$$(4.4) \quad \mathcal{G}_\chi(h) \underset{h \rightarrow 0^+}{\sim} d_\chi \frac{(2\pi h)^{-d}}{2\pi} \int_G \int_{\mathbb{R}_t} \int_{\mathbb{R}^{2d}} \overline{\chi(g)} \chi_1(z) \hat{f}(t) \psi(H(z)) \det_+^{-\frac{1}{2}}\left(\frac{A+iB-i(C+iD)}{2}\right) e^{\frac{i}{h} \varphi_E(t, z, g)} dt dz dg,$$

where  $\chi_1$  is a smooth function compactly supported in  $\mathbb{R}^{2d}$ , equal to 1 on  $\Sigma_E = \{H = E\}$ . Note that problems of entire powers of  $h$  and of shifting in powers of  $h$  are solved with exactly the same method as in [7]. We have now to apply the generalised stationary phase theorem (in the version of [10], Theorem 3.3), to each term  $J(h)$ . In this section, we describe the minimal hypotheses to be fulfilled for applying this theorem. These will be called  $G$ -clean flow assumptions. We then give the theoretical asymptotics in Theorem 4.6. We will compute leading terms of the expansion in two particular cases in sections 5 and 6. For the proof of Theorems 4.6, 5.2, 1.2, we recall that, by an averaging argument, we can suppose that the group  $G$  is made of isometries (see [7]). We will often denote  $M(g)$  by  $g$ , in order to simplify notations. Finally, we suppose that hypotheses of symplectic reduction are fulfilled in  $\Omega_0 \cap U$ , where  $U := H^{-1}([E - \delta E, E + \delta E])$ .

**4.1. Computation of the critical set.** Let

$$\mathcal{C}_E := \{a \in \mathbb{R} \times \mathbb{R}^{2d} \times G : \Im(\varphi_E(a)) = 0, \nabla \varphi_E(a) = 0\}.$$

**Proposition 4.1.** *The critical set is:*

$$(4.5) \quad \mathcal{C}_E = \{(t, z, g) \in \mathbb{R} \times \mathbb{R}^{2d} \times G : z \in (\Omega_0 \cap \Sigma_E), M(g)\Phi_t(z) = z\}.$$

*Proof:* as in [7], using that  $\|W_t\|_{\mathcal{L}(\mathbb{C}^d)} < 1$ , we get that  $\Im(\varphi_E(t, z, g)) = 0 \iff M(g)\Phi_t(z) = z$ . We first need some formulae coming from the symmetry that will be helpful for the computation: We recall that  $F_z(t) = \partial_z(\Phi_t(z))$ . By differentiating formula (1.2), we get:

$$(4.6) \quad \nabla H(M(g)z) = {}^t M(g^{-1}) \nabla H(z), \quad \forall z \in \mathbb{R}^{2d}, \forall g \in G.$$

This formula implies that we have also:

$$(4.7) \quad \Phi_t(M(g)z) = M(g)\Phi_t(z), \quad \forall z \in \mathbb{R}^{2d}, \forall g \in G, \forall t \in \mathbb{R} \text{ such that the flow exists at time } t.$$

Moreover we recall that, since  $M(g)$  is symplectic, we have:

$$(4.8) \quad JM(g) = {}^t M(g^{-1})J \quad \text{and} \quad M(g)J = J^t M(g^{-1}).$$

Finally, if  $t$  and  $z$  are such that  $M(g)\Phi_t(z) = z$ , then we have:

$$(4.9) \quad (M(g)F_z(t) - I)J\nabla H(z) = 0 \quad \text{and} \quad ({}^t F_z(t) {}^t M(g) - I)\nabla H(z) = 0.$$

The second identity comes from the first one since  $M(g)F_z(t)$  is symplectic. For this first identity, one can differentiate at  $s = t$  the equation:

$$\Phi_t(\widehat{M}(g)\Phi_s(z)) = \Phi_s(z).$$

Moreover, we recall (3.17), and claim that we have:

$$(4.10) \quad \forall A \in \mathcal{G}, \forall z \in \Omega_0, \quad \langle Az, \nabla H(z) \rangle = 0.$$

Formula (4.10) is equivalent to say that  $\{H, F_A\} = 0$ .

• *Computation of the gradient of  $\varphi_1$* : if  $A \in \mathcal{G}$ , then we have:

$$(4.11) \quad \begin{cases} \partial_t \varphi_1(t, z, g) = E - H(z) - \frac{1}{2} \langle (z_t - M(g^{-1})z); J\dot{z}_t \rangle \\ \nabla_z \varphi_1(t, z, g) = \frac{1}{2}(M(g) + {}^t F_z(t))J(z_t - M(g^{-1})z) \\ \partial_g \varphi_1(t, z, g)(Ag) = \frac{1}{2} \langle JAz; M(g)z_t \rangle \end{cases}$$

• *Computation of the gradient of  $\varphi_2$* :

$$(4.12) \quad \begin{cases} 4\partial_t \varphi_2(t, z, g) = 2 \langle (I - \widehat{W}_t)(z_t - g^{-1}z); \dot{z}_t \rangle - \langle \partial_z(\widehat{W}_t)(z_t - g^{-1}z); (z_t - g^{-1}z) \rangle \\ 4\nabla_z \varphi_2(t, z, g) = 2({}^t F_z(t) - g)(I - \widehat{W}_t)(z_t - g^{-1}z) - {}^t [\partial_z(\widehat{W}_t)(z_t - g^{-1}z)](z_t - g^{-1}z) \\ 4\partial_g \varphi_2(t, z, g)(Ag) = 2 \langle (I - \widehat{W}_t)g^{-1}Az; (z_t - g^{-1}z) \rangle, \quad \text{if } A \in \mathcal{G}. \end{cases}$$

The only difficulty lies in the computation of  $\nabla_z \varphi_1(t, z, g)$ : we have:  $\nabla_z \varphi_1(t, z, g) =$

$$-t\nabla H(z) + \frac{1}{2}(gJ + {}^t(gJ)) - \frac{1}{2} \int_0^t {}^t \partial_z(\Phi_s(z) - g^{-1}z)J\dot{z}_s ds - \frac{1}{2} \int_0^t {}^t (J\partial_z(\dot{z}_s))(z_s - g^{-1}z) ds.$$

We note that:  $\partial_z(\dot{z}_s) = \partial_z \partial_s(\Phi(s, z)) = \frac{d}{ds}(F_z(s))$ . By an integration by parts, we obtain:

$$\int_0^t {}^t (J\partial_z(\dot{z}_s))(z_s - g^{-1}z) ds = -[{}^t F_z(s)J(z_s - g^{-1}z)]_0^t + \int_0^t {}^t F_z(s)J\dot{z}_s ds.$$

The end of the calculus is straightforward, if we note that:

$${}^t F_z(s)J\dot{z}_s = -{}^t F_z(s)\nabla H(\Phi_s(z)) = -\nabla(H(\Phi_s(z))) = -\nabla H(z).$$

Therefore, we remark that  $(t, z, g) \in \mathcal{C}_E$  if and only if  $\Phi_t(z) = M(g^{-1})z$ ,  $H(z) = E$  and for all  $A$  in  $\mathcal{G}$ ,  $\langle JAz; z \rangle = 0$ , i.e.  $z \in \Omega_0$ . This ends the proof of proposition 4.1.  $\square$

**4.2. Computation of the Hessian  $\text{Hess } \varphi_E(t, z, g)$ .** The space of complex matrices is endowed with the scalar product (3.21). If  $(t_0, z_0, g_0) \in \mathcal{C}_E$ , then we choose the chart of  $\mathbb{R} \times \mathbb{R}^{2d} \times G$  at  $(t_0, z_0, g_0)$  to be:

$$(4.13) \quad \varphi(t_0, z_0, g_0) = \varphi : \begin{cases} \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^{2d} \times \mathbb{R}^p \rightarrow \mathbb{R} \times \mathbb{R}^{2d} \times G \\ (t, z, s) \mapsto (t, z, \exp(\sum_{i=1}^p s_i A_i) \cdot g_0). \end{cases}$$

where  $(A_1, \dots, A_p)$  is the orthonormal basis of  $\mathcal{G}$  defined such that:

$$(4.14) \quad (A_1, \dots, A_{k_0}) \text{ is the orthonormal basis of } [T_{Id}(\text{Stab}(z_0))]^\perp.$$

$$(4.15) \quad (A_{k_0+1}, \dots, A_p) \text{ is the orthonormal basis of } T_{Id}(\text{Stab}(z_0)).$$

Then we define:

$$\text{Hess } \varphi_E(t_0, z_0, g_0) := \left( \left( \frac{\partial^2(\varphi_E \circ \varphi)}{\partial x_i \partial x_j}(t_0, z_0, 0) \right) \right)_{1 \leq i, j \leq p} \in M_{2d+p+1}(\mathbb{C}).$$

**Proposition 4.2.** *Let  $(t, z, g) \in \mathcal{C}_E$ . Then  $\text{Hess } \varphi_E(t, z, g) =$*

$$\left( \begin{array}{c|c|c|c} \frac{i}{2} \langle g(I - \widehat{W}_t)g^{-1}J\nabla H; J\nabla H \rangle & -{}^t\nabla H & \frac{i}{2} \langle J\nabla H; g(I - \widehat{W}_t)g^{-1}A_j z \rangle & 0 \\ \hline +\frac{i}{2} {}^t[({}^tF - g)(I - \widehat{W}_t)g^{-1}J\nabla H] & \frac{1}{2}[JgF - {}^t(gF)J] & \frac{1}{2}({}^tF + g)Jg^{-1}A_j z & 0 \\ \hline +\frac{i}{2}({}^tF - g)(I - \widehat{W}_t)g^{-1}J\nabla H & +\frac{i}{2}({}^tF - g)(I - \widehat{W}_t)(F - g^{-1}) & +\frac{i}{2}({}^tF - g)(I - \widehat{W}_t)g^{-1}A_j z & 0 \\ \hline \frac{i}{2} \langle J\nabla H; g(I - \widehat{W}_t)g^{-1}A_i z \rangle & \frac{1}{2} {}^t[({}^tF + g)Jg^{-1}A_i z] & \frac{i}{2} \langle g(I - \widehat{W}_t)g^{-1}A_i z; A_j z \rangle & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right)$$

where, we denoted  $F_z(t)$  by  $F$ , and  $\nabla H(z)$  by  $\nabla H$ , and each line index  $i$  and column index  $j$  is repeated  $k_0$  times.

*Proof:* use (4.11) and (4.12) together with (4.6), (4.7) and (4.9). For  $\partial_{s_i}\partial_{s_j}\varphi_E$ , use (3.16) to write:

$$\begin{aligned} \frac{\partial^2}{\partial_{s_i}\partial_{s_j}}\varphi_1(t, z, g) &= \frac{1}{4} \langle z; J(A_i A_j + A_j A_i)gz \rangle + \frac{1}{4} \int_0^t \langle g^{-1}(A_i A_j + A_j A_i)z, J\dot{z}_s \rangle ds \\ &= \frac{1}{4} \langle (A_i A_j + A_j A_i)z, Jz \rangle = -\frac{1}{2} \langle A_j z, JA_i z \rangle = 0, \end{aligned}$$

in view of (3.17). Besides, we have:  $4\partial_{s_i}\partial_{s_j}\varphi_2(t, z, g) =$

$$\begin{aligned} &\frac{d}{ds_j} \Big|_{s_j=0} \left[ \langle (I - \widehat{W}_t) \frac{d}{ds_i} \Big|_{s_i=0} (z_t - g^{-1}z); z_t - g^{-1}z \rangle + \langle (I - \widehat{W}_t)(z_t - g^{-1}z); \frac{d}{ds_i} \Big|_{s_i=0} (z_t - g^{-1}z) \rangle \right] \\ &= \langle (I - \widehat{W}_t) \frac{d}{ds_i} \Big|_{s_i=0} (z_t - g^{-1}z); \frac{d}{ds_j} \Big|_{s_j=0} (z_t - g^{-1}z) \rangle + \langle (I - \widehat{W}_t) \frac{d}{ds_j} \Big|_{s_j=0} (z_t - g^{-1}z); \frac{d}{ds_i} \Big|_{s_i=0} (z_t - g^{-1}z) \rangle \\ &= 2 \langle (I - \widehat{W}_t)g^{-1}A_i z, g^{-1}A_j z \rangle. \end{aligned}$$

If one remembers (3.24), then the proof of the proposition is clear.  $\square$

**4.3. Computation of the real kernel of the Hessian.** We denote here by  $\ker_{\mathbb{R}} \text{Hess } \varphi_E(t, z, g)$  the set of  $(\tau, \alpha, Ag)$  in  $\mathbb{R} \times \mathbb{R}^{2d} \times \mathfrak{G}g$  such that  $A = \sum_{i=1}^p s_i A_i$  ( $s_i \in \mathbb{R}$ ) and  $(t, z, s_1, \dots, s_p) \in \ker(\text{Hess } \varphi_E(t, z, g))$ . We recall that, if  $(t, z, g) \in \mathcal{C}_E$ , then:

$$\varphi_E''(t, z, g)|_{\mathcal{N}_{(t, z, g)} e_E} \text{ is non degenerate} \iff \ker_{\mathbb{R}}(\text{Hess } \varphi_E(t, z, g)) \subset T_{(t, z, g)} \mathcal{C}_E.$$

**Proposition 4.3.** *Let  $(t, z, g) \in \mathcal{C}_E$ . Then  $\ker_{\mathbb{R}} \text{Hess } \varphi_E(t, z, g) =$*

$$(4.16) \quad \{(\tau, \alpha, Ag) \in \mathbb{R} \times \mathbb{R}^{2d} \times \mathfrak{G}g : \alpha \perp \nabla H(z), \alpha \perp J(\mathfrak{G}z), \tau J\nabla H(z) + (M(g)F_z(t) - Id)\alpha + Az = 0\}.$$

*Proof:* we set  $\widehat{W}_t =: \widehat{W}_1 + i\widehat{W}_2$ , where  $\widehat{W}_1$  and  $\widehat{W}_2$  are the real and (resp.) imaginary parts of  $\widehat{W}_t$  ( $\widehat{W}_1$  and  $\widehat{W}_2$  are symmetric matrices). We denote  $F_z(t)$  by  $F$ . Then, by proposition 4.2, if we set  $x := \tau J\nabla H(z) + (gF - I)\alpha + Az$ , then  $(\tau, \alpha, Ag) \in \ker_{\mathbb{R}} \text{Hess } \varphi_E(t, z, g)$  if and only if:

$$(4.17) \quad \begin{cases} \langle g\widehat{W}_2 g^{-1} J\nabla H(z); x \rangle = 2 \langle \nabla H(z); \alpha \rangle. \\ \langle g(I - \widehat{W}_1)g^{-1} J\nabla H(z); x \rangle = 0. \\ ({}^tF - g)(I - \widehat{W}_1)g^{-1}x = 0. & (\star) \\ \langle g\widehat{W}_2 g^{-1} A_i z; x \rangle + \langle JA_i z; (gF + I)\alpha \rangle = 0, \text{ for } i = 1, \dots, k_0. \\ \langle g(I - \widehat{W}_1)g^{-1} A_i z; x \rangle = 0 \text{ for } i = 1, \dots, k_0. \end{cases}$$

and

$$(4.18) \quad \left. \begin{aligned} -2\tau \nabla H(z) + [JgF - {}^t(gF)J]\alpha + ({}^tF - g)\widehat{W}_2[(F - g^{-1})\alpha + Az] \\ + \tau({}^tF - g)\widehat{W}_2 g^{-1} J\nabla H(z) + ({}^tF + g)Jg^{-1}Az = 0. \end{aligned} \right\}$$

We note that  $\widehat{W}_2 = J\widehat{W}_1$ ,  $[g, J] = 0$  and that  $gF$  is a symplectic matrix. By multiplying the last equality by  $gFJ$  and using the fact that:  $gFJ\nabla H(z) = J\nabla H(z)$ , we get:

$$(4.18) \iff (gF - I)(I - g\widehat{W}_1 g^{-1})x = -2x.$$

We set  $y := (I - g\widehat{W}_1g^{-1})x$ . Then:

$$\begin{cases} (\star) \iff y \in \ker[(gF) - I] = \text{Im}(gF - I)^\perp. \\ (4.18) \iff (gF - I)y = -2x. \end{cases}$$

If  $(\tau, \alpha, gA) \in \ker_{\mathbb{R}} \text{Hess } \varphi_E(t, z, g)$  then  $x \perp y$ , i.e.

$$\langle (I - g\widehat{W}_1g^{-1})x; x \rangle = 0, \quad \text{i.e. } |x|^2 = \langle \widehat{W}_1g^{-1}x; g^{-1}x \rangle.$$

Since  $\|\widehat{W}_1\| < 1$ , we get  $x = 0$ . Then, using (4.17), we get  $\alpha \perp \nabla H(z)$  and  $(gF + I)\alpha \in J(\mathcal{G}z)^\perp$ . In view of (4.10) and (3.17),  $J\nabla H(z)$  and  $Az$  are in  $(J\mathcal{G}z)^\perp$ , and  $x = 0$ , thus:  $(gF - I)\alpha \in (J\mathcal{G}z)^\perp$ . Therefore, we have  $\alpha \in (J\mathcal{G}z)^\perp$ . The converse is clear.  $\square$

**4.4. Asymptotics under  $G$ -clean flow conditions.** We now give a simple geometrical criterion to have an asymptotic expansion of  $\mathcal{G}_\chi(h)$  if  $T > 0$  is such that  $\text{Supp}(f) \subset ]-T, T[$ :

$$\text{Let } \Psi := \begin{cases} ]-T, T[ \times (\Sigma_E \cap \Omega_0) \times G \rightarrow \mathbb{R}^{2d} \\ (t, z, g) \mapsto M(g)\Phi_t(z) - z \end{cases}$$

**Definition 4.4.** We say that the flow is  $G$ -clean on  $]-T, T[ \times (\Sigma_E \cap \Omega_0)$  if zero is a weakly regular value of  $\Psi$ , i.e. :

- $\Psi^{-1}(\{0\}) =: \mathcal{C}_{E,T}$  is a finite union of submanifolds of  $\mathbb{R} \times \mathbb{R}^{2d} \times G$ .
- $\forall (t, z, g) \in \mathcal{C}_{E,T}$ , we have  $T_{(t,z,g)}\mathcal{C}_{E,T} = \ker d_{(t,z,g)}\Psi$ .

If there is no critical point of  $\tilde{H}$  on  $\tilde{\Sigma}_E$  (see (1.11)), then the  $G$ -clean flow hypothesis is somehow optimal to apply the generalised stationary phase theorem. Indeed, if  $(t, z, g) \in \mathcal{C}_{E,T}$ , then:

$$(4.19) \quad \ker d_{(t,z,g)}\Psi = \ker_{\mathbb{R}} \text{Hess } \varphi_E(t, z, g).$$

The justification is the following: we first note that, if  $(\tau, \beta, A) \in \mathbb{R} \times T_z(\Omega_0 \cap \Sigma_E) \times \mathcal{G}$ , then:

$$d_{(t,z,g)}\Psi(\tau, \beta, Ag) = \tau J\nabla H(z) + (M(g)F_z(t) - Id_{\mathbb{R}^{2d}})\beta + Az.$$

Moreover, we have the following lemma:

**Lemma 4.5.** We recall that hypotheses of symplectic reduction are fulfilled on  $U \cap \Omega_0$  where  $U := H^{-1}(]E - \delta E, E + \delta E[)$ . Then following assertions are equivalent:

- (1)  $\Omega_0 \cap U$  and  $\Sigma_E$  are transverse submanifolds of  $\mathbb{R}^{2d}$ .
- (2) There is no critical point of  $\tilde{H}$  on  $\tilde{\Sigma}_E$ .

Therefore, by Proposition 4.3, we have (4.19).

*Proof of the lemma:* the negation of (1) is:  $\exists z \in (\Omega_0 \cap U) \cap \Sigma_E$ ,  $T_z\Omega_0 \subset (\mathbb{R}\nabla H(z))^\perp$ , i.e., since  $(J\mathcal{G}z)^\perp = T_z\Omega_0$ :  $\mathbb{R}\nabla H(z) \subset J\mathcal{G}z$  that is  $J\nabla H(z) \in \mathcal{G}z$ . Finally, if  $\pi : \Omega_0 \rightarrow \Omega_{red}$  denotes the canonical projection on the quotient, then we have (since  $\ker d_z\pi = \mathcal{G}z$ ):  $d_{\pi(z)}\tilde{H} = 0$  if and only if  $J\nabla H(z) \in \mathcal{G}z$ .  $\square$

We get the following theorem:

**Theorem 4.6.** Let  $G$  be a compact Lie group of  $Gl(d, \mathbb{R})$  and  $H : \mathbb{R}^{2d} \rightarrow \mathbb{R}$  be a smooth  $G$ -invariant Hamiltonian satisfying (3.2). Let  $E \in \mathbb{R}$  be such that  $H^{-1}(]E - \delta E, E + \delta E[)$  is compact for some  $\delta E > 0$ , and that  $\tilde{\Sigma}_E = \{\tilde{H} = E\}$  has no critical points. We suppose that hypotheses of reduction are satisfied on  $\Omega_0 \cap U$ , where  $U := H^{-1}(]E - \delta E, E + \delta E[)$ . Let  $f$  and  $\psi$  be real functions in  $\mathcal{S}(\mathbb{R})$  with  $\text{Supp}(\psi) \subset ]E - \delta E, E + \delta E[$  and  $\hat{f}$  compactly supported in  $]-T, T[$  where  $T > 0$ . Moreover, we suppose that the  $G$ -clean flow conditions are satisfied on  $]-T, T[ \times (\Omega_0 \cap \Sigma_E)$ . We denote by:

$$\mathcal{C}_{E,T} := \{(t, z, g) \in ]-T, T[ \times \mathbb{R}^{2d} \times G : z \in (\Omega_0 \cap \Sigma_E), M(g)\Phi_t(z) = z\},$$

and by  $[\mathcal{C}_{E,T}]$  the set of its connected components. Then the quantity  $\int_0^t p_s \dot{q}_s ds$  ( $(q_t, p_t) := \Phi_t(z)$ ) is constant on each element  $Y$  of  $[\mathcal{C}_{E,T}]$ , denoted by  $S_Y$  and  $\mathcal{G}_\chi(h)$  has the following asymptotic

expansion modulo  $O(h^{+\infty})$ :

$$\mathfrak{G}_\chi(h) = \sum_{Y \in [\mathbb{C}_E]} (2\pi h)^{\frac{1-\dim Y+p}{2}} e^{\frac{i}{h} S_Y} \psi(E) \frac{d_\chi}{2\pi} \left( \int_Y \hat{f}(t) \overline{\chi(g)} d(t, z, g) d\sigma_Y(t, z, g) + \sum_{j \geq 1} h^j a_{j,Y} \right).$$

where the  $a_{j,Y}$  are distributions in  $\hat{f}$ , and the density  $d(t, z, g)$  is defined by:

$$d(t, z, g) := \det_+^{-\frac{1}{2}} \left( \frac{\varphi_E''(t, z, g)|_{\mathcal{N}(t, z, g)^Y}}{i} \right) \det_+^{-\frac{1}{2}} \left( \frac{A + iB - i(C + iD)}{2} \right).$$

We recall that  $\varphi_E''(t, z, g)$  is given by proposition 4.2 and that  $A, B, C, D$  are given by:  $\partial_z(\Phi_t(z)) = F_z(t) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ .

Proof of the theorem: we can apply the stationary phase theorem to each coefficient (4.3). Then one can use (4.4) to calculate the first term, and remark that, on  $\mathbb{C}_{E,T}$ ,  $\varphi_E$  is constant with value  $\varphi_E(t, z, g) = \int_0^t p_s q_s ds$ , if  $(t, z, g) \in \mathbb{C}_E$ .  $\square$

## 5. THE WEYL PART

In this section, we plan to give asymptotics of  $\mathfrak{G}_\chi(h)$  when  $\text{Supp}(\hat{f}) \cap \mathcal{L}_{red}(E) = \{0\}$ , where:

$$(5.1) \quad \mathcal{L}_{red}(E) = \mathcal{L}_{red} := \{t \in \mathbb{R} : \exists g \in G, \exists z \in \Omega_0 \cap \Sigma_E : M(g)\Phi_t(z) = z\}.$$

Note that  $\mathcal{L}_{red}(E)$  is the set of periods of periodic orbits of  $\tilde{\Sigma}_E$ . In particular, this assumption is fulfilled if  $\hat{f}$  is supported close enough to zero, which will lead to Theorem 1.1.

**Proposition 5.1.** *Suppose that  $\tilde{\Sigma}_E$  is non critical. Then 0 is isolated in  $\mathcal{L}_{red}(E)$ . Moreover, if  $\text{Supp}(\hat{f}) \cap \mathcal{L}_{red}(E) = \{0\}$ , then  $\mathbb{C}_E \cap (\text{Supp}(\hat{f}) \times \mathbb{R}^{2d} \times G) = \{0\} \times W_0$ , where*

$$(5.2) \quad W_0 := \{(z, g) \in (\Omega_0 \cap \Sigma_E) \times G : M(g)z = z\}.$$

$W_0$  is a submanifold of  $\mathbb{R}^{2d} \times G$ ,  $\dim W_0 = 2d - 2k_0 + p - 1$  and if  $(z, g) \in W_0$ , then

$$(5.3) \quad T_{(z,g)}W_0 = \{(\alpha, Ag) \in \mathbb{R}^{2d} \times \mathfrak{G}g : (M(g) - I)\alpha + Az = 0, \alpha \perp \nabla H(z) \text{ and } \alpha \in T_z\Omega_0\}.$$

Proof:  $\tilde{\Sigma}_E$  is a compact and non critical energy level. So it is a well known fact that it has a minimal strictly positive period. If  $\text{Supp}(\hat{f}) \cap \mathcal{L}_{red}(E) = \{0\}$ , by the non stationary phase theorem, our critical set becomes  $\mathbb{C}_E \cap (\text{Supp}(\hat{f}) \times \mathbb{R}^{2d} \times G) = \{0\} \times W_0$ . We note that  $W_0 = \Gamma_0 \cap [(\Sigma_E \cap \Omega_0) \times G]$ , where  $\Gamma_0$  is given by (3.10). By Lemma 3.2 (but here with  $U = H^{-1}([E - \delta E, E + \delta E])$ ), we know that  $\Gamma_0$  is a submanifold of  $\mathbb{R}^{2d} \times G$ . We have to show that  $\Gamma_0$  and  $(\Sigma_E \cap \Omega_0) \times G$  are transverse submanifolds of  $(\Omega_0 \cap U) \times G$ : in the case of the contrary, let  $(z, g) \in W_0$  such that  $T_z\Gamma_0 \subset (T_z\Sigma_E \cap T_z\Omega_0) \times \mathfrak{G}g$ . If  $\alpha \in T_z\Omega_0$ , we have seen in section 3 (3.13) that  $(M(g) - I)\alpha \in \mathfrak{G}z$ . Hence, there exists  $A \in \mathfrak{G}$  such that  $(\alpha, Ag) \in T_{(z,g)}\Gamma_0$ . Thus, we have  $T_z\Omega_0 \subset T_z\Sigma_E$ , which is in contradiction with lemma 4.5.  $\square$

**Theorem 5.2.** *Let  $G$  be a compact Lie group of  $Gl(d, \mathbb{R})$  and  $H : \mathbb{R}^{2d} \rightarrow \mathbb{R}$  be a smooth  $G$ -invariant Hamiltonian satisfying (3.2). Let  $E \in \mathbb{R}$  be such that  $H^{-1}([E - \delta E, E + \delta E])$  is compact for some  $\delta E > 0$ , and that  $\tilde{\Sigma}_E = \{\tilde{H} = E\}$  has no critical points. We suppose that hypotheses of reduction are satisfied on  $\Omega_0 \cap U$ , where  $U := H^{-1}([E - \delta E, E + \delta E])$ . Let  $f$  and  $\psi$  be real functions in  $\mathcal{S}(\mathbb{R})$  with  $\text{Supp}(\psi) \subset [E - \delta E, E + \delta E]$  and such that  $\hat{f}$  is compactly supported.*

*If  $\text{Supp}\hat{f} \cap \mathcal{L}_{red}(E) = \emptyset$ , then  $\mathfrak{G}_\chi(h) = O(h^{+\infty})$  as  $h \rightarrow 0^+$ .*

*If  $\text{Supp}\hat{f} \cap \mathcal{L}_{red}(E) = \{0\}$ , then  $\mathfrak{G}_\chi(h)$  has a complete expansion in powers of  $h$ , whose coefficients are distributions in  $\hat{f}$  with support in  $\{0\}$ , and:*

$$(5.4) \quad \mathfrak{G}_\chi(h) = (2\pi h)^{k_0 - d + 1} \hat{f}(0) \psi(E) \frac{d_\chi}{2\pi} \int_{\tilde{\Sigma}_E} dL_{\tilde{H}, E} \left[ \rho_{\chi|_{H_0}} : \mathbb{1} \right] + O(h^{k_0 - d + 2}).$$

where  $dL_{\tilde{H},E} = \frac{d\tilde{\Sigma}_E}{d\tilde{H}}$  is the Liouville measure associated to  $\tilde{H}$  on  $\tilde{\Sigma}_E$ ,  $k_0$  is the common dimension of  $G$ -orbits of  $\Omega_0 \cap U$ , and  $[\rho_{\chi|_{H_0}} : \mathbb{1}]$  is as in Theorem 3.1.

Namely, if  $f$  is smooth with compact support in  $(\Omega_0 \cap U)/G \subset \Omega_{red}$ , then the Liouville measure satisfies:

$$(5.5) \quad \int_{\tilde{\Sigma}_E} f(x) dL_{\tilde{H},E} = \int_{\Sigma_E \cap \Omega_0} \frac{f(\pi(z))}{\text{Vol}(G(z))} \frac{d\sigma_{\Sigma_E \cap \Omega_0}(z)}{\|\Pi_{T_z \Omega_0}(\nabla H(z))\|_{\mathbb{R}^{2d}}}.$$

where  $d\sigma_{\Sigma_E \cap \Omega_0}$  is the Lebesgue measure on  $\Sigma_E \cap \Omega_0$ , and  $\Pi_{T_z \Omega_0}$  denotes the orthogonal projector on  $T_z \Omega_0$  in  $\mathbb{R}^{2d}$ .

*Remark:* as a consequence of this theorem, we note that if  $\hat{f}$  is supported close enough to zero, then  $\text{Supp} \hat{f} \cap \mathcal{L}_{red}(E) = \{0\}$  (because of the existence of a minimal period on  $\tilde{\Sigma}_E$ ). Using a classical Tauberian argument (see [25]), this leads to Theorem 1.1.

*Proof:* the case where  $\text{Supp} \hat{f} \cap \mathcal{L}_{red}(E) = \emptyset$  is straightforward by using a non-stationary phase theorem, since the intersection of the critical set with the support of the amplitude is empty. Suppose that  $\text{Supp} \hat{f} \cap \mathcal{L}_{red}(E) = \{0\}$ . Let  $(0, z, g) \in \mathcal{C}_E \cap (\text{Supp}(\hat{f}) \times \mathbb{R}^{2d} \times G)$ . Let  $(\tau, \alpha, Ag) \in \ker_{\mathbb{R}} \text{Hess} \varphi_E(0, z, g)$ , i.e., by Proposition 4.3,  $\alpha \perp \nabla H(z)$ ,  $\alpha \in (J\mathcal{G}z)^\perp = T_z \Omega_0$  and

$$(5.6) \quad \tau J\nabla H(z) + (M(g) - I)\alpha + Az = 0,$$

Since  $M(g)z = z$ , we have  $(M(g) - I)\alpha \in \mathcal{G}z$ . Thus, if  $\tau \neq 0$ , then  $J\nabla H(z) \in \mathcal{G}z$ , which is in contradiction with Lemma 4.5. Thus  $\tau = 0$ . Therefore, in view of Proposition 5.1, we have  $\ker_{\mathbb{R}} \text{Hess} \varphi_E(0, z, g) = T_{(0,z,g)} \mathcal{C}_E$ , and we can apply the stationary phase theorem.

Now, we wish to compute the leading term of the expansion of  $\mathcal{G}_\chi(h)$ . We are going to use weak asymptotics given in Theorem 3.1. Note that we could have made the calculus using the result of the stationary phase theorem, but the computation is more technical (see [8]).

Since  $\mathcal{G}_{\chi,h}(E) := \text{Tr} \left( \psi(\hat{H}_\chi) f \left( \frac{E - \hat{H}_\chi}{h} \right) \right)$  is continuous with respect to variable  $E$ , and since hypotheses of Theorem 5.2 are available for energies in a neighbourhood of  $E$ , we have a continuous function  $\lambda \mapsto a(\lambda)$  in the neighbourhood of  $E$  such that:

$$\mathcal{G}_{\chi,h}(\lambda) = a(\lambda) h^{k_0 - d + 1} + O(h^{k_0 - d + 2}),$$

with uniform rest with respect to  $\lambda$  near  $E$ . Let  $\varphi$  be a smooth compactly supported function in this neighbourhood. We have:

$$(5.7) \quad \int_{\mathbb{R}} \varphi(\lambda) \mathcal{G}_{\chi,h}(\lambda) d\lambda = h^{k_0 - d + 1} \int_{\mathbb{R}} \varphi(\lambda) a(\lambda) d\lambda + O(h^{k_0 - d + 2}).$$

Moreover, if  $x \in \mathbb{R}$ , then  $\int_{\mathbb{R}} \varphi(\lambda) f \left( \frac{\lambda - x}{h} \right) d\lambda = h \int_{\mathbb{R}} \varphi(x + th) f(t) dt$ . Thus:

$$(5.8) \quad \int_{\mathbb{R}} \varphi(\lambda) \mathcal{G}_{\chi,h}(\lambda) d\lambda = h \int_{\mathbb{R}} \text{Tr} \left( \psi(\hat{H}_\chi) \varphi(\hat{H}_\chi + th) \right) f(t) dt.$$

If  $x$  and  $\lambda$  are real, we have  $|\varphi(x + th) - \varphi(x)| \leq \|\varphi'\|_\infty |th|$ . Therefore:

$$\begin{aligned} \left| \text{Tr} \left( \psi(\hat{H}_\chi) [\varphi(\hat{H}_\chi + th) - \varphi(\hat{H}_\chi)] \right) \right| &\leq \left\| \psi(\hat{H}_\chi) \right\|_{\text{Tr}} \left\| \varphi(\hat{H}_\chi + th) - \varphi(\hat{H}_\chi) \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \\ &\leq \left\| \psi(\hat{H}_\chi) \right\|_{\text{Tr}} \|\varphi'\|_\infty |th| \end{aligned}$$

$\psi$  being non negative, we have  $\left\| \psi(\hat{H}_\chi) \right\|_{\text{Tr}} = \text{Tr}(\psi(\hat{H}_\chi)) = O(h^{k_0 - d})$  in view of the weak asymptotics. Thus

$$\int_{\mathbb{R}} \text{Tr} \left( \psi(\hat{H}_\chi) [\varphi(\hat{H}_\chi + th) - \varphi(\hat{H}_\chi)] \right) dt = O(h^{k_0 - d + 1}).$$

Using (5.8), we get:

$$\int_{\mathbb{R}} \varphi(\lambda) \mathcal{G}_{\chi,h}(\lambda) d\lambda = h \hat{f}(0) \text{Tr} \left( \psi(\hat{H}_\chi) \varphi(\hat{H}_\chi) \right) + O(h^{k_0 - d + 2}).$$

Thus, by Theorem 3.1, we have:

$$\int_{\mathbb{R}} \varphi(\lambda) \mathcal{G}_{\chi, h}(\lambda) d\lambda = h^{k_0-d+1} (2\pi)^{k_0-d} d_{\chi} \hat{f}(0) \int_{\Omega_{red}} (\psi\varphi)(\tilde{H}(x)) d\sigma_{red}(x) \left[ \rho_{\chi|_{H_0}} : \mathbb{1} \right] + O(h^{k_0-d+2})$$

which implies that (in view of (5.7)):

$$\int_{\mathbb{R}} \varphi(\lambda) a(\lambda) d\lambda = (2\pi)^{k_0-d} d_{\chi} \hat{f}(0) \int_{\Omega_{red}} (\psi\varphi)(\tilde{H}(x)) d\sigma_{red}(x) \left[ \rho_{\chi|_{H_0}} : \mathbb{1} \right].$$

Thus, by (3.37):

$$(5.9) \quad \int_{\mathbb{R}} \varphi(\lambda) a(\lambda) d\lambda = (2\pi)^{k_0-d} d_{\chi} \hat{f}(0) \int_{\Omega_0} \frac{(\psi\varphi)(H(z))}{Vol(G(z))} d\sigma_{\Omega_0}(x) \left[ \rho_{\chi|_{H_0}} : \mathbb{1} \right].$$

Using Lemma 3.4, one can show that, if  $f$  is smooth, compactly supported in  $\Omega_0 \cap U$ , then:

$$\int_{\Omega_0} f(z) d\sigma_{\Omega_0}(z) = \int_{\mathbb{R}} \int_{\Omega_0 \cap \Sigma_{\lambda}} f(z) \frac{d\sigma_{\Omega_0 \cap \Sigma_{\lambda}}(z)}{\|\Pi_{T_z \Omega_0}(\nabla H(z))\|_{\mathbb{R}^{2d}}} d\lambda,$$

where  $\Pi_{T_z \Omega_0}$  denotes the orthogonal projector on  $T_z \Omega_0$ . Finally, one can apply this last formula to (5.9) to get:

$$\int_{\mathbb{R}} \varphi(\lambda) a(\lambda) d\lambda = (2\pi)^{k_0-d} d_{\chi} \hat{f}(0) \left[ \rho_{\chi|_{H_0}} : \mathbb{1} \right] \int_{\mathbb{R}} \varphi(\lambda) \psi(\lambda) \int_{\Omega_0} \frac{1}{Vol(G(z))} \frac{d\sigma_{\Omega_0 \cap \Sigma_{\lambda}}(z)}{\|\Pi_{T_z \Omega_0}(\nabla H(z))\|_{\mathbb{R}^{2d}}} d\lambda,$$

and we get the expression of  $a(E)$ .  $\square$

## 6. THE OSCILLATORY PART

This section is dedicated to the proof of Theorem 1.2. We suppose that  $0 \notin \text{Supp}(\hat{f})$  and that periodic orbits of  $\tilde{\Sigma}_E$  having a period in  $\text{Supp}(\hat{f})$  are *non degenerate*, i.e., if  $x \in \tilde{\Sigma}_E$  and  $\tilde{T} \in \text{Supp}(\hat{f})$  satisfy  $\tilde{\Phi}_{\tilde{T}}(x) = x$ , then 1 is not an eigenvalue of the differential of the Poincaré map at  $x$  at time  $\tilde{T}$  restricted to the energy level  $\tilde{\Sigma}_E$ . If  $x \in \tilde{\Sigma}_E$ , we denote by  $\tilde{F}_x(\tilde{T})$  the differential of the reduced flow  $x \mapsto \tilde{\Phi}_{\tilde{T}}(x)$  with respect to variable  $x \in \Omega_{red}$ . The non degenerate hypothesis is equivalent to say that, if  $\tilde{\Phi}_{\tilde{T}}(x) = x$ , with  $\tilde{T} \in \text{Supp}(\hat{f})$  then:

$$(6.1) \quad \dim \ker[(\tilde{F}_x(\tilde{T}) - Id_{T_x \Omega_{red}})^2] \leq 2.$$

Note that this is always the case when  $\dim(\Omega_{red}) = 2$ , i.e.  $d = k_0 + 1$ . In particular this happens for the spherical symmetry  $G = SO(d)$  (see examples in section 2).

**6.1. The stationary phase problem.** Since  $\tilde{\Sigma}_E$  is compact without critical points, then by the cylinder theorem (see [1] 8.2.2), there is a finite number of periodic orbits in  $\tilde{\Sigma}_E$  with period in  $\text{Supp}(\hat{f})$  and  $\mathcal{L}_{red} \cap \text{Supp}(\hat{f})$  is a finite set. If  $t_0 \in \mathcal{L}_{red} \cap \text{Supp}(\hat{f})$ , we set:

$$(6.2) \quad W_{t_0} := \{(z, g) \in (\Omega_0 \cap \Sigma_E) \times G : M(g)\Phi_{t_0}(z) = z\}.$$

By Proposition 4.1, we get that:

$$(6.3) \quad \mathcal{C}_E \cap (\text{Supp}(\hat{f}) \times \mathbb{R}^{2d} \times G) = \bigcup_{t_0 \in \mathcal{L}_{red} \cap \text{supp}(\hat{f})} \{t_0\} \times W_{t_0}$$

**Proposition 6.1.** *We suppose that periodic orbits of  $\tilde{\Sigma}_E$  are non degenerate on  $\text{Supp}(\hat{f})$  in the sense given above. Let  $t_0 \in \mathcal{L}_{red} \cap \text{Supp}(\hat{f})$ . Then  $W_{t_0}$  is a submanifold of  $\mathbb{R}^{2d} \times G$ ,  $\dim W_{t_0} = p + 1$ , and if  $(z, g) \in W_{t_0}$ , then:*

$$(6.4) \quad T_{(z, g)} W_{t_0} = \{(\alpha, Ag) \in T_z \Omega_0 \times \mathcal{G}g : \alpha \in \mathbb{R}J\nabla H(z) + \mathcal{G}z \text{ and } (M(g)F_z(t_0) - I)\alpha + Az = 0\}.$$

*Proof:* let  $\pi_1 : \Omega_0 \cap \Sigma_E \rightarrow \tilde{\Sigma}_E$  be the restriction of  $\pi$  to  $\Sigma_E \cap \Omega_0$ . We claim that  $\pi_1$  is a submersion. Indeed, if  $z \in \Omega_0 \cap \Sigma_E$ , and if  $u \in T_{\pi(z)}\tilde{\Sigma}_E$ , then,  $\pi$  being a submersion, there exists  $\alpha \in T_z\Omega_0$  such that  $u = d_z\pi(\alpha)$ . By the Theorem 2.2, we have:

$$d_{\pi(z)}\tilde{H}(d_z\pi(\alpha)) = \omega_{red}(\pi(z))(d_z\pi(\alpha), X_{\tilde{H}}(\pi(z))) = \langle J\alpha, J\nabla H(z) \rangle,$$

since  $d_z\pi(J\nabla H(z)) = X_{\tilde{H}}(\pi(z))$ . Remembering that  $T_{\pi(z)}\tilde{\Sigma}_E = \ker d_{\pi(z)}\tilde{H}$ , we get that  $\alpha \perp \nabla H(z)$ , and  $\alpha \in (T_z\Omega_0) \cap (T_z\Sigma_E) = T_z(\Omega_0 \cap \Sigma_E)$ .

If  $z \in \Omega_0 \cap U$ , if  $g \in G$ , differentiating with respect to  $z \in \Omega_0$  the identity  $\pi(M(g)\Phi_{t_0}(z)) = \tilde{\Phi}_{t_0}(\pi(z))$ , we get that, if  $z = M(g)\Phi_{t_0}(z)$ , then  $M(g)F_z(t_0)T_z\Omega_0 \subset T_z\Omega_0$ , and we have on  $T_z\Omega_0$ :

$$(6.5) \quad d_z\pi \circ M(g)F_z(t_0) = \tilde{F}_{\pi(z)}(t_0) \circ d_z\pi.$$

Let  $R_{t_0} := \{(z, M(g)\Phi_{t_0}(z)) : z \in \Sigma_E \cap \Omega_0, g \in G\}$ .

**Lemma 6.2.**  *$R_{t_0}$  is a submanifold of  $(\Sigma_E \cap \Omega_0)^2$ ,  $\dim R_{t_0} = 2d - 1$ . If  $z = M(g)\Phi_{t_0}(z)$  then*

$$T_{(z,z)}R_{t_0} = \{(\alpha, M(g)F_z(t_0)\alpha + Az) : \alpha \in T_z(\Omega_0 \cap \Sigma_E) \text{ and } A \in \mathcal{G}\}.$$

*Proof:* let  $\mathcal{A}_{t_0} := \{(x, \tilde{\Phi}_{t_0}(x)) : x \in \tilde{\Sigma}_E\}$ .  $\mathcal{A}_{t_0}$  is a submanifold of  $[(\Omega_0 \cap U)/G]^2$  and  $\dim \mathcal{A}_{t_0} = \dim \tilde{\Sigma}_E = 2d - 2k_0 - 1$ . Moreover  $R_{t_0} = (\pi_1 \times \pi_1)^{-1}(\mathcal{A}_{t_0})$ . Thus,  $\pi_1 \times \pi_1$  being submersion,  $R_{t_0}$  is a submanifold  $(\Omega_0 \cap U)^2$  and  $\dim R_{t_0} = 2d - 1$ . Besides,  $T_{(x, \tilde{\Phi}_{t_0}(x))}\mathcal{A}_{t_0} = \{(u, \tilde{F}_x(t_0)u) : u \in T_x\tilde{\Sigma}_E\}$ . Thus, if  $z = M(g)\Phi_{t_0}(z)$ , then:

$$T_{(z,z)}R_{t_0} = \{(\alpha, \beta) \in T_{(z,z)}(\Omega_0 \cap \Sigma_E)^2 : d_z\pi(\beta) = \tilde{F}_{\pi(z)}(t_0)d_z\pi(\alpha)\}.$$

In view of (6.5), the proof is clear if we remember that  $\ker d_z\pi = \mathcal{G}z$ .  $\square$

We denote by  $\mathcal{P}_{red}(E, t_0)$  the set of periodic orbits of  $\tilde{\Sigma}_E$  with period  $t_0$ . We have:

$$(6.6) \quad W_{t_0} = \bigsqcup_{\bar{\gamma} \in \mathcal{P}_{red}(E, t_0)} \Lambda_{\bar{\gamma}, t_0}.$$

where

$$(6.7) \quad \Lambda_{\bar{\gamma}, t_0} := \{(z, g) \in (\Omega_0 \cap \Sigma_E) \times G : z = M(g)\Phi_{t_0}(z) \text{ and } \pi(z) \in \bar{\gamma}\}.$$

Let  $\bar{\gamma} \in \mathcal{P}_{red}(E, t_0)$  and  $\varphi_0 : (\Omega_0 \cap \Sigma_E) \times G \rightarrow R_{t_0}$  defined by  $\varphi_0(z, g) := (z, M(g)\Phi_{t_0}(z))$ . We have  $\Lambda_{\bar{\gamma}} = \varphi_0^{-1}(P_{\bar{\gamma}})$ , where  $P_{\bar{\gamma}} := \{(z, z) \in (\Sigma_E \cap \Omega_0)^2 : \pi(z) \in \bar{\gamma}\} \subset R_{t_0}$ . We have:  $P_{\bar{\gamma}} = \text{diag}(\pi_1^{-1}(\bar{\gamma}))$ , thus,  $\pi_1$  being a submersion,  $P_{\bar{\gamma}}$  is a submanifold of  $R_{t_0}$ ,  $\dim(P_{\bar{\gamma}}) = k_0 + 1$ , and  $T_{(z,z)}P_{\bar{\gamma}} = (\mathcal{G}z + \mathbb{R}J\nabla H(z))^2$ , since we have  $d_z\pi(J\nabla H(z)) = X_{\tilde{H}}(\pi(z))$  and  $\ker d_z\pi = \mathcal{G}z$ . The proof of the proposition is clear if we show that  $\varphi_0$  is a submersion at all points of  $W_{t_0}$ , which is given by Lemma 6.2.  $\square$

Now we have to show that the transversal Hessian of  $\varphi_E$  is non degenerate on points of  $\{t_0\} \times W_{t_0}$ . Let  $(z, g) \in W_{t_0}$ . We have to show that  $\ker_{\mathbb{R}} \text{Hess } \varphi_E(t_0, z, g) = \{0\} \times T_{(z,g)}W_{t_0}$ . Let  $\tau \in \mathbb{R}$ ,  $\alpha \in T_z(\Omega_0 \cap \Sigma_E)$ , and  $A \in \mathcal{G}$  such that:

$$(6.8) \quad \tau J\nabla H(z) + (M(g)F_z(t_0) - Id)\alpha + Az = 0.$$

Applying  $d_z\pi$ , we get (using (6.5)):

$$(6.9) \quad \tau X_{\tilde{H}}(\pi(z)) + (\tilde{F}_{\pi(z)}(t_0) - Id)d_z\pi(\alpha) = 0.$$

Thus,  $d_z\pi(\alpha) \in \ker(\tilde{F}_{\pi(z)}(t_0) - Id_{\Omega_{red}})^2$ . Using by (6.1), we have a basis  $(u_1, u_2)$  of  $\ker(\tilde{F}_{\pi(z)}(t_0) - Id_{\Omega_{red}})^2$  with:  $u_1 = X_{\tilde{H}}(\pi(z))$  et  $w_{red}(\pi(z))(u_1, u_2) \neq 0$  (as in the case without symmetry). Thus, there exists  $\lambda_1, \lambda_2$  in  $\mathbb{R}$  such that  $d_z\pi(\alpha) = \lambda_1 u_1 + \lambda_2 u_2$ . Therefore:

$$w_{red}(\pi(z))(d_z\pi(\alpha), u_1) = d_{\pi(z)}\tilde{H}(d_z\pi(\alpha)) = \langle \nabla H(z), \alpha \rangle = 0.$$

Thus  $\lambda_2 = 0$  and  $d_z\pi(\alpha) = \lambda_1 u_1$ . Using (6.9), we obtain  $\tau = 0$  since there is no critical points of  $\tilde{H}$  on  $\tilde{\Sigma}_E$ . Besides, since  $d_z\pi(J\nabla H(z)) = X_{\tilde{H}}(\pi(z))$ , we have  $\alpha \in \mathbb{R}J\nabla H(z) + \mathcal{G}z$ , and, in view

of (6.8) and Proposition 6.1, we get that  $(0, \alpha, Ag) \in T_{(t_0, z, g)}\mathcal{C}_E$ .

Thus, we have shown that, when periodic orbits of the reduced space are non degenerate, then we can apply the stationary phase theorem, and get an asymptotic expansion of  $\mathcal{G}_\chi(h)$ .

**6.2. Computation of first terms.** We apply the stationary phase formula to (4.4). The set  $\mathcal{C}_E \cap (\text{Supp}(\hat{f}) \times \mathbb{R}^{2d} \times G)$  splits into disjoint sets  $\{t_0\} \times \Lambda_{\bar{\gamma}, t_0}$  given by (6.3) and (6.6), where  $\Lambda_{\bar{\gamma}, t_0}$  is given by (6.7).  $\Lambda_{\bar{\gamma}, t_0}$  may not be a connected set, but one can show that the quantity  $S(t_0)(z) := \int_0^{t_0} pdq$  doesn't depend on  $z \in \pi^{-1}(\bar{\gamma})$  (see [2]). We denote it by  $S_{\bar{\gamma}}(t_0)$ . Then, using that  $\dim \Lambda_{\bar{\gamma}, t_0} = \dim W_{t_0} = p + 1$ , we get that:

$$\mathcal{G}_\chi(h) = \psi(E)d_\chi \sum_{t_0 \in \mathcal{L}_{red}(E) \cap \text{Supp} \hat{f}} \hat{f}(t_0) \sum_{\bar{\gamma} \in \mathcal{P}_{red}(E, t_0)} e^{\frac{i}{h} S_{\bar{\gamma}}(t_0)} \frac{1}{2\pi} \int_{\Lambda_{\bar{\gamma}, t_0}} \overline{\chi(g)} d(t_0, z, g) d\sigma_{\Lambda_{\bar{\gamma}}}(z, g) + O(h),$$

where  $d(t_0, z, g)$  is given by:

$$d(t, z, g) := \det_+^{-\frac{1}{2}} \left( \frac{\varphi_E''(t, z, g)|_{\mathcal{N}(t, z, g)^{c_E}}}{i} \right) \det_+^{-\frac{1}{2}} \left( \frac{A + iB - i(C + iD)}{2} \right).$$

In this case, the computation of  $\det_+^{-\frac{1}{2}} \left( \frac{\varphi_E''(t, z, g)|_{\mathcal{N}(t, z, g)^{c_E}}}{i} \right)$  seems to be a non trivial calculus.

We didn't succeed in interpreting it geometrically as in the case of the Weyl term. However, one should be able to make appear the primitive period of  $\bar{\gamma}$  and the differential of its Poincaré map, as we did in the case of a finite group in [7].

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