TECHNISCHE UNIVERSITÄT CHEMNITZ

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The Bruhat order on involutions and pattern avoidance

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Erklärung

Ich erkläre an Eides Statt, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Chemnitz,

Abstract

The symmetric group, the group of signed permutations and the group of signed permutations with even number of negative entries are Coxeter groups and can be seen as partially ordered sets with respect to the Bruhat order. A result of Tenner (2006) shows that the elements of those posets which have a boolean lower order ideal are exactly those avoiding certain sets of patterns.

The theory of twisted involutions was developed by Richardson and Springer (1990) and Hultman (2004). We show that a twisted involution having a boolean lower order ideal can be characterized in terms of reduced twisted expressions. We also consider the special case of involutions of the groups mentioned above and show that those are again characterized by the avoidance of certain sets of patterns.

We enumerate the boolean involutions of the said groups recursively. For the involutions of the symmetric group we can also give recursions with respect to some well-known statistics and a bijection with a certain class of Motzkin paths.

Contents

1 Introduction

The study of Coxeter groups and Bruhat order has produced a wide variety of results concerning combinatorial, geometric and algebraic questions. For instance, the following questions have been studied more or less recently. First, properties of lower intervals in Coxeter groups of type A, B and D have been characterized by pattern avoidance. Second, the theory of twisted involutions including the special case of involutions has been developed. This work will try to connect both ideas and study connections between booleanness of lower ideals in the partially ordered set of involutions and pattern avoidance.

In chapter 2 we present the notation used throughout this work with particular emphasis on partially ordered sets and permutations. Chapter 3 introduces the reader to the theory of Coxeter groups. We provide basic definitions and properties of Coxeter groups and the Bruhat order and acquaint the reader with the theory of twisted involutions. We try to include all material necessary to understand the work without any prerequisites about Coxeter groups. However, the interested reader may be willing to find more detailed treatises in [4] and [10]. Finally, chapter 4 introduces the notion of permutation patterns and gives a short overview over known connections between Bruhat order and pattern avoidance. All the material covered in those first chapters is known and does not contain any contribution from the author.

The author's results are presented in chapter 5 and 6. In chapter 5 we study connections between boolean involutions and pattern avoidance. Section 5.1 gives a simple result characterizing boolean twisted involutions in terms of reduced twisted expressions. The result has been stated for a special case without proof already in [22]. In section 5.2 we apply the said result in order to characterize the boolean involutions of the symmetric group in terms of pattern avoidance. A connection between twisted reduced expressions of involutions in the symmetric group and the avoidance of certain patterns is proved in section 5.3. This is very similar to a result in [23] and our proof just adjusts the ideas used there to our situation. In section 5.4 the results of section 5.2 are used to characterize boolean involutions in other Coxeter groups in terms of pattern avoidance. This includes the group of signed permutations, the group of even signed permutations and the affine permutation group.

Chapter 6 is devoted to the enumeration of boolean involutions. We count the number of boolean involutions of the symmetric group with respect to some statistics in 6.1 using the combinatorial description presented earlier. A more general approach is applied in section 6.2 which enables us to count the boolean involutions of other Coxeter groups, as well.

Finally, we summarize the work and ask some more questions in chapter 7.

2 Notation

The set of non-negative integers is $\mathbb{N} = \{0, 1, 2, \ldots\}$. We write

$$[n] = \{1, \ldots, n\}$$

and

$$[\pm n] = \{-n, \ldots, -1, 1, \ldots, n\}.$$

A word over the letters $\Gamma = \{s_1, \ldots, s_n\}$ is an element of the free monoid generated by Γ. By the word $s_1 \ldots \hat{s}_k \ldots s_n$ we mean the word having letters $s_1, \ldots, s_n$ with the letter $\hat{s}_k$ left out.

The unit element of a group will always be denoted by e.

The used notation for partially ordered sets and permutations follows [21] and [4].

Partially ordered sets

A *partially ordered set* or *poset* $(P, \leq)$ is a set P together with an order relation $\leq$ on P which satisfies the following axioms:

(i) $p \leq p$ for all $p \in P$ (reflexivity).

(ii) $p \leq q$ and $q \leq p \Rightarrow p = q$ for all $p, q \in P$ (anti-symmetry).

(iii) $p \leq q$, $q \leq r \Rightarrow p \leq r$ for all $p, q, r \in P$ (transitivity).

An *interval* $[p, q]$ in a poset P is of the form $[p, q] = \{r \in P : p \leq r \leq q\}$. Note that $[p, q]$ is non-empty if and only if $p \leq q$. We say that $q \in P$ *covers* $p \in P$ if $p < q$ and there is no $r \in P$ such that $p < r < q$. The covering relation is denoted by $p \lhd q$. A *chain* of length k is a totally ordered subset $x_0 < x_1 < \ldots < x_k$ of P. A chain is called *saturated* if $x_i \lhd x_{i+1}$ for all i and it is called maximal if it is maximal with respect to inclusion.

The *Hasse diagram* of a poset P is a graph with vertex set P and edges between p and q where q is drawn above p for all covering relations $p \lhd q$.

A poset P is called *bounded*, if P has a unique minimal and a unique maximal element, which in that case are denoted by $\hat{0}$ and $\hat{1}$ in general. A poset is *graded* if every maximal chain is of the same length. In that case, there exists a *rank function* $r : P \to \mathbb{N}$ with $r(p) = 0$ for all minimal elements $p \in P$ and $r(q) = r(p) + 1$ for all covering relations $p \lhd q$.

An element $z \in P$ is an *upper bound* (respectively lower bound) for $x, y \in P$ if $z \geq x$ and $z \geq y$ (respectively $z \leq x$ and $z \leq y$). A *lattice* is a poset P in which any two elements $x, y \in P$ have a unique minimal upper bound, denoted by $x \vee y$, and a unique maximal lower bound, denoted by $x \wedge y$.

The *Möbius function* of a locally finite poset P maps every ordered pair $p \leq q$ to an integer $\mu(p, q)$ according to the recursion

$$\mu(p,q) = \begin{cases} 1 & \text{if } p = q, \\ -\sum_{p \leq r < q} \mu(p,r) & \text{if } p < q. \end{cases}$$

The *boolean lattice* B_n is the poset $(2^{[n]}, \subseteq)$ containing all subsets of $[n]$ partially ordered with respect to inclusion.

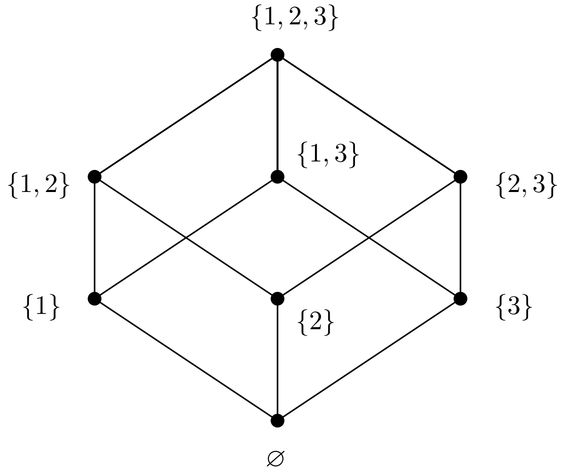


Figure 2.1: Hasse diagram of B_3

The *lower order ideal* of $p \in P$ is $B(p) := \{q \in P : q \leq p\}$. We call $p \in P$ *boolean* if $B(p)$ is isomorphic to B_n for some $n \in \mathbb{N}$, otherwise we call p non-boolean. A boolean p is *maximal boolean* if all $q > p$ are non-boolean.

Assume that $B(p)$ is graded with $r(p) = n$. Then, p is called *rank-symmetric* if $|\{q \in B(p) : r(q) = k\}| = |\{q \in B(p) : r(q) = n - k\}|$ for all $k = 1, \ldots, n$.

A *special matching* on the poset P is an involutive bijection $m : P \to P$ such that $m(p) \lhd p$ or $p \lhd m(p)$ for all $p \in P$ and if $p \lhd q$ and $m(p) \neq q$ then $m(p) < m(q)$ for all $p, q \in P$.

Permutations

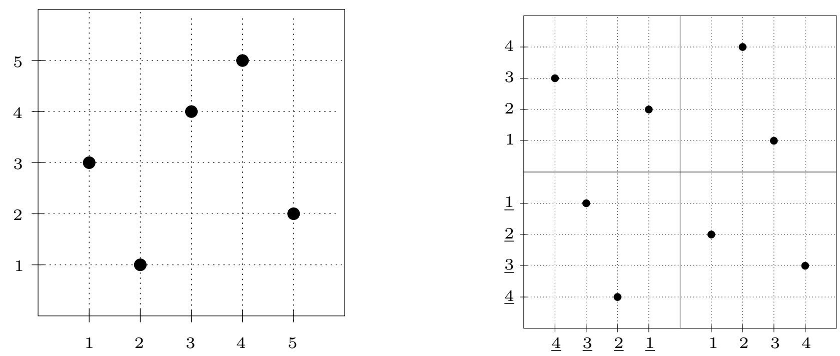
A *permutation* of a set M is a bijection $\pi : M \to M$. The *symmetric group* S_n is the group consisting of all permutations of $[n]$ with composition as the multiplication. The identity element e of S_n is the identity map $k \mapsto k$.

We will use two different notations for a permutation $\pi \in S_n$. In the *one-line notation* we denote $\pi \in S_n$ by $\pi(1) \ldots \pi(n)$. A *cycle* of π is of the form $(i, \pi(i), \pi^2(i), \ldots, \pi^{p-1}(i))$ where $p \in \mathbb{N}$ is minimal such that $\pi^p(i) = i$. If $(i, j, k, \ldots, l)$ is a cycle of π then $\pi(i) = j$, $\pi(j) = k$ and so on until finally $\pi(l) = i$. A cycle with k elements is called *k-cycle*.

In the *cycle notation*, π is denoted by writing up all cycles of π after each other. It is clear that, in contrast to the one-line notation, the cycle notation is not unique. A cycle containing only one element, called one-cycle, is a fixed point of π. We will usually omit to write the one-cycles.

Thus, (i, j) with $i, j \in [n]$ and $i \neq j$ denotes the permutation that maps i to j and j to i and leaves all other elements as fixed points. Such a permutation is called a *transposition*. An *adjacent transposition* is given by $(i, i+1)$ for some $i \in [n-1]$.

A permutation $\pi \in S_n$ can be illustrated by drawing points with coordinates $(i, \pi(i))$ for all $i \in [n]$ in the plane. We call that the *diagram representation* of π.



(a) $31452 \in S_5$ (b) $\underline{2}41\underline{3} \in S_4^B$

Figure 2.2: Two diagram representations

An *inversion* of a permutation π is a pair (i, j) with $i < j$ and $\pi(i) > \pi(j)$. The number of inversions of a permutation π is denoted by $\mathrm{inv}(\pi)$.

A *signed permutation* π is a permutation of the set $[\pm n]$ such that $\pi(-i) = -\pi(i)$ for all $i \in [n]$. The group of all signed permutations (with composition as multiplication) is denoted by S_n^B. The symmetric group S_n can be identified as a subgroup of S_n^B in a natural way and S_n^B can be identified as a subgroup of $S([\pm n])$ in a natural way.

Negative values in signed permutations will be denoted by underlining the absolute value, i.e. $-i$ is denoted by $\underline{i}$. The one-line notation $\pi(-n) \ldots \pi(-1)\pi(1) \ldots \pi(n)$ of a signed permutation is also called complete notation. A signed permutation is already uniquely defined by the values $\pi(1) \ldots \pi(n)$. We can thus use the *window notation* $[\pi(1) \ldots \pi(n)]$ in order to denote a signed permutation π. When referring to an entry of a signed permutation we mean an entry in the window notation.

A signed permutation $\pi \in S_n^B \subseteq S([\pm n])$ can also be represented by a diagram, drawing points with coordinates $(i, \pi(i))$ for all $i \in [\pm n]$. By definition the diagram of a signed permutation will be symmetric with respect to $(0, 0)$.

The subgroup of S_n^B consisting of all signed permutations having an even number of negative entries is denoted by S_n^D.

3 Coxeter groups and the Bruhat order

Coxeter groups are motivated by and studied in different areas of mathematics such as geometry, algebra and combinatorics. They generalize the theory of finite reflection groups (see [10]) that provide the examples which the later part of this work will focus on.

After giving the basic definitions as well as some examples of Coxeter groups that appear interesting or will be needed later we will introduce the Bruhat order and some important properties.

3.1 Fundamentals

This section gives a short introduction to Coxeter groups. All facts are taken from [4] and [10] which we also point out as references for further information.

3.1.1 Coxeter groups

Definition 3.1. *A Coxeter system is a pair (W, S) of a group W with set of generators $S \subset W$ satisfying only relations of the form $(ss')^{m(s,s')} = e$ where $m(s, s) = 1$ for all $s \in S$ and $m(s, s') = m(s', s) \geq 2$ for $s \neq s'$ in S. By convention we let $m(s, s') = \infty$ if the pair s and s' does not give rise to any relation. The group W is called a Coxeter group.*

Formally W is isomorphic to the quotient F/N where F is the free group generated by S and F is the normal subgroup generated by all elements $(ss')^{m(s,s')}$ for all $s, s' \in S$ with $m(s, s') < \infty$.

In general W can have different generating sets S, which do not even need to have the same cardinality (see [4, Exercise 1.2]). Anyway, we write W instead of (W, S) when the choice of S is clear from the context.

A Coxeter system can be represented by the corresponding *Coxeter graph* with vertex set S and undirected edges $\{s, s'\}$ for all $s, s' \in S$ with $m(s, s') > 2$. The edge $\{s, s'\}$ is labeled with $m(s, s')$ whenever $m(s, s') > 3$. We note that the correspondence between Coxeter systems and Coxeter graphs is one-to-one. A Coxeter system is called *irreducible* if the corresponding Coxeter graph is connected, otherwise it is called reducible. Reducible Coxeter systems decompose uniquely as products of irreducible ones.

For $J \subseteq S$, the subgroup W_J of W generated by the elements of J is called a *standard parabolic subgroup*. The pair (W_J, J) is again a Coxeter system with Coxeter graph isomorphic to the subgraph of the Coxeter graph of W induced by the vertices corresponding to J.

Example 3.2 ([4, Example 1.2.7])**.** Fix some $m \in \mathbb{N}$ and consider the finite reflection group $I_2(m)$ given by the symmetry group of a regular m-gon in the Euclidean plane. Let r_k denote the reflection with axis at angle $\frac{k\pi}{m}$ with respect to a fixed line passing through the center and one vertex of the polygon and let R_k denote the rotation by angle $\frac{2k\pi}{m}$ with respect to the center of the polygon. Then

$$I_2(m) = \{r_k : k = 0, \dots, m-1\} \cup \{R_k : k = 0, \dots, m-1\}$$

and $I_2(m)$ is generated by r_0 and r_1, for example. Furthermore we have $R_0 = e$ as the identity element of $I_2(m)$, $r_0^2 = r_1^2 = e$ and $(r_0 r_1)^m = e$. We call $I_2(m)$ the *dihedral group* of order $2m$.

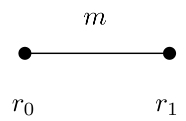


Figure 3.1: Coxeter graph of $I_2(m)$

Example 3.3. For $n \geq 3$ the affine symmetric group $\tilde{S}_n \subset S[\mathbb{Z}]$ is the group of permutations $w : \mathbb{Z} \to \mathbb{Z}$ of the integers such that $w(i+n) = w(i) + n$ for all $i \in \mathbb{Z}$ and $\sum_{i=1}^{n} w(i) = \binom{n+1}{2}$. It is generated by the periodic adjacent transpositions $\tilde{s}_i = \prod_{k \in \mathbb{Z}} (i + kn, i + 1 + kn)$ for $i = 1, \dots, n$. Then, $\tilde{S}_n$ is a Coxeter group of type $\tilde{A}_n$. The corresponding Coxeter graph is shown in Figure 3.2.

We remark that in contrast to $I_2(m)$ and the Coxeter groups presented later, $\tilde{S}_n$ is infinite. An affine permutation w is uniquely defined by the values $w(1), \dots, w(n)$ and we use the window notation $[w(1) \dots w(n)]$ when representing an affine permutation.

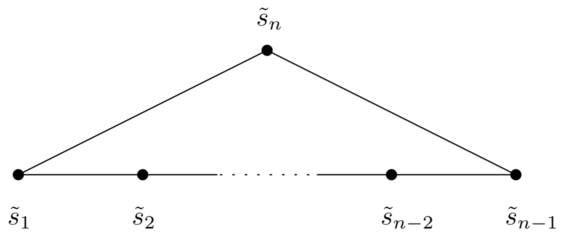


Figure 3.2: Coxeter graph of $\tilde{S}_n$

Every element $w \in W$ of a Coxeter group W can (not necessarily uniquely) be written as product of generators $s_1 \cdots s_k$ with $s_1, \dots, s_k \in S$ (not necessarily different). We call $s_1 \cdots s_k$ a *word* or *expression* for w. If k is minimal such that w can be written as a product of k generators then k is called the *length* of w and denoted by $l(w)$. In that case $s_1 \dots s_k$ is called a *reduced word* or a *reduced expression* for w.

We will now present two important properties of Coxeter groups. Actually the statements of propositions 3.4 and 3.5 are equivalent and do even characterize Coxeter systems in the sense that every group W with generating set S consisting of elements of order 2 that satisfies the statement of one of the following propositions will satisfy both of them and be a Coxeter system.

Proposition 3.4 **(Exchange property).** *Let $w = s_1 \dots s_k$ be a reduced expression for $w \in W$ and let $s \in S$. Then $l(sw) < l(w)$ if and only if $sw = s_1 \dots \hat{s}_i \dots s_k$ for some $i \in [k]$.*

Proposition 3.5 **(Deletion property).** *If $w = s_1 \dots s_k$ and $l(w) < k$ then $w = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$ for some $1 \leq i < j \leq k$.*

Let $s_1 \dots s_k$ be an expression for some element $w \in W$. We call an expression of the form $s_{i_1} \dots s_{i_l}$ with $1 \leq i_1 < \dots < i_l \leq k$ a *subexpression* or *subword* of $s_1 \dots s_k$. We can formulate a direct consequence of the deletion property as follows: every expression for $w \in W$ contains a reduced expression for w as a subword.

There exists a stronger version of the exchange property which also holds in all Coxeter systems. Let $T := \{w^{-1}sw : w \in W, s \in S\}$ be the set of *reflections*, i.e. elements conjugate to some generator. The exchange property actually holds for all reflections $t \in T$ and not only for all $s \in S \subseteq T$.

Proposition 3.6 **(Strong exchange property).** *Let $w = s_1 \dots s_k$ be a reduced expression for $w \in W$ and let $t \in T$. Then $l(tw) < l(w)$ if and only if $tw = s_1 \dots \hat{s}_i \dots s_k$ for some $i \in [k]$.*

The weaker as well as the stronger version of the Exchange Property are stated in their left versions here. There also exists a right version which holds for all Coxeter systems, just write wt instead of tw in all statements.

3.1.2 Bruhat order

We will now introduce the Bruhat order which turns a Coxeter group into a partially ordered set with interesting properties. In particular, this partial order is compatible with the length function. The Bruhat order is motivated by the connection to algebraic geometry, where it is defined on the Weyl group of a semi-simple algebraic group via inclusion of certain cell decompositions.

Definition 3.7. *Let $v, w \in W$. Write $v \to w$ if there exists some $t \in T$ with $w = tv$ and $l(w) > l(v)$. Define $v \leq w$ if $v = w_0 \to w_1 \to \dots \to w_m = w$ for some $w_0, \dots, w_m \in W$. The resulting partial order on W is called* Bruhat order*.*

Again, the definition is stated as a one-sided version and seems to favor left multiplication. However, if $w = tv$ for some $t \in T$, we have $w = vt'$ with $t' = v^{-1}tv \in T$. Thus, the Bruhat order could equivalently be defined using right multiplication. From now on, we will understand W as a Coxeter group partially ordered with respect to the Bruhat order without further mentioning.

It turns out that the Bruhat order can be described by the notion of subwords in a very natural way.

Proposition 3.8 (**Subword property**). *Let $w = s_1 \dots s_k \in W$ be a reduced expression and let $u \in W$. Then $u \leq w$ if and only if there exists a reduced expression for u which is a subword of $s_1 \dots s_k$.*

The subword property can be used to prove the so-called *chain property* saying that for every $u, v \in W$ with $u < v$ there is a chain $u = x_0 < x_1 < \dots < x_k = v$ such that $l(x_i) = l(u) + i$ for $i = 1, \dots, k$. Thus, a Coxeter system with Bruhat order is a graded poset with the length function as rank function. In particular, every interval $[u, v] \subseteq W$ is graded and finite.

The following property of the Bruhat order is a useful technical tool. In a sense it even characterizes the Bruhat order in Coxeter systems.

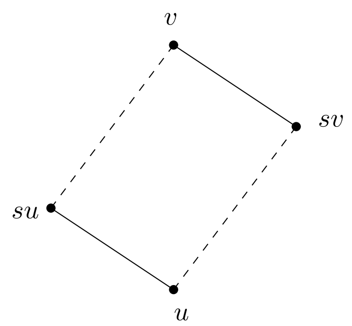


Figure 3.3: The lifting property of the Bruhat order

Proposition 3.9 (**Lifting property**). *Let $u, v \in W$ and $s \in S$ be such that $l(sv) < l(v)$ and $l(su) > l(u)$. Then the following are equivalent:*

(i) $u < v$.

(ii) $u \leq sv$.

(iii) $su \leq v$.

Example 3.10. Remember the dihedral group $I_2(m)$ generated by $a, b \in I_2(m)$ with $(ab)^m = e$. It has exactly two elements of length $k \in [2, m-1]$ with reduced expressions $aba \dots$ and $bab \dots$ (each having k letters). It holds that $u < v$ (respectively $u \lhd v$) for some $u, v \in I_2(m)$ if and only if $l(v) > l(u)$ (respectively $l(v) = l(u) + 1$).

Let's make some short comments on the structure of intervals in Coxeter groups ordered by Bruhat order (see [4, Section 2.7 and 2.8] for more details). Besides many topological properties of intervals and the corresponding order complexes, it is known that the Möbius function in a Coxeter group is given by $\mu(u, v) = (-1)^{l(v)-l(u)}$ for all $u \leq v$. Furthermore, every interval of length 2 is isomorphic to B_2 and intervals of length 3 are isomorphic to so-called k-crowns.

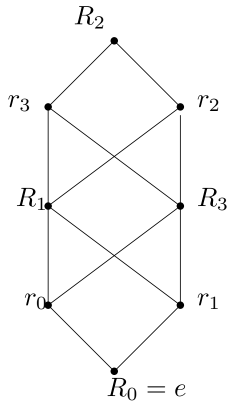


Figure 3.4: The Hasse diagram of $I_2(4)$

3.2 Finite Coxeter groups

The finite Coxeter groups are exactly the finite reflection groups (see [10]). Here a reflection is a linear operator on a Euclidean space that sends some nonzero vector to its negative and fixes the hyperplane orthogonal to that vector pointwise. A *finite reflection group* is a finite group generated by reflections.

In the finite case, there always exists a greatest element, which will be denoted by w_0. Thus, we have $w_0 \geq w$ for all $w \in W$. This maximal element fulfills $w_0^2 = e$ and $l(w_0) = |T|$.

The finite and irreducible Coxeter groups have been classified (see [4, Appendix A1]). We will present the classes that will be used frequently in this work.

3.2.1 The symmetric group S_n

The Coxeter system of type A_{n-1} is determined by the Coxeter graph in Figure 3.5. The corresponding finite reflection group is the symmetry group of the n-dimensional simplex in $\mathbb{R}^n$ generated by the reflections with fixed hyperplanes $x_i = x_{i+1}$ for $i = 1, \ldots, n-1$.

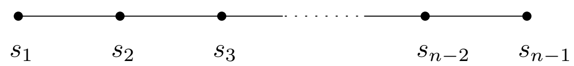


Figure 3.5: Coxeter graph of S_n

In particular, the symmetric group S_n of all permutations on n elements generated by the adjacent transpositions $s_i = (i, i+1)$ for $i \in [n-1]$ is of type A_{n-1}. Indeed, we have $s_i^2 = e$ for all $i \in [n-1]$ as well as $(s_i s_{i+1})^3 = (s_{i+1} s_i)^3 = e$ for all $i \in [n-2]$ and $(s_i s_j)^2 = (s_j s_i)^2 = e$ for all $i, j \in [n-1]$ such that $|i-j| > 1$. The set of reflections T of S_n is exactly the set of all transpositions $t_{i,j} := (i, j)$ for $i, j \in [n]$ with $i \neq j$.

We can describe the Bruhat order on S_n in a combinatorial way. The rank and length function of S_n is given by $l(w) = \mathrm{inv}(w)$ for all $w \in S_n$. The smallest element is the identity $e = 12 \dots n$ and the maximal element is $w_0 = n(n-1)\dots 21$ with $l(w_0) = \binom{n}{2}$.

From the definition it follows, that $u \to w$ for permutations $u, w \in S_n$ if and only if $w = u \cdot t_{i,j}$ for some $i, j \in [n]$ such that $i < j$ and $u(i) < u(j)$. Thus, u is covered by w if and only if we have $w = u \cdot t_{i,j}$ with $i, j \in [n]$ as above and there is no $k \in [n]$ with $i < k < j$ and $u(i) < u(k) < u(j)$.

There is a simple way to check if two permutations $u, v \in S_n$ are comparable. Define $w[i,j] := |\{k \in [n] : k \leq i, w(k) \geq j\}|$ for $w \in S_n$. Then, $u \leq v$ if and only if $u[i,j] \leq v[i,j]$ for all $i, j \in [n]$. We remark that using the diagram representation of u and v, $u[i,j]$ counts the number of drawn points in the part of the plane given by $x \leq i$ and $y \geq j$.

The reader may note that the generators of the symmetric group S_n are denoted by $s_1, \dots, s_n$. This can lead to some confusion when using indices for generators in expressions later. Hopefully this can be avoided by the following convention: when speaking about a Coxeter group in general, then s_i will denote any generator and $s_i = s_j$ is possible for $i \neq j$. If we are speaking about the symmetric group or similar groups as presented below, then s_i will always denote a certain generator and in particular $s_i = s_j$ will imply $i = j$.

3.2.2 The signed symmetric groups S_n^B and S_n^D

The Coxeter graph in Figure 3.6 determines the Coxeter group of type B_n. It is the symmetry group of the n-dimensional hyperoctahedron in $\mathbb{R}^n$ generated by the reflections with fixed hyperplanes $x_i = x_{i+1}$ for $i = 1, \dots, n-1$ as well as $x_1 = 0$.

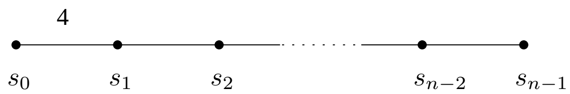


Figure 3.6: Coxeter graph of S_n^B

The *group of signed permutations* S_n^B is of type B_n. Its canonical set of generators is given by $s_i = (i, i+1)(-i, -i-1)$ for $i \in [n-1]$ and the sign change $s_0 = (1, -1)$. We note that S_n^B can be obtained as the semi-direct product of S_n with $(\mathbb{Z}/2\mathbb{Z})^n$ and contains $2^n n!$ elements. We have $m(s_i, s_j) \in \{1, 2, 3\}$ for $i, j \in [n-1]$ as before and we have $(s_0 s_1)^4 = e$ as well as $(s_0 s_j)^2 = e$ for all $2 \leq j \leq n-1$.

The length function l_B of S_n^B is given by

$$l^B(w) = \mathrm{inv}_B(w) := \mathrm{inv}(w(1), \dots, w(n)) + \mathrm{neg}(w(1), \dots, w(n)) + \mathrm{nsp}(w(1), \dots, w(n))$$

where $\mathrm{neg}(w(1), \dots, w(n)) := |\{i \in [n] : w(i) < 0\}|$ is the number of negative entries of w and $\mathrm{nsp}(w(1), \dots, w(n)) := |\{\{i, j\} \in \binom{[n]}{2} : w(i) + w(j) < 0\}|$ is the number of negative sum pairs of w. The minimal and maximal elements of S_n^B are $e = [12\dots n]$ and $w_0 = [\underline{1}\dots\underline{n}]$ with $l^B(w_0) = n^2$.

The set of reflections of S_n^B is given by

$$\{(i,j)(-i,-j) : 1 \leq i < |j| \leq n\} \cup \{(i,-i) : i \in [n]\}.$$

Again we can check if two elements $u, v \in S_n^B$ are comparable. Let $w[i,j] := |\{a \in [-n,n] : a \leq i, w(a) \geq j\}|$ where $w(0) := 0$. Then, $u \leq v$ if and only if $u[i,j] \leq v[i,j]$ for all $i, j \in [-n,n]$. The interpretation using the diagram representation of this fact is the same as for permutations in S_n.

The subgroup S_n^D of S_n^B is itself a Coxeter group with generators $s_1, \dots, s_{n-1}$ as before and $s_0' = s_0 s_1 s_0$. The corresponding Coxeter diagram determining the Coxeter group of type D_n is shown in Figure 3.7.

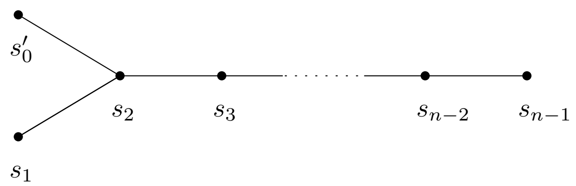


Figure 3.7: Coxeter graph of S_n^D

3.3 Twisted involutions

3.3.1 Combinatorics of twisted involutions

Consider a Coxeter group W and a bijection $\theta : W \to W$. We call θ

(i) *order-preserving automorphism* if $u \leq v \Leftrightarrow \theta(u) \leq \theta(v)$ for all $u, v \in W$.

(ii) *group automorphism* if it respects the group structure of W, i.e. $\theta(uv) = \theta(u)\theta(v)$ for all $u, v \in W$.

(iii) *graph automorphism* if it is a group automorphism induced by an automorphism of the Coxeter graph of W, i.e. a permutation of S which preserves the relations between elements of S.

(iv) *involutive* if $\theta^2 = \mathrm{id}$.

From the definition it is clear, that every graph automorphism is order-preserving. The converse is not true, as for example the bijection $w \mapsto w^{-1}$ is order-preserving but not a graph automorphism in general (because it fixes all $s \in S$, but not necessarily all $w \in W$).

Example 3.11. Remember the dihedral group $I_2(m)$. Two elements u and v of the dihedral group $I_2(m)$ satisfy $u \leq v$ if and only if $l(u) \leq l(v)$. There are exactly two elements of length k for every $k = 1, \dots, m-1$. Thus, the group of order-preserving automorphisms of $I_2(m)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{m-1}$.

In [24] the automorphisms of the Bruhat order of a Coxeter group W are classified. Interesting for us is the irreducible and non-dihedral case.

Proposition 3.12 ([24])**.** *Let (W, S) be an irreducible Coxeter system with $|S| \geq 3$. Then every automorphism of the Bruhat order is either a graph automorphism or a graph automorphism followed by group inversion.*

We will now present the theory of twisted involutions. In [20], [17] and [18] the combinatorics of twisted involutions is developed for finite W and from an algebraical point of view. For our purposes we will follow [8], where the theory is developed combinatorially for general Coxeter groups.

From now on, $\theta : W \to W$ denotes an involutive group automorphism such that $\theta(S) = S$. In particular, θ is order-preserving and because of Proposition 3.12 it is induced by an automorphism of the Coxeter graph of W.

Definition 3.13. *The set of* twisted involutions *of W with respect to θ is*

$$\mathcal{I}(W, \theta) := \{w \in W : \theta(w) = w^{-1}\}.$$

For $w \in W$ and $s \in S$ define the action of the symbol $\underline{s}$ from the right on w by

$$w\underline{s} = \begin{cases} ws & \text{if } \theta(s)ws = w, \\ \theta(s)ws & \text{otherwise.} \end{cases}$$

Define $w\underline{s}_1 \dots \underline{s}_k := (\dots(w\underline{s}_1)\underline{s}_2 \dots)\underline{s}_k$ and write $\underline{s}_1 \dots \underline{s}_k$ instead of $e\underline{s}_1 \dots \underline{s}_k$. The following proposition justifies the use of the above expressions.

Proposition 3.14 ([8, Proposition 3.5])**.** *Let $w \in W$. Then the following are equivalent:*

(i) $w \in \mathcal{I}(W, \theta)$.

(ii) There exist $s_1, \dots, s_k \in S$ such that $w = \underline{s}_1 \dots \underline{s}_k$.

We call $w = \underline{s}_1 \dots \underline{s}_k$ a *twisted expression* for $w \in \mathcal{I}(W, \theta)$. If k is minimal such that $w = \underline{s}_1 \dots \underline{s}_k$ for some $s_1, \dots, s_k \in S$, then we write $\rho(w) = k$ and call $\underline{s}_1 \dots \underline{s}_k$ a *reduced twisted expression.*

This reminds of the expressions for $w \in W$ and there are indeed very useful similarities in the behavior of expressions for Coxeter elements and twisted expressions for twisted involutions as the next propositions illustrate. However, the set of twisted involutions is in general not a Coxeter group.

Lemma 3.15 ([8, Lemma 3.8]). *Let $w \in \mathcal{I}(W,\theta)$ and $s \in S$. Then $\rho(w\underline{s}) < \rho(w)$ if and only if $l(ws) < l(w)$.*

Proposition 3.16 (**Exchange property in $\mathcal{I}(\mathbf{W},\theta)$**, [8, Proposition 3.10]). *Let $\underline{s}_1 \dots \underline{s}_k$ be a reduced twisted expression and assume that $s \in S$ fulfills $\rho(\underline{s}_1 \dots \underline{s}_k \underline{s}) < k$. Then $\underline{s}_1 \dots \underline{s}_k \underline{s} = \underline{s}_1 \dots \hat{\underline{s}}_j \dots \underline{s}_k$ for some $j \in [k]$.*

Proposition 3.17 (**Deletion property in $\mathcal{I}(\mathbf{W},\theta)$**, [8, Proposition 3.11]). *Let $\underline{s}_1 \dots \underline{s}_k$ be a twisted expression with $\rho(\underline{s}_1 \dots \underline{s}_k) < k$. Then $\underline{s}_1 \dots \underline{s}_k = \underline{s}_1 \dots \hat{\underline{s}}_i \dots \hat{\underline{s}}_j \dots \underline{s}_k$ for some $i, j \in [k]$ with $i \neq j$.*

We use the notion of subwords of twisted expressions in the same way as for expressions in W. Again, every twisted expression for $w \in \mathcal{I}(W,\theta)$ contains a reduced twisted expression for w as a subword.

From now on, consider $\mathcal{I}(W,\theta) \subseteq W$ as a partially ordered set with respect to the Bruhat order in W, i.e. $u \leq v$ in $\mathcal{I}(W,\theta)$ if and only if $u \leq v$ in W for all $u, v \in \mathcal{I}(W,\theta)$. This makes it possible to use the symbol $\leq$ in both partially ordered sets W and $\mathcal{I}(W,\theta)$ without being ambiguous.

Again, a lifting property similar to Proposition 3.9 holds and is used to prove the subword property which describes the partial order on $\mathcal{I}(W,\theta)$ in terms of subwords.

Proposition 3.18 (**Subword property in $\mathcal{I}(W,\theta)$, [9, Theorem 2.8]**). *Let $v = \underline{s}_1 \dots \underline{s}_k \in \mathcal{I}(W,\theta)$ be a reduced twisted expression and let $u \in \mathcal{I}(W,\theta)$. Then $u \leq v$ if and only if there exists a reduced twisted expression for u which is a subword of $\underline{s}_1 \dots \underline{s}_k$.*

The subword property for twisted expressions will play an essential role in chapter 5 in order to examine the structure of lower order ideals in $\mathcal{I}(W,\theta)$. The following proposition will give some first information about those.

Proposition 3.19 ([8, Theorem 4.5]). *Let $w \in \mathcal{I}(W,\theta)$ and $s \in S$ be such that $\rho(w\underline{s}) < \rho(w)$. Then, the map $v \mapsto v\underline{s}$ is a special matching on $B(w) = [e, w] \subseteq \mathcal{I}(W,\theta)$.*

From the existence of special matchings for all intervals $[e, w]$ with $w \in \mathcal{I}(W,\theta)$ and $w \neq e$ it follows that $\mathcal{I}(W,\theta)$ is a zircon. Zircons are posets generalizing the notion of Coxeter groups partially ordered by Bruhat order, see [16] for the exact definition. It is known that in that case $[e, w]$ is a lattice if and only if $w \in \mathcal{I}(W,\theta)$ is boolean ([16, Corollary 5.5]).

In [7], Hultman shows that $\mathcal{I}(W,\theta)$ is graded and describes the rank function. Those results were proved by Incitti in [11, 12, 13] for Coxeter groups of type A, B and D in the case $\theta = \mathrm{id}$ and conjectured to hold for arbitrary Coxeter groups. In [17] they are proved for Weyl groups and arbitrary θ. In [9], the twisted absolute length function l^θ is described as follows: if $w = \underline{s}_1 \dots \underline{s}_k \in \mathcal{I}(W,\theta)$ is a reduced twisted expression, then $l^\theta(w) = |\{i \in [k] : \underline{s}_1 \dots \underline{s}_{i-1}\underline{s}_i = \underline{s}_1 \dots \underline{s}_{i-1} s_i\}|$.

Proposition 3.20 ([7]). *The poset $\mathcal{I}(W,\theta)$ is graded. The rank of $w \in \mathcal{I}(W,\theta)$ is given by $\rho(w) = \frac{l(w)+l^\theta(w)}{2}$.*

The following example illustrates that twisted involutions and their properties can be considered as generalizations of Coxeter groups.

Example 3.21 ([17, 8])**.** Let W be any Coxeter group. Then $W \times W$ is a Coxeter group with componentwise multiplication and generating set $(S \times \{e\}) \cup (\{e\} \times S)$. Consider the involutive automorphism $\theta : W \times W \to W \times W$ given by $(v, w) \mapsto (w, v)$. Then $\mathcal{I}(W \times W, \theta) = \{(w, w^{-1}) : w \in W\}$ is isomorphic to W and the corresponding bijection $w \mapsto (w, w^{-1})$ is order-preserving. Furthermore, $s_1 \ldots s_k$ is a (reduced) expression for $w \in W$ if and only if $\underline{(s_1, e)} \ldots \underline{(s_k, e)}$ is a (reduced) twisted expression for $(w, w^{-1}) \in \mathcal{I}(W \times W, \theta)$. If $w \in W$ has length k, then (w, w^{-1}) has length $2k$ as an element of $W \times W$ and rank k as an element of $\mathcal{I}(W \times W, \theta)$, i.e. in that case $l^\theta((w, w^{-1})) = 0$ for all $w \in W$.

3.3.2 The case $\theta = \mathrm{id}$

Let us consider $\theta = \mathrm{id}$. Then $\mathcal{I}(W, \mathrm{id})$ is the set of (ordinary) involutions in W. We will restate some of the theory from above for $\theta = \mathrm{id}$, for example the action of $\underline{s}$ on $w \in W$ for $s \in S$.

Lemma 3.22. *Let $\theta = id$, $w \in W$ and $s \in S$. Then, the action of the symbol $\underline{s}$ from the right on w as defined in the general case is as follows:*

$$w\underline{s} = \begin{cases} ws & \text{if } ws = sw, \\ sws & \text{otherwise.} \end{cases}$$

Proof. This is clear by definition, because $\theta(s) = s$ for all $s \in S$. □

We conclude $\underline{s} = s$ for all $s \in S$. In particular, $\underline{s_1} = \underline{s_2}$ holds for some $s_1, s_2 \in S$ if and only if $s_1 = s_2$.

Lemma 3.23. *Let $w \in \mathcal{I}(W, id)$. Then, every reduced twisted expression for w contains the same set of letters.*

Proof. Let $s \in S$ be such that $\underline{s}$ is a letter of some reduced twisted expression for w. In particular, we have $\underline{s} \leq w$. Let $\underline{s_1} \ldots \underline{s_k}$ be any reduced twisted expression for w. From the subword property it follows, that $\underline{s_i} = \underline{s}$ for some $i \in [k]$, i.e. $s_i = s$ because of our previous remark. □

We want to state a simple property of twisted expressions for elements of $\mathcal{I}(W, \mathrm{id})$ concerning the commutativity of its letters. This will later be used without further remarks.

Lemma 3.24. *Let (W, S) be a Coxeter system, $\theta = id$ and let $s_1, s_2 \in S$ with $s_1 \neq s_2$. If s_1 and s_2 commute, then $w\underline{s_1}\underline{s_2} = w\underline{s_2}\underline{s_1}$ holds for all $w \in W$. Furthermore $\underline{s_1}\underline{s_2} = \underline{s_2}\underline{s_1}$ if and only if $m(s_1, s_2) \leq 3$.*

Proof. The first part follows from Definition 3.13 about the action of $\underline{s}_1$ and $\underline{s}_2$ on elements of W. Note that under our assumptions s_2 commutes with w if and only if it commutes with $w\underline{s}_1$ and conversely.

The second part follows from the first part for $m(s_1, s_2) = 2$ and from $\underline{s}_1\underline{s}_2 = s_2 s_1 s_2$ and $\underline{s}_2\underline{s}_1 = s_1 s_2 s_1$ for $m(s_1, s_2) \geq 3$. $\square$

3.3.3 The involution set I_n

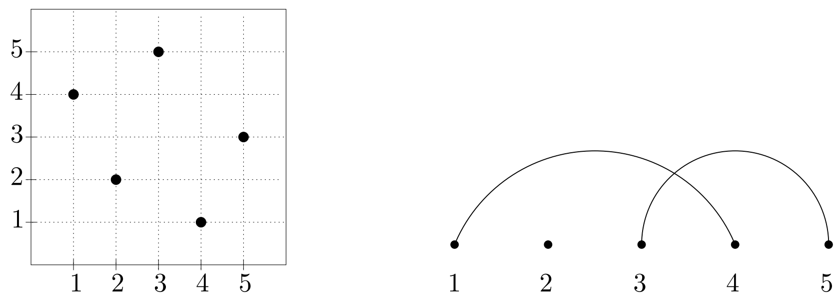
Let's have a closer look at the involutions of the symmetric group S_n.

Definition 3.25. *Define*

$$I_n := \mathcal{I}(S_n, id) \subseteq S_n$$

as the set of involutions of S_n partially ordered by the Bruhat order.

The involutions are exactly those permutations whose diagram representation is symmetric with respect to the main diagonal $x = y$. Aside from the diagram representation for permutations we can represent an involution $w \in I_n$ by an undirected graph on vertex set $[n]$ as well, where the vertices $i, j \in [n]$ are joined by an edge if and only if $w(i) = j$. We call this the *graph representation* of w. The graph representation is well-defined if w is an involution.



(a) Diagram representation (b) Graph representation

Figure 3.8: The involution $42513 \in I_5$

In [12], the poset I_n is characterized in a purely combinatorial way. In particular, the paper gives an explicit description of the covering relations in I_n as follows.

Let $w \in I_n$. A pair $(i, j) \in [n]^2$ is called a *free rise* of w if $i < j$ and $w(i) < w(j)$ and if there is no $k \in [n]$ with $i < k < j$ and $w(i) < w(k) < w(j)$. Depending on if i and j are fixed points (f), excedances (e) or deficiencies (d), the rise (i, j) is called of type (f,f), (f,e), (f,d) and so on. A free rise is suitable if it is of type (f,f), (f,e), (e,f), (e,e) or (e,d). For every suitable rise (i, j), Incitti defines the covering transformation $\mathrm{ct}_{(i,j)}(w) \in I_n$ as in table 3.1. In the last column of the table both w and $\mathrm{ct}_{(i,j)}(w)$ are represented in a diagram: the black dots indicate w and the white dots indicate $\mathrm{ct}_{(i,j)}(w)$. There are no other (black or white) dots in the gray shaded area due to the condition that (i,j) is a free rise.

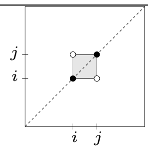
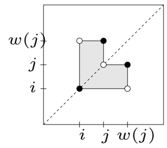
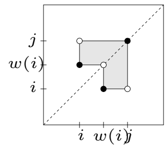
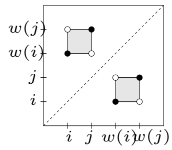
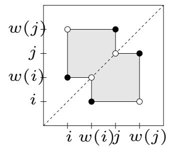
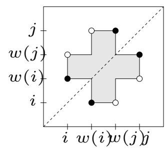
Type of (i,j)	$\mathrm{ct}_{(i,j)}(w)$	Diagram
(f,f)-rise	$w\,(i,j)$	
(f,e)-rise	$w\,(i,j,w(j))$	
(e,f)-rise	$w\,(i,j,w(i))$	
Non-crossing (e,e)-rise	$w\,(i,j)\,(w(i),w(j))$	
Crossing (e,e)-rise	$w\,(i,j,w(j),w(i))$	
(e,d)-rise	$w\,(i,j)\,(w(i),w(j))$	

Table 3.1: The covering transformation, [12, Table 1]

The covering relation in I_n is completely described by the covering transformations.

Proposition 3.26 ([12, Theorem 5.1])**.** *Let $u, v \in I_n$. Then $u \lhd v$ if and only if $v = ct_{(i,j)}(u)$ for some suitable rise (i,j) of u.*

Furthermore, the rank function of I_n is formulated combinatorially.

Proposition 3.27 ([12, Theorem 5.2])**.** *The set of involutions I_n partially ordered with respect to the Bruhat order is graded with rank function ρ given by*

$$\rho(w) = \frac{inv(w) + exc(w)}{2}$$

for all $w \in I_n$. Here $exc(w) = |\{i \in [n] : w(i) > i\}|$ denotes the number of excedances in w.

4 Permutation patterns

4.1 Fundamentals

Definition 4.1. *Let $w \in S_n$ and $p \in S_k$ for $k \leq n$. The permutation w contains the* pattern *p if there exist $i_1 < \ldots < i_k$ such that $w(i_1)\ldots w(i_k)$ is in the same relative order as $p(1)\ldots p(k)$, i.e. if $w(i_l) < w(i_m) \Leftrightarrow p(l) < p(m)$ for all $l, m \in [k]$. If w does not contain the pattern p, then w avoids p or is p-avoiding.*

The next piece of notation follows [23]. If w contains the pattern p with $\{i_1, \ldots, i_k\}$ as in Definition 4.1, then $w(i_1)\ldots w(i_k)$ is called an occurrence of p in w. By $\langle p(j)\rangle$ and $\langle p\rangle$ we will denote the values $w(i_j)$ and $w(i_1)\ldots w(i_k)$, respectively.

Definition 4.1 can easily be extended to signed permutations where we only have to add a clause concerning the signs.

Definition 4.2. *Let $w \in S_n^B$ and $p \in S_k^B$ for $k \leq n$. The signed permutation w contains the* signed pattern *p if there exist $i_1 < \ldots < i_k$ such that $|w(i_1)|\ldots|w(i_k)|$ is in the same relative order as $|p(1)|\ldots|p(k)|$, i.e. if $|w(i_l)| < |w(i_m)| \Leftrightarrow |p(l)| < |p(m)|$ for all $l, m \in [k]$, and such that $w(i_l)$ and $p(l)$ have the same sign for all $l \in [k]$. The notions of an occurrence and avoiding the pattern p are as before.*

There is a variety of studies concerning permutation patterns or closely connected topics. In particular, one is interested in counting the number of permutations with a certain number of occurrences of a pattern or a list of patterns. Pattern avoidance is a special case of that question. We refer to [14] as a survey on that topic presenting a detailed overview of known results and methods.

4.2 Pattern avoidance and Bruhat order

Several properties of permutations have been characterized by pattern avoidance and containment (see [2] for more references). In [15] Lakshmibai and Sandhya showed, that the Schubert variety corresponding to some permutation $w \in S_n$ is smooth if and only if p avoids 3412 and 4231. The connection to Bruhat order is given by the fact, that $B(w)$ encodes the cell incidence structure of that variety. Gasharov proved in [6] that for $w \in S_n$ the Poincaré polynomial of $B(w) = [e, w]$ factors into certain polynomials if and only if w avoids 3412 and 4231. A consequence is the following proposition.

Proposition 4.3 ([6, 3])**.** *Let $w \in S_n$. Then, $B(w) = [e, w]$ is rank-symmetric if and only if w avoids the patterns* 3412 *and* 4231*.*

Similar results were shown by Billey in [1] for S_n^B and S_n^D. In particular, also the elements of S_n^B and S_n^D having rank-symmetric lower order ideals can be characterized by avoiding a list of signed patterns (see [3]). However, the list of patterns is rather long for S_n^B and S_n^D.

A stronger condition on w is to have a lower order ideal isomorphic to a boolean lattice. Remember that we call those w boolean. It turns out that booleanness can be characterized in terms of patterns avoidance in S_n, S_n^B and S_n^D, as well. This is proved by Tenner in [22]. Her results are stated in the following propositions. Again, the list of patterns grows rather long for S_n^B and S_n^D.

Proposition 4.4 ([22, Theorem 5.3]). *The permutation $w \in S_n$ is boolean if and only if w avoids the patterns* 321 *and* 3412.

Proposition 4.5 ([22, Theorem 8.4]). *The signed permutation $w \in S_n^B$ is boolean if and only if w avoids the signed patterns* $\underline{12}$, $\underline{21}$, 321, 3412, $32\underline{1}$, $34\underline{1}2$, $\underline{3}21$, $\underline{3}412$, $1\underline{2}$ *and* $3\underline{2}1$.

Proposition 4.6 ([22, Theorem 8.7]). *The signed permutation $w \in S_n^D$ is boolean if and only if w avoids the signed patterns* $\underline{123}$, $\underline{132}$, $\underline{213}$, $\underline{231}$, $\underline{312}$, $\underline{321}$, 321, 3412, $32\underline{1}$, $3\underline{12}$, $34\underline{1}2$, $34\underline{21}$, $\underline{3}21$, $\underline{23}1$, $\underline{3}412$, $\underline{43}12$, $1\underline{2}$, $3\underline{2}1$, $\underline{3}2\underline{1}$ *and* $\underline{3}4\underline{1}2$.

Tenner also enumerates the boolean elements in S_n, S_n^B and S_n^D by counting certain reduced expressions.

Another result of Tenner is used in her proofs of the Propositions 4.5 and 4.6. It connects pattern avoidance with properties of reduced expressions. A *factor* is a consecutive subword of a reduced expression. A *shift* of a reduced expression for some $w \in S_n$ is the reduced expression obtained by shifting all indices of the letters of the expression by the same value.

Proposition 4.7. *Let $p \in S_k$ be* 2143*-avoiding. If the permutation $w \in S_n$ contains a p-pattern then there exists a reduced expression for w containing some shift of a reduced expression for p as a factor.*

Tenner's results on boolean elements in S_n, S_n^B and S_n^D motivate our study of boolean elements in the posets of twisted involutions of Coxeter groups. It will turn out that we can make quite similar statements.

5 Boolean involutions and pattern avoidance

Throughout this section let (W, S) be a Coxeter system and $\theta : W \to W$ an involutive graph automorphism. Remember, that this implies that θ is order-preserving and a group automorphism. Furthermore, it holds that $\theta(S) = S$.

If $w \in \mathcal{I}(W, \theta)$ is boolean, then it obviously holds that $[e, w] \cong B_{\rho(w)}$. Furthermore, the set of boolean elements forms a lower order ideal of $\mathcal{I}(W, \theta)$, i.e. if $w \in \mathcal{I}(W, \theta)$ is boolean and $v \in \mathcal{I}(W, \theta)$ is such that $v \leq w$, then v is boolean, too. This shows that booleanness is quite a convenient property.

5.1 Reduced twisted expressions of boolean elements

The next proposition classifies the boolean elements in $\mathcal{I}(W, \theta)$ in terms of their reduced expressions. The statement is valid for all W and θ and thus the most general we can make on boolean elements. It also provides a useful tool for further investigations about boolean elements in more specific situations.

We have to make a comment on the equality of subwords here. Two subwords $\underline{s}_{i_1} \dots \underline{s}_{i_l}$ and $\underline{s}_{j_1} \dots \underline{s}_{j_m}$ of some reduced twisted expression $\underline{s}_1 \dots \underline{s}_k$ are considered to be equal if and only if $l = m$ and $i_o = j_o$ for all $o \in [l]$. In particular, different subwords can be twisted expressions for the same element in general.

Proposition 5.1. *Let $w \in \mathcal{I}(W, \theta)$ and let $\underline{s}_1 \dots \underline{s}_k$ be a reduced expression for w. Then the following are equivalent:*

(i) w is boolean.

(ii) Every subword of $\underline{s}_1 \dots \underline{s}_k$ is reduced and different subwords are reduced expressions for different elements.

(iii) $\underline{s}_p \neq \underline{s}_q$ for all $p, q \in [k]$ with $p \neq q$.

Proof. *(i)* $\Rightarrow$ *(iii)*. Suppose that there are $p \neq q$ with $\underline{s}_p = \underline{s}_q$. Then w is of rank k in $B(w)$ but $B(w)$ has at most $k-1$ elements of rank 1. Thus $B(w)$ cannot be isomorphic to a boolean lattice and w is not boolean.

(iii) $\Rightarrow$ *(ii)*. Assume that there are two different subwords of $\underline{s}_1 \dots \underline{s}_k$ which are reduced expressions for the same element $v \in \mathcal{I}(W, \theta)$. Choose $\underline{s}_p$ as a letter which is contained in one of the words but not in both (with respect to the index). It holds that $\underline{s}_p \leq v$ and using the subword property for $\mathcal{I}(W, \theta)$ (Proposition 3.18) it follows that there is some $\underline{s}_q$ in the word that does not contain $\underline{s}_p$ such that $\underline{s}_p = \underline{s}_q$. By the choice of $\underline{s}_p$ we have $p \neq q$.

Assume that there is a subword $\underline{s}_{i_1} \dots \underline{s}_{i_l}$ of $\underline{s}_1 \dots \underline{s}_k$ which is not reduced. Let $m \in [l]$ be minimal such that $\underline{s}_{i_1} \dots \underline{s}_{i_m}$ is not reduced. The exchange property for $\mathcal{I}(W, \theta)$ (Proposition 3.16) implies that $\underline{s}_{i_1} \dots \underline{s}_{i_{m-1}} = \underline{s}_{i_1} \dots \hat{\underline{s}}_{i_p} \dots \underline{s}_{i_m}$. Both twisted expressions are reduced and we found two different subwords of $\underline{s}_1 \dots \underline{s}_k$ which are reduced expressions for the same element in $\mathcal{I}(W,\theta)$. Again, the existence of $p \neq q$ with $\underline{s}_p = \underline{s}_q$ follows.

$(ii) \Rightarrow (i)$. Consider the mapping $\varphi : 2^{[k]} \to B(w)$ that maps a subset $\{i_1, \dots, i_l\} \subseteq [k]$ with $i_1 \leq \dots \leq i_l$ to $\underline{s}_{i_1} \dots \underline{s}_{i_l}$. The subword property for $\mathcal{I}(W,\theta)$ and (ii) ensure that this mapping is well-defined, order-preserving and bijective. Thus, we have $B(w) \cong B_k$. $\square$

In [22, Section 5], Tenner mentions that $w \in S_n$ is boolean if and only if no reduced expression for w has repeated letters. We note that this notion of a boolean permutation can be seen as special case of our considerations in the following way:

Example 5.2. Remember Example 3.21 with $\theta : W \times W \to W \times W$ and $\theta(v,w) = (w,v)$. From what we already have seen, it follows that $W \supseteq [e,w] \cong [e,(w,w^{-1})] \subseteq W \times W$ and thus $w \in W$ is boolean if and only if $(w,w^{-1}) \in \mathcal{I}(W \times W, \theta)$ is boolean. But $w = s_1 \dots s_k$ is a reduced twisted expression for w if and only if $\underline{(s_1,e)} \dots \underline{(s_1,e)}$ is a reduced twisted expression for (w,w^{-1}). We have $\underline{(s_p,e)} = \underline{(s_q,e)}$ if and only if $s_p = s_q$ and this implies that $w \in W$ is boolean if and only if $s_1 \dots s_k$ has no repeated letters.

In section 3.3.2 we studied twisted expressions in the special case $\theta = \mathrm{id}$. Applying Proposition 5.1 to this special situation yields a more specific characterization of boolean involutions.

Corollary 5.3. *Let $w \in \mathcal{I}(W, id)$. Then w is boolean if and only if no reduced twisted expression for w has repeated letters. This is the case if and only if there is a twisted expression for w without repeated letters.*

Proof. The first claim follows directly from Proposition 5.1 and the fact that $\underline{s}_1 = \underline{s}_2$ if and only if $s_1 = s_2$ for all $s_1, s_2 \in S$. The second part follows directly from Lemma 3.23. $\square$

5.2 Boolean involutions in I_n

We turn our attention to $I_n = \mathcal{I}(S_n, \mathrm{id})$ which was introduced in section 3.3.3. After some preparing considerations we will be able to characterize booleanness in I_n in terms of pattern avoidance.

Firstly, have a look at the considerably small example I_4 with Hasse diagram as in Figure 5.1. We note that there is only one non-boolean involution in I_4, namely $w_0 = 4321$. Thus, 4321 is a minimal non-boolean involution of I_4. We will soon see, that 4321 in some sense is the only minimal non-boolean involution.

Definition 5.4. *Let $w \in I_n$. The positions $i, j \in [n]$ are called*

(i) directly connected *in w if $i < j$ and $w(i) > w(j)$ or $i > j$ and $w(i) < w(j)$.*

(ii) connected *if there exists a sequence $i = i_0, i_1, \dots, i_k = j$ such that i_{l-1} and i_l are directly connected for all $l = 1, \dots, k$.*

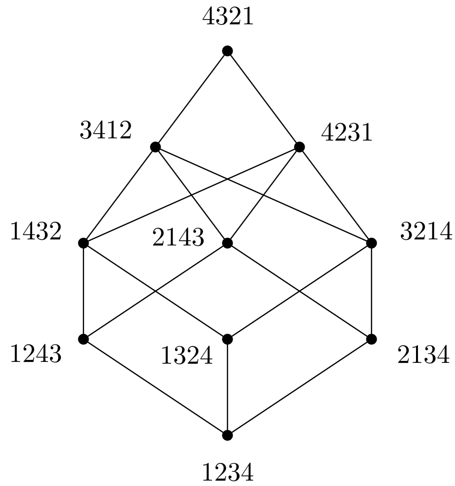


Figure 5.1: Hasse diagram of I_4

Observe that this notion of connectedness induces an equivalence relation on $[n]$. We call the equivalence classes with respect to this relation *connected components* of w and denote the set of connected components of w by

$$\mathcal{C}(w) = \{C \subseteq [n] : C \text{ is a connected component of } w\}.$$

We claim that the connected components of $w \in I_n$ are in fact intervals.

Lemma 5.5. *Let $w \in I_n$ and let $i, j, k \in [n]$ with $i < j < k$ be such that i and k are in the same connected component of w. Then j is in that connected component, too.*

Proof. From the definition of connectedness it follows that there are $p, q \in [n]$ such that $p < j < q$, p and q are connected with i and k and such that p and q are directly connected. In particular, this implies $w(p) > w(q)$. But then $w(j) < w(p)$ or $w(j) > w(q)$ will hold, i.e. j is directly connected to p or q and thus in the same connected component as i and k. □

For $w \in S_n$ and any subset $D \subseteq [n]$ we define the restriction w_D of w to D by

$$w_D(i) := \begin{cases} w(i) & \text{if } i \in D, \\ i & \text{otherwise.} \end{cases}$$

It is now easy to check that w_C is an involution if C is a connected component (or the union of connected components) of $w \in I_n$: if (i, j) is a cycle of w, then i and j are directly connected and thus in the same connected component.

Let $w \in I_n$ and $\mathcal{C}(w) = \{C_1, \ldots, C_k\}$. Then w_{C_i} belongs to the standard parabolic subgroup of S_n generated by $s_{a_i}, \ldots, s_{b_i}$ where $C_i = [a_i, b_i + 1]$. In particular, those subgroups are pairwise disjoint and generators of different subgroups commute. This implies that the concatenation of reduced twisted expressions for w_{C_i} and w_{C_j} is a reduced twisted expression for $w_{C_i \cup C_j}$ for all $i, j \in [k]$ with $i \neq j$. The following lemma is now immediate.

Lemma 5.6. *Let $w \in I_n$ with $\mathcal{C}(w) = \{C_1, \dots, C_k\}$. Then the following holds*

(i) If $\underline{w}_i$ is a reduced twisted expression for w_{C_i} for all $i \in [k]$, then the permuted concatenation of those expressions $\underline{w}_{\pi(1)}\underline{w}_{\pi(2)} \cdots \underline{w}_{\pi(k)}$ is a reduced twisted expression for w for any $\pi \in S_k$.

(ii) $[e, w] \cong [e, w_{C_1}] \times \ldots \times [e, w_{C_k}]$.

(iii) w is boolean if and only if w_{C_i} is boolean for all $i \in [k]$.

□

We call $w \in I_n$ connected if $[n]$ is the unique connected component, i.e. if $\mathcal{C}(w) = \{[n]\}$. Using Lemma 5.6 we can restrict our further investigations to connected $w \in I_n$.

We want to classify the boolean involutions in I_n. After a useful technical definition we make a sequence of propositions which construct the statements of the final theorem step by step.

Definition 5.7. *Let $w \in I_n$ and $i, j \in [n]$. The pair (i, j) is* long-crossing *in w if $i < j < w(j)$ and $w(i) > j + 1$.*

It is easy to see, that the elements i and j of a long-crossing pair (i, j) in some $w \in I_n$ are connected and thus in the same connected component of w.

Proposition 5.8 (A sufficient criterion). *Let $w \in I_n$. If there is no long-crossing pair (i, j) in w then w is boolean.*

Proof. From Lemma 5.6 we know, that w is a boolean involution if and only if all connected components of w are boolean. Our above remark tells us that there is a long-crossing pair (i, j) in w if and only if there is one in some connected component of w. Therefore, we can assume that w is connected (otherwise consider each connected component separately).

Assume that $\{(i_l, w(i_l)) : l = 1, \dots, k\}$ is the set of 2-cycles of w and that $i_l < w(i_l)$ for all $l \in [k]$ and $i_1 < i_2 < \ldots < i_k$. The non-existence of long-crossing pairs yields directly that $w(i_1) < \ldots < w(i_k)$. If $w(i_l) < i_{l+1}$ for some $l \in [k-1]$ then w is not connected. Thus $w(i_l) > i_{l+1}$ for all $l \in [k-1]$. But (i_l, i_{l+1}) is a long-crossing pair if $w(i_l) > i_{l+1} + 1$ and it follows from our assertion that $w(i_l) = i_{l+1} + 1$ for all $l \in [k-1]$. Connectedness of w implies $i_1 = 1$ and $w(i_k) = n$.

Let $C_m = \cup_{l=1}^{m} \{i_l, w(i_l)\}$ and for ease of notation $i_{k+1} = w(i_k) - 1 = n - 1$. Then w_{C_m} is an involution for all $m \in [k]$. We claim that

$$\underline{s}_1 \underline{s}_2 \cdots \underline{s}_{i_2 - 1} \underline{s}_{i_2 + 1} \underline{s}_{i_2} \underline{s}_{i_2 + 2} \cdots \underline{s}_{i_m - 1} \underline{s}_{i_m + 1} \underline{s}_{i_m} \underline{s}_{i_m + 2} \cdots \underline{s}_{i_{m+1}}$$

is a twisted expression for w_{C_m} for all $m \in [k]$ where the twisted expression contains all letters $\underline{s}_1, \dots \underline{s}_{i_{m+1}}$ in lexicographically increasing order except s_{i_l} and s_{i_l+1} which are switched for all $l = 2, \dots, m$. We will prove that inductively.

We check that $w_{C_1} = t_{i_1,w(i_1)} = \underline{s}_{i_1}\underline{s}_{i_1+1} \cdots \underline{s}_{i_2}$. This settles the case $m = 1$. For $m > 1$ we have $w_{C_m} = w_{C_{m-1}} t_{i_m,w(i_m)}$. For brevity let $w'_{C_{m-1}} = w_{C_{m-1}}\underline{s}_{i_m}$, i.e.

$$w'_{C_{m-1}} = \underline{s}_{i_1}\underline{s}_{i_1+1} \cdots \underline{s}_{i_2-1}\underline{s}_{i_2+1}\underline{s}_{i_2}\underline{s}_{i_2+2} \cdots \underline{s}_{i_{m-1}-1}\underline{s}_{i_{m-1}+1}\underline{s}_{i_{m-1}}\underline{s}_{i_{m-1}+2} \cdots \underline{s}_{i_m-1}.$$

We compute

$$\begin{aligned}
w_{C_m} &= w_{C_{m-1}} t_{i_m,w(i_m)} \\
&= w'_{C_{m-1}}\underline{s}_{i_m} t_{i_m,w(i_m)} \\
&= s_{i_{m+1}} \cdots s_{i_m+2} s_{i_m} w'_{C_{m-1}} s_{i_m} s_{i_m+1} s_{i_m} s_{i_m+1} \cdots s_{i_{m+1}} \\
&= s_{i_{m+1}} \cdots s_{i_m+2} s_{i_m} w'_{C_{m-1}} s_{i_m+1} s_{i_m} \cdots s_{i_{m+1}} \\
&= w'_{C_{m-1}}\underline{s}_{i_m+1}\underline{s}_{i_m} \cdots \underline{s}_{i_{m+1}} \\
&= \underline{s}_{i_1}\underline{s}_{i_1+1} \cdots \underline{s}_{i_2-1}\underline{s}_{i_2+1}\underline{s}_{i_2}\underline{s}_{i_2+2} \cdots \underline{s}_{i_m-1}\underline{s}_{i_m+1}\underline{s}_{i_m}\underline{s}_{i_m+2} \cdots \underline{s}_{i_{m+1}}.
\end{aligned}$$

Setting $m = k$ we conclude that there exists a twisted expression for w which does not have repeated letters. Applying Corollary 5.3 this implies that w is a boolean involution. $\square$

Actually, the above criterion is also necessary for booleanness. We will prove this using some knowledge about the combinatorics of I_n. Let $w \in I_n$ and let $i \in [n]$ not be a fixed point in w, i.e. $(i, w(i))$ is a 2-cycle of w. Then, we can delete that cycle by multiplication of w with $t_{i,w(i)}$ from the right. This does not change the entries of w except in the positions i and $w(i)$ and we have $v = w\, t_{i,w(i)} < w$.

If w is as before such that $i \in [n]$ is an excedance, i.e. $w(i) > i$, and $j \in [n]$ is a fixed point with $i < j < w(i)$, then we can shrink the cycle $(i, w(i))$ by conjugation with $t_{j,w(i)}$ without changing w except in the positions i, j and $w(i)$. We get $v = t_{j,w(i)}\, w\, t_{j,w(i)} < w$, (i, j) and $w(i)$ are a cycle respectively a fixed point of v.

Proposition 5.9 (A necessary criterion). *Let $w \in I_n$. If there is a long-crossing pair (i, j) in w then w is not boolean.*

Proof. Fix $i, j \in [n]$ such that (i, j) is a long-crossing pair in w. Following our remarks above we can delete all cycles except $(i, w(i))$ and $(j, w(j))$ and get an involution $v \le w$ whose only non-fixed points are $i, j, w(i), w(j)$. Now we can shrink the remaining two cycles such that we finally get an involution x with cycles $(j-1, j+2)$ and $(j, j+1)$ in the following way: conjugation of v with $t_{j+1,w(j)}$ yields $u \le v$ with $u(j) = j + 1$. Then we can conjugate u with $t_{i,j-1}$ and $t_{j+2,w(i)}$ and get $x \le u$ having the 2-cycles $(j - 1, j + 2)$ and $(j, j + 1)$ and fixed points in all other positions. (Multiplication or conjugation with $t_{k,k}$ for any $k \in [n - 1]$ is just the identity map.) A reduced twisted expression for x is given by $\underline{s}_{j-1}\underline{s}_j\underline{s}_{j+1}\underline{s}_j$ and thus x is not boolean. But we have $x \le u \le v \le w$ and therefore w is not boolean either. $\square$

The proof shows that for every non-boolean element $w \in I_n$, the lower order ideal $B(w)$ contains some $v \in I_n$ which just is a shift of $\underline{s}_1\underline{s}_2\underline{s}_3\underline{s}_2 = 4321 \in I_4$. This explains our remark about 4321 in some sense being the unique minimal non-boolean involution.

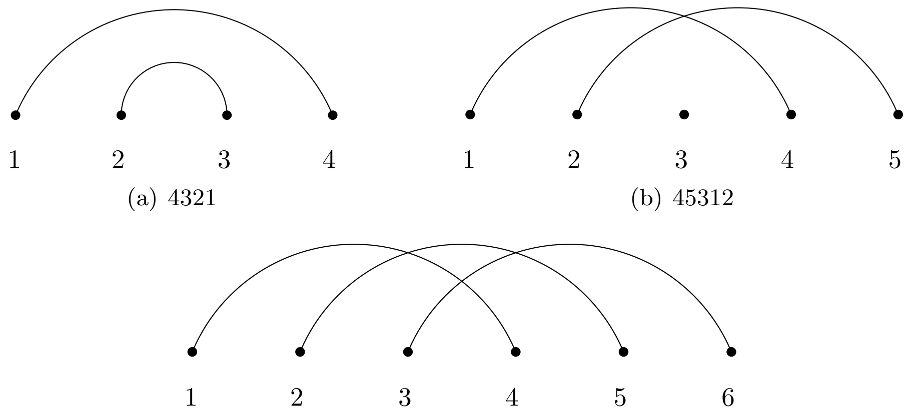
Proposition 5.10 (A pattern criterion). *Let $w \in I_n$. There is a long-crossing pair (i, j) in w if and only if w contains of one the patterns* 4321, 45312 *and* 456123.

Proof. "$\Rightarrow$". Let (i, j) be a long-crossing pair in w. If w contains the pattern 4321 we are done. Thus, we can assume, that w avoids 4321. In particular, this implies $w(i) < w(j)$. If $j + 1$ is a fixed point then w contains the pattern 45312. Otherwise, we have $w(j + 1) < i$ or $w(j + 1) > w(j)$ because we assumed w to be 4321-avoiding. But then w contains 456123.

"$\Leftarrow$". We distinguish three cases. First, assume that w contains 4321 and that $\langle 4321 \rangle$ is an occurrence. Then, $\langle 3 \rangle$ or $\langle 2 \rangle$ is not a fixed point of w, denote that value by k. If $w(k) > k$, then $w(\langle 1 \rangle) > w(k) > k > \langle 1 \rangle$ and $(\langle 1 \rangle, k)$ is a long-crossing pair in w. Otherwise, it follows that $w(\langle 4 \rangle) < w(k) < k < \langle 4 \rangle$ and $(w(\langle 4 \rangle), w(k))$ is such a pair.

The second case is that w avoids 4321 but contains 45312. Let $\langle 45312 \rangle$ be an occurrence. Then $\langle 3 \rangle$ is a fixed point, because otherwise w will contain 4321 by similar arguments as in the first case. This implies that $(\langle 1 \rangle, \langle 2 \rangle)$ is a long-crossing pair.

Finally, assume that w avoids 4321 and 45312 and let $\langle 456123 \rangle$ be an occurrence of 456123 in w. The fact, that w avoids 45312 implies that none of $\langle 1 \rangle, \langle 2 \rangle, \ldots \langle 6 \rangle$ is a fixed point. Furthermore, if $\langle 1 \rangle$, $\langle 2 \rangle$ or $\langle 3 \rangle$ is a deficiency, denote that value by k. Then $w(\langle 4 \rangle) < w(k) < k < \langle 4 \rangle$ and w contains 4321 in contradiction to our assumption. Thus, $\langle 1 \rangle$, $\langle 2 \rangle$ and $\langle 3 \rangle$ are excedances. If $w(\langle 1 \rangle) > \langle 3 \rangle$ then $(\langle 1 \rangle, \langle 2 \rangle)$ are a long-crossing pair in w. Otherwise, $(w(\langle 5 \rangle), \langle 3 \rangle)$ are one. $\square$



(a) 4321 (b) 45312

(c) 456123

Figure 5.2: Bad patterns for I_n

All preparation is done and we can summarize what we learned about boolean involutions in I_n in the following theorem.

Theorem 5.11. *Let $w \in I_n$. The following are equivalent:*

(i) w is boolean.

(ii) There is a twisted expression for w without repeated letters.

(iii) No reduced twisted expression for w has repeated letters.

(iv) There is no long-crossing pair (i, j) in w.

(v) w is 4321-,45312- and 456123-avoiding.

Proof. Follows from the propositions 5.3, 5.8, 5.9 and 5.10. □

5.3 Patterns and factors in I_n

In Proposition 4.7 we presented a result of Tenner, that connects the occurrence of a 2143-avoiding pattern p in some permutation $w \in S_n$ with the existence of a reduced expression for w containing some shift of a reduced expression for p as a factor. We will prove a similar result for the set of involutions I_n in this section. For that we need to introduce the notion of an induced pattern as well as some more notation defined in [23].

Definition 5.12. *Let $p \in I_k$ and $w \in I_n$ with $k \leq n$. We call an occurrence $\langle p \rangle$ in w* induced *in the case that $p(i) = j$ if and only if $w(\langle p(i) \rangle) = w(\langle j \rangle)$ for all $i, j \in [k]$. Then, we say that w contains the induced pattern p or is induced-p-containing if there exists an induced occurrence of p in w. Otherwise we call w induced-p-avoiding.*

We remark, that an involution containing an induced p-pattern also is p-containing by definition but the converse is not true in general.

Definition 5.13 ([23, Definition 2.6])**.** *If $\langle p \rangle$ is an occurrence of the pattern p in w and $w(j) \in \langle p \rangle$, then $w(j)$ is called a pattern entry in w. Otherwise $w(j)$ is a non-pattern entry. A non-pattern entry is said to be inside the pattern if it lies between two pattern entries in the one-line notation of w.*

When saying that any value $w(j_1)$ is to the left or right of $w(j_2)$ for some $j_1, j_2 \in [n]$, we always refer to the one-line notation of w.

Definition 5.14 ([23, Definition 2.7])**.** *Let $\langle p \rangle$ be an occurrence of the pattern $p \in S_k$ in $w \in S_n$ and suppose that x is a non-pattern entry inside the pattern, that $\langle m \rangle < x < \langle m+1 \rangle$ for some $m \in [k-1]$ and that the values $\langle m \rangle, x, \langle m+1 \rangle$ appear in increasing order in the one-line notation of w. Let $a, b \in \mathbb{N}$ be maximal so that the values*

$$\{\langle m-a \rangle, \langle m-a+1 \rangle, \langle m \rangle, x, \langle m+1 \rangle, \ldots, \langle m+b-1 \rangle, \langle m+b \rangle\}$$

appear in increasing order. The entry x is called obstructed to the left if a pattern entry smaller than $\langle m-a \rangle$ appears between $\langle m-a \rangle$ and x in w. Likewise, x is obstructed to the right if a pattern entry larger than $\langle m+b \rangle$ appears between x and $\langle m+b \rangle$ in w.

We remark, that if w contains the 2143-avoiding pattern p and x is as in Definition 5.14, then x cannot simultaneously be obstructed to the left and to the right ([23, Proposition 3.3]).

We will now present the algorithm IPACC (Induced Pattern As Connected Component) which takes an involution $w \in I_n$ containing an induced 2143-avoiding pattern $p \in S_k$ and constructs an involution $\tilde{w} \in I_n$ satisfying

(P1) $\tilde{w} = w\underline{s}_{i_1} \dots \underline{s}_{i_r}$ for some $r \in \mathbb{N}$ and $i_1, \dots, i_r \in [n-1]$.

(P2) $\rho(\tilde{w}) = \rho(w) - r$.

(P3) $\tilde{w}(1+M) \dots \tilde{w}(k+M)$ is an induced p-occurrence for some $M \in [n-k]$ and $\{1+M, \dots, k+M\}$ is a connected component (or a union of connected components) in w.

IPACC has very close connections to the algorithm VEX presented by Tenner in [23]. VEX is constructed for permutations, reduced expressions and permutation patterns in general. In particular, VEX will not necessarily return an involution when given some $w \in I_n$. Our algorithm applies the ideas used there to involutions, their twisted expressions and induced patterns. IPACC will be the main tool in the proof of our actual result.

Algorithm 5.15. IPACC

Input: $w \in I_n$ with an induced occurrence $\langle p \rangle$ of $p \in I_k$, where p is 2143-avoiding
Output: $\tilde{w} \in I_n$ satisfying (P1), (P2) and (P3)

$w_{[0]} := w$ and $i := 0$
if there is an entry $x > \langle k \rangle$ which is not to the right of $\langle p \rangle$ **then**
  Let $S = \{y > \langle k \rangle : y \text{ is not to the right of } \langle p \rangle\}$.
  Consider the elements of S in decreasing order. Let $\underline{s}_{w(y)}\underline{s}_{w(y)+1} \cdots \underline{s}_{\langle p(k) \rangle - 1}$ act from the right on $w_{[i]}$ to move each element y to the right of $\langle p(k) \rangle$.
  Let $w_{[i+1]}$ be the resulting involution. Set $i := i + 1$.
end if
if there is an entry $x < \langle 1 \rangle$ which is not to the left of $\langle p \rangle$ **then**
  Let $S = \{y < \langle 1 \rangle : y \text{ is not to the left of } \langle p \rangle\}$.
  Consider the elements of S in increasing order. Let $\underline{s}_{w(y)-1}\underline{s}_{w(y)-2} \cdots \underline{s}_{\langle p(1) \rangle}$ act from the right on $w_{[i]}$ to move each element y to the right of $\langle p(k) \rangle$.
  Let $w_{[i+1]}$ be the resulting involution. Set $i := i + 1$.
end if
if there is no entry inside the pattern **then**
  return $w_{[i]}$
else
  Choose $x_{[i]}$ inside the pattern.
end if
Let $m \in [k-1]$ be the unique value such that $\langle m \rangle < x_{[i]} < \langle m+1 \rangle$.

18: **if** $\langle m+1\rangle$ is to the left of $x_{[i]}$ **then**
19: Let $\underline{s}_l$ act from the right on $w_{[i]}$ repeatedly for appropriate $l \in [x_{[i]}, \langle m+1\rangle)$ to increase $x_{[i]}$ and decrease $\langle m+1\rangle$ until $x_{[i]} > \langle m+1\rangle$.
20: Let $w_{[i]}$ be the resulting involution. Set $i := i+1$ and **goto** line 2.
21: **else if** $\langle m\rangle$ is to the right of $x_{[i]}$ **then**
22: Let $\underline{s}_l$ act from the right on $w_{[i]}$ repeatedly for appropriate $l \in [\langle m\rangle, x_{[i]})$ to decrease $x_{[i]}$ and increase $\langle m\rangle$ until $x_{[i]} < \langle m\rangle$.
23: Let $w_{[i]}$ be the resulting involution. Set $i := i+1$ and **goto** line 2.
24: **else if** $\langle m\rangle$, $x_{[i]}$ and $\langle m+1\rangle$ appear in increasing order in $w_{[i]}$ **then**
25: Define a and b as in Definition 5.14.
26: **if** $x_{[i]}$ is unobstructed to the right **then**
27: Let $\underline{s}_l$ act from the right on $w_{[i]}$ repeatedly for appropriate $l \in [w(x_{[i]}), \langle m+b\rangle)$ until $x_{[i]}$ is immediately to the right of $\langle m+b\rangle$ or the right neighbor of $x_{[i]}$ is $y > x_{[i]}$. In the latter case interchange the roles of $x_{[i]}$ and y and continue moving this new $x_{[i]}$ to the right. Note that this interchange can redefine the induced occurrence $\langle p\rangle$ if y is a pattern entry.
28: Let $w_{[i+1]}$ be the resulting permutation with the occurrence $\langle p\rangle$ possibly redefined after the interchange of roles and let $x_{[i+1]}$ be the non-pattern entry after the final move and any interchange of roles.
29: **else if** $x_{[i]}$ is unobstructed to the left **then**
30: Let $\underline{s}_l$ act from the right on $w_{[i]}$ repeatedly for appropriate $l \in [\langle m-a\rangle, w(x_{[i]}))$ until $x_{[i]}$ is immediately to the left of $\langle m-a\rangle$ or the left neighbor of $x_{[i]}$ is $y < x_{[i]}$. In the latter case interchange the roles of $x_{[i]}$ and y and continue moving this new $x_{[i]}$ to the left. Note that this interchange can redefine the induced occurrence $\langle p\rangle$ if y is a pattern entry.
31: Let $w_{[i+1]}$ be the resulting permutation with the occurrence $\langle p\rangle$ possibly redefined after the interchange of roles and let $x_{[i+1]}$ be the non-pattern entry after the final move and any interchange of roles.
32: **end if**
33: **if** $\langle 1\rangle < x_{[i+1]} < \langle k\rangle$ and $x_{[i]}$ is inside the pattern **then**
34: **goto** line 17 with $i := i+1$
35: **else**
36: **goto** line 2 with $i := i+1$.
37: **end if**
38: **end if**

Lemma 5.16. *The algorithm IPACC is correct.*

Proof. Firstly, we observe that IPACC stops by returning some $\tilde{w}$ if and only if the pattern p occurs at consecutive positions $1+M, \ldots, k+M$ for some $M \in [n-k]$ such that there is no $x \in [n]$ with x to the left of $\langle p\rangle$ and $\tilde{w}(x)$ to the right of $\langle p\rangle$. Because p is an induced pattern, $1+M, \ldots, k+M$ will be a connected component of $\tilde{w}$ (or a union of such if p is not connected). Thus, $\tilde{w}$ satisfies $(P3)$.

$(P1)$ holds by construction of $\tilde{w}$. Furthermore, every action of some $\underline{s}_i$ from the right on $w_{[i]}$ reduces the number of inversions and thus $(P2)$ holds.

We claim, that $w_{[i]}$ has an induced occurrence of p for every i. Every time we let some $\underline{s}_i$ act from the right on $w_{[i]}$ we exchange position or values of a non-pattern entry with an adjacent or lexicographically consecutive entry, respectively. However, this does not destroy the occurrence of the pattern p. When interchanging the roles of $x_{[i]}$ and y in line 27 or line 30 and when y is a pattern entry, then the condition $\langle m \rangle < x_{[i]} < \langle m+1 \rangle$ ensures that the occurrence of p is not destroyed.

Finally, IPACC is finite. Every time the conditions of line 2 or line 7 are satisfied, the algorithm reduces the number of non-pattern entries which are in a connected component together with some pattern entry. All other possible steps do not increase that number and can only be applied a finite number of times before the conditions of line 2 or line 7 are fulfilled again or the algorithm stops at line 13. □

Before using IPACC to prove a connection between induced patterns and reduced twisted expressions we want to demonstrate how the algorithm works for a simple example.

Example 5.17. Let $w = \mathbf{7}8\mathbf{9}6\mathbf{5}4\mathbf{123} \in I_9$ and $p = 45312 \in I_5$. Then IPACC may proceed as described in table 5.1. Throughout the example, the current occurrence of p is marked in bold in the one-line notation and with crosses in the corresponding figures.

IPACC returns $\tilde{w} = 453127698$ such that $\tilde{w}(1)\ldots\tilde{w}(5)$ is an induced occurrence of $p = 45312$. Furthermore, the algorithm tells us that

$$w = \tilde{w}\,\underline{s}_5\underline{s}_4\underline{s}_3\underline{s}_6\underline{s}_5\underline{s}_7\underline{s}_6\underline{s}_5\underline{s}_4\underline{s}_3.$$

Tenner's algorithm VEX is used to prove a result about the existence of certain reduced twisted expressions for a permutation $w \in S_n$ if w contains a 2143-avoiding pattern. We presented that result in Proposition 4.7. With the help of our algorithm IPACC we can now make a similar statement about induced 2143-avoiding patterns and reduced twisted expressions of involutions.

Proposition 5.18. *Let $p \in I_k$ be 2143-avoiding and let $w \in I_n$ contain an induced p-pattern. Then there exists a reduced twisted expression for w which begins with a shift of some reduced expression for p.*

Proof. Let the algorithm IPACC run on w and get $\tilde{w}$ satisfying (P1), (P2) and (P3). It follows from (P3) and Lemma 5.6 that there is a reduced twisted expression for $\tilde{w}$ beginning with a shift of a reduced twisted expression for p. (P1) and (P2) yield that this also is true for w. □

Finally, we want to remark the following. The involutions 4321, 45312 and 456123 are all 2143-avoiding and have reduced twisted expressions $\underline{s}_1\underline{s}_2\underline{s}_3\underline{s}_2$, $\underline{s}_1\underline{s}_2\underline{s}_4\underline{s}_3\underline{s}_2$ and $\underline{s}_1\underline{s}_3\underline{s}_5\underline{s}_2\underline{s}_4\underline{s}_3$, respectively. Thus, we can directly conclude from the previous proposition, that every involution having an induced 4321-, 45312- or 456123-pattern is not boolean.

line 1	$w_{[0]} = \mathbf{7}8\mathbf{9}6\mathbf{5}4\mathbf{1}2\mathbf{3}$
line 15	$x_{[0]} = 8$
line 17	$\langle m \rangle = 7$, $m = 4$
line 27	$a = 0$, $b = 1$
line 29	interchange roles of 8 and 9
line 30	$w_{[1]} = \mathbf{78}96\mathbf{5}4\mathbf{12}3$
	$x_{[1]} = 9$
line 3	$S = \{9\}$
line 5	$w_{[2]} = w_{[1]}\underline{s}_3\underline{s}_4\underline{s}_5\underline{s}_6\underline{s}_7$
	$w_{[2]} = \mathbf{67}5\mathbf{4312}98$
line 15	$x_{[2]} = 5$
line 17	$\langle m \rangle = 7$, $m = 4$
line 20	$w_{[3]} = w_{[2]}\underline{s}_5$
	$w_{[3]} = \mathbf{57}6\mathbf{41}3\mathbf{2}98$
line 15	$x_{[3]} = 6$
line 17	$\langle m \rangle = 5$, $m = 4$
line 20	$w_{[4]} = w_{[3]}\underline{s}_6$
	$w_{[4]} = \mathbf{567412}398$
line 3	$S = \{7\}$
line 5	$w_{[5]} = w_{[4]}\underline{s}_3\underline{s}_4\underline{s}_5$
	$w_{[5]} = \mathbf{45312}7698$
line 13	return $\mathbf{45312}7698$

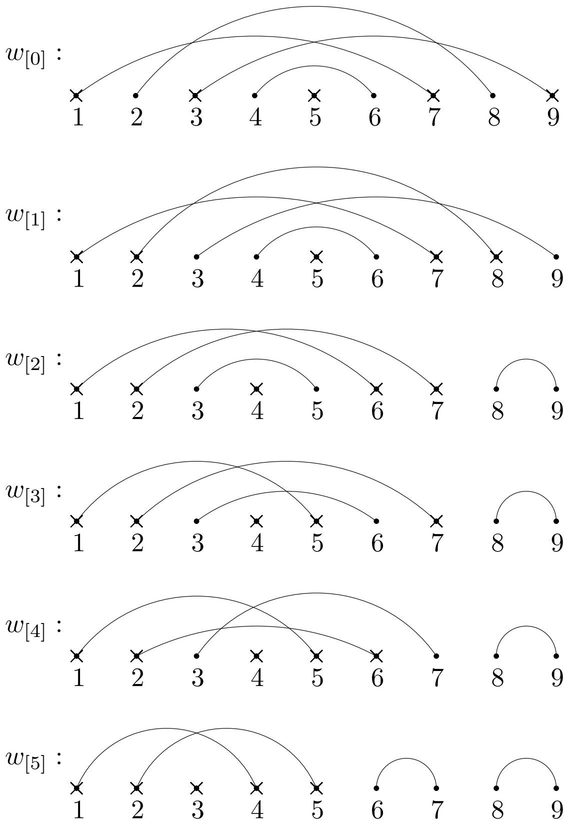


Table 5.1: IPACC runs for $w = 789654123$

In fact, in the proof of Proposition 5.10 we showed, that having an induced occurrence of one of those patterns is equivalent to having any occurrence of one of those (although not necessarily the same) patterns. Thus, the previous theorem can also be used as a tool for proving the characterization of boolean involutions in I_n.

5.4 Other Coxeter groups

The knowledge we gained in section 5.2 about boolean involutions in I_n can be used to classify boolean involutions in $\mathcal{I}(W, \mathrm{id})$ for some other W.

5.4.1 Signed boolean involutions

Consider $I_n^B := \mathcal{I}(S_n^B, \mathrm{id})$. Because S_n^B is a subgroup of the permutation group $S([\pm n])$, we can consider I_n^B as a subset of $I_{\pm n} := \mathcal{I}(S([\pm n]), \mathrm{id})$ in a natural way. We will denote the inclusion map from I_n^B into $I_{\pm n}$ by ϕ and the canonical generators of $S([\pm n])$ by $s'_i = (i, i+1)$ and $s'_{\underline{i}} = (-i, -i-1)$ for $i \in [n-1]$ as well as $s'_0 = (1, -1)$.

Lemma 5.19. *Let $w \in I_n^B$ and $w' = \phi(w) \in I_{\pm n}$. Then $\phi(w\underline{s}_0) = w'\underline{s}'_0$ and for $i \in [n-1]$*

$$\phi(w\underline{s}_i) = \begin{cases} w'\underline{s}'_i & \text{if } w'\underline{s}'_i = w'\underline{s}'_{\underline{i}}, \\ w'\underline{s}'_i\underline{s}'_{\underline{i}} & \text{otherwise.} \end{cases}$$

Proof. We have $\phi(\underline{s}_0) = \phi(s_0) = s'_0 = \underline{s}'_0$ and $\phi(\underline{s}_i) = \phi(s_i) = s'_i s'_{\underline{i}} = \underline{s}'_i \underline{s}'_{\underline{i}}$. Now the claim follows directly from Definition 3.13 about the action of the symbols $\underline{s}_i$ and $\underline{s}'_i$ on w and w', respectively. □

This means, that we can translate twisted expressions for $w \in I_n^B$ into certain twisted expressions for $w' \in I_{\pm n}$. From [4, Corollary 8.1.9] we know that ϕ is order-preserving. Thus, it is even true, that a reduced twisted expression for w is translated into a reduced twisted expression for w'. We can conclude the following

Lemma 5.20. *The involution $w \in I_n^B$ is boolean if and only if $\phi(w) = w' \in I_{\pm n}$ is boolean.*

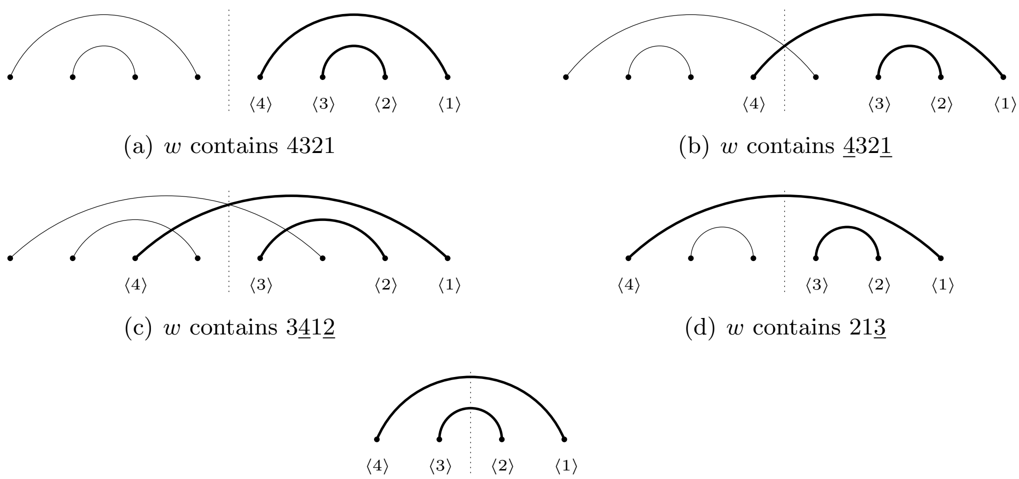
Proof. Consider any reduced twisted expression for w and the corresponding 'translation', which is a reduced twisted expression for w'. From the previous lemma we deduce that one of them contains repeated letters if and only if the other one contains repeated letters. □

Finally, we want to translate the notion of patterns in w' into signed patterns in w. We can already characterize the booleanness of $w' \in I_{\pm n} \cong I_{2n}$ in terms of avoiding the patterns 4321, 45312 and 456123. The following lemma will be the final step toward our desired result.

Lemma 5.21. *Let $w \in I_n^B$ and $w' = \phi(w) \in I_{\pm n}$. The involution w' avoids the patterns* 4321, 45312 *and* 456123 *if and only if w avoids all of the following signed patterns.*

4321	45312	456123
$\underline{1}\underline{2}$	$1\underline{3}\underline{2}$	$\underline{3}\underline{2}\underline{1}$
$21\underline{3}$	$42\underline{3}1$	$4\underline{3}\underline{2}1$
$3\underline{4}1\underline{2}$	$\underline{4}53\underline{1}2$	$45\underline{3}12$
$\underline{4}32\underline{1}$	$5\underline{4}3\underline{2}1$	$\underline{4}56\underline{1}23$
		$5\underline{4}6\underline{2}13$

Proof. "$\Leftarrow$". We know that w' contains 4321, 45312 or 456123 if and only if it has an induced occurrence of one of those three patterns. We will show that such an induced occurrence implies that w has an induced occurrence of one of the signed patterns listed in the lemma. Assume that w' contains an induced 4321-pattern. An occurrence of this pattern causes an induced pattern according to one of the graph representations in Figure 5.3, because the graph representation of w' is symmetric with respect to the perpendicular bisector of the segment between 1 and -1. In the figures the original induced occurrence is drawn with thick lines. Thus, w contains one of the signed patterns 4321, $\underline{4}32\underline{1}$, $3\underline{4}1\underline{2}$, $21\underline{3}$ or $\underline{12}$. Similarly, one can check that if w' does not contain 4321 but



(a) w contains 4321

(b) w contains $\underline{4}32\underline{1}$

(c) w contains $3\underline{4}1\underline{2}$

(d) w contains $21\underline{3}$

(e) w contains $\underline{12}$

Figure 5.3: Possible graph representations for w' which contains 4321

contains 45312 as induced pattern then w contains one of the signed patterns 45312, $1\underline{32}$, $42\underline{3}1$ and $\underline{4}53\underline{1}2$. Finally, we can deduce the existence of a signed $\underline{321}$-, $4\underline{32}1$-, $45\underline{3}12$-, $\underline{4}56\underline{1}23$ or $5\underline{4}6\underline{2}13$-pattern in w in the case that w' avoids 4321 and 45312 but contains 456123 as induced patterns.

"$\Rightarrow$". It is easy to check, that if w contains any of the signed patterns listed in the lemma then w' contains 4321, 45312 or 456123. We show this only for one such pattern, it follows in the same way for the others. Assume that w contains $21\underline{3}$. Then it follows from the definition of ϕ that w' contains $3\underline{12}21\underline{3}$, which then again contains 4321. $\square$

Proposition 5.22. *Let $w \in I_n^B$. Then w is boolean if and only if it avoids the signed patterns listed in Lemma 5.21.*

Proof. w is boolean $\Leftrightarrow$ there is a reduced twisted expression for w with no repeated letters $\Leftrightarrow$ there is a reduced twisted expression for w' with no repeated letters $\Leftrightarrow$ w' is boolean $\Leftrightarrow$ w' avoids the patterns 4321, 45312 and 456123 $\Leftrightarrow$ w avoids the signed patterns from Lemma 5.21. $\square$

5.4.2 Boolean involutions in I_n^D

Consider $I_n^D := \mathcal{I}(S_n^D, \mathrm{id}) \subset I_n^B$. This is the set of signed involutions having an even number of negative values in the window notation, of course partially ordered by the Bruhat order. Unfortunately, we cannot use the same methods as for I_n^B, because the corresponding natural injection $I_n^D \to I_{\pm n}$ does not map boolean elements of I_n^D to boolean ones in $I_{\pm n}$ in general. For example, $\underline{s}_0'\underline{s}_1 = [\underline{12}]$ is boolean in I_n^D but $\underline{s}_1\underline{s}_0\underline{s}_1 = 21\underline{12}$ is non-boolean in $I_{\pm 2}$.

Instead, we examine the structure of boolean involutions in I_n^D and in particular their window notations. From that we can conclude a list of signed patterns that have to be avoided by a boolean involution in I_n^D. Finally, we show that containing a signed pattern from that list is necessary for being non-boolean, too.

Proposition 5.23. *Let $w \in I_n^D$ be such that $\underline{s}_0' \le w$. Then w is boolean if and only if it has one of the following forms (in window notation).*

(D1) $[\underline{k}2\dots(k-1)\underline{1}\dots]$ *for* $k \ge 2$ *or* $[\underline{k}2\dots(k-2)l\underline{1}\dots]$ *for* $l > k \ge 3$.

(D2) $[1\underline{k}3\dots(k-1)\underline{2}\dots]$ *for* $k \ge 3$ *or* $[1\underline{k}3\dots(k-2)l\underline{2}\dots]$ *for* $l > k \ge 4$.

(D3) $[\underline{k32}4\dots(k-1)\underline{1}\dots]$ *for* $k \ge 4$ *or* $[\underline{k32}4\dots(k-2)l\underline{1}\dots]$ *for* $l > k \ge 5$.

(D4) $[\underline{1}2\dots(k-1)\underline{k}\dots]$ *for* $k \ge 2$ *or* $[\underline{1}2\dots(k-2)l\underline{k}\dots]$ *for* $l > k \ge 3$.

(D5) $[k\underline{32}4\dots(k-1)1\dots]$ *for* $k \ge 4$ *or* $[k\underline{32}4\dots(k-2)l1\dots]$ *for* $l > k \ge 5$.

Here the last ... always mean that the following entries are arbitrary but with no more negative entries than already indicated and such that w is 4321-, 45312 *and* 456123-*avoiding.*

Proof. "$\Rightarrow$". Let $w \in I_n^D$ be boolean and such that $\underline{s}_0' \le w$. Because $\underline{s}_0'$ occurs exactly once in every reduced expression for w and $\underline{s}_2$ is the only letter which does in general not commute with $\underline{s}_0'$ in twisted expressions, there is a reduced twisted expression for w which has $\underline{s}_0'$ either as first or as last letter.

Assume first, that w has a reduced expression with $\underline{s}_0'$ as its last letter. Then $w\underline{s}_0' \lhd w$ is a boolean involution in I_n and $w\underline{s}_0'$ has one of the following forms (in one-line notation where ... in the end means the same as in the statement of the proposition):

(i) $w\underline{s}_0' = 1k3\dots(k-1)2\dots$ for some $k \ge 2$ or $w\underline{s}_0' = 1k3\dots(k-2)l2\dots$ for some $l > k \ge 3$, in this case w has a window-notation as in case (D1) (remember Definition 3.13 about the action of $\underline{s}_0'$ from the right).

(ii) $w\underline{s}_0' = 21\dots$. Then w has a form as in case (D4) with $k = 2$.

(iii) $w\underline{s}_0' = k2\dots(k-1)1\dots$ for some $k \ge 3$ or $w\underline{s}_0' = k2\dots(k-2)l1\dots$ for some $l > k \ge 4$. Then w is as in case (D2).

(iv) $w\underline{s}_0' = 3k14\dots(k-1)2\dots$ for some $k \geq 4$ or $w\underline{s}_0' = 3k14\dots(k-2)l2\dots$ for some $l > k \geq 5$, which yields that w is as in case (D3).

If on the other hand w does not have a reduced twisted expression with $\underline{s}_0'$ in the end, then it will have a reduced twisted expression beginning with one of the following factors:

(i) $\underline{s}_0'\underline{s}_1\underline{s}_2$ or $\underline{s}_0'\underline{s}_1\underline{s}_3\underline{s}_2$. Then w has a form as in case (D4) for $k \geq 3$.

(ii) $\underline{s}_0'\underline{s}_3\underline{s}_2$ and $\underline{s}_1 \not\leq w$. Then w is as in case (D1) for $k = 3$ and $l > k$.

(iii) $\underline{s}_0'\underline{s}_3\underline{s}_2\underline{s}_1$. Then w is as in case (D5).

In all cases we see that w has one of the claimed forms.

"$\Leftarrow$". Assume that w is of the form (D1)-(D5). We actually exposed a reduced twisted expression for every case already in the first part and all of those expressions contain every letter at most once. Thus, in any case, w is boolean. $\square$

We can now state a list of signed patterns that a boolean $w \in I_n^D$ will avoid.

Proposition 5.24. *Let $w \in I_n^D$. Then w is boolean if and only if w avoids all of the following signed patterns.*

4321	45312	456123			
$1\underline{2}\underline{3}$	$12\underline{4}\underline{3}$	$\underline{1}32\underline{4}$	$21\underline{4}\underline{3}$	$3\underline{2}1\underline{4}$	$3\underline{4}1\underline{2}$
$4\underline{2}\underline{3}1$	$\underline{4}32\underline{1}$	$\underline{1}53\underline{4}2$	$15\underline{4}\underline{3}2$	$\underline{4}53\underline{1}2$	$52\underline{4}\underline{3}1$
$5\underline{4}3\underline{2}1$	$\underline{1}56\underline{4}23$	$\underline{4}56\underline{1}23$	$5\underline{4}6\underline{2}13$	$56\underline{4}\underline{3}12$	
$\underline{1234}$	$\underline{1243}$	$\underline{1324}$	$\underline{1342}$	$\underline{1423}$	$\underline{1432}$
$\underline{2134}$	$\underline{2143}$	$\underline{2314}$	$\underline{2341}$	$\underline{2413}$	$\underline{2431}$
$\underline{3124}$	$\underline{3142}$	$\underline{3214}$	$\underline{3241}$	$\underline{3412}$	$\underline{3421}$
$\underline{4123}$	$\underline{4132}$	$\underline{4213}$	$\underline{4231}$	$\underline{4312}$	
$1\underline{5432}$	$\underline{5}2\underline{431}$	$\underline{54}3\underline{21}$	$\underline{54}6\underline{21}3$	$\underline{5}6\underline{431}2$	$6\underline{5432}1$
$\underline{654321}$					

Proof. "$\Rightarrow$". Suppose that w is boolean. If $\underline{s}_0' \not\leq w$, then w will be boolean in I_n as well and thus avoid the patterns 4321, 45312 and 456123. It avoids all remaining patterns from above because it does not have any negative entries in the window-notation. If $\underline{s}_0' \leq w$, then w has one of the forms presented in Proposition 5.23. The reader may check, that indeed none of the listed patterns occurs as a pattern in any of the possible forms.

"$\Leftarrow$". Suppose that w avoids all patterns from the above list. By definition, $w \in S_n^D$ has an even number of negative entries. If w does not have any negative entries at all, then $w \in I_n$ and w is boolean, because it avoids 4321, 45312 and 456123.

Assume now that w has exactly two negative entries $\underline{i}$ and $\underline{j}$ with $i < j$ and that $\underline{i}$ is to the left of $\underline{j}$. It follows that $\underline{i}$ and $\underline{j}$ are in positions i and j because there are no other negative entries. As w avoids the signed patterns $1\underline{23}$, $3\underline{2}1\underline{4}$ and $4\underline{23}1$, we have $i = 1$. But w also avoids $\underline{1}32\underline{4}$, $\underline{1}53\underline{4}2$ and $\underline{1}56\underline{4}23$ which leads to the conclusion that w has a form as in (D4).

All other cases with w having exactly two negative entries can be treated similarly and always lead to w having one of the forms (D1)-(D5) from Proposition 5.23.

If w has exactly four negative entries, then those form the signed pattern $\underline{4321}$. Furthermore, w avoids $21\underline{43}$, $3\underline{4}1\underline{2}$, $\underline{4}32\underline{1}$, $1\underline{5432}$, $\underline{5}2\underline{431}$, $\underline{54}3\underline{21}$, $\underline{54}6\underline{21}3$, $\underline{5}6\underline{431}2$ and $6\underline{5432}1$ which yields that the first three entries $w(1)$, $w(2)$ and $w(3)$ of w are negative and w has a form as in (D3).

Finally, w does not have more than 4 negative entries. □

Our list of bad signed patterns is rather long and indeed we can shorten it a little. The list contains all signed patterns with exactly four entries and all of them negative except $\underline{4321}$. A signed permutation which avoids all those patterns does not contain any signed pattern with three and all of them negative entries except possibly $\underline{321}$ either. Conversely, if a signed permutation $w \in S_n^D$ avoids the patterns $\underline{123}$, $\underline{132}$, $\underline{213}$, $\underline{231}$ and $\underline{312}$ then w also avoids the patterns with four and all of them negative entries except possibly $\underline{4321}$. Thus, we can substitute the 23 signed patterns with four and all negative entries in our list by the five signed patterns with three and all negative entries from above.

Using similar arguments we can replace

- $3\underline{2}1\underline{4}$, $\underline{1}32\underline{4}$ and $21\underline{43}$ by $21\underline{3}$.
- $\underline{1}53\underline{4}2$, $3\underline{4}1\underline{2}$ and $52\underline{43}1$ by $31\underline{2}$.
- $\underline{4}32\underline{1}$, $\underline{4}53\underline{1}2$, $5\underline{4}3\underline{2}1$ by $32\underline{1}$.
- $\underline{1}56\underline{4}23$, $5\underline{4}6\underline{2}13$ and $56\underline{43}12$ by $4\underline{3}12$.
- $\underline{54}6\underline{21}3$ and $\underline{5}6\underline{431}2$ by $\underline{3}4\underline{21}$.

This reduces the list of signed patterns characterizing booleanness in I_n^D to the following.

Corollary 5.25. *Let $w \in I_n^D$. Then w is boolean if and only if it avoids all of the following signed patterns.*

4321	45312	456123	$1\underline{23}$	$21\underline{3}$	$31\underline{2}$
$32\underline{1}$	$\underline{123}$	$\underline{132}$	$\underline{213}$	$\underline{231}$	$\underline{312}$
$12\underline{43}$	$\underline{3}4\underline{21}$	$4\underline{23}1$	$4\underline{3}12$	$15\underline{43}2$	$1\underline{5432}$
$\underline{5}2\underline{431}$	$\underline{54}3\underline{21}$	$\underline{4}56\underline{1}23$	$6\underline{5432}1$	$\underline{654321}$	

5.4.3 Affine boolean involutions

As one more example, consider $\tilde{I}_n := \mathcal{I}(\tilde{S}_n, \mathrm{id})$, the set of affine involutions. This is an infinite poset but we will still be able to characterize booleanness of its elements by avoidance of finitely many patterns. However, the window notation introduced in Example 3.3 is not sufficient to decide if an element is boolean by pattern avoidance as the following example shows.

Example 5.26. Consider $w = [0\underline{1}65] = \underline{\tilde{s}}_4\tilde{s}_3\underline{\tilde{s}}_1\underline{\tilde{s}}_4 \in \tilde{I}_4$, which is minimal non-boolean. Then $w(1)w(2)w(3)w(4)$ contains exactly the same patterns as $v(1)v(2)v(3)v(4)$ where $v = [2143] = \underline{\tilde{s}}_1\underline{\tilde{s}}_3$ is boolean.

Let $w \in \tilde{I}_n$ and let $\underline{\tilde{s}}_{i_1} \ldots \underline{\tilde{s}}_{i_k}$ be a reduced twisted expression for w. We define the projection $\phi_m(w)$ of w into I_m (with respect to the chosen expression) for $m > n$ large by

$$\phi_m(w) := \prod_{l \in [k]} \prod_{0 \le p < \frac{m - i_l}{n}} \underline{\tilde{s}}_{i_l + pn}$$

where the product here means concatenation of the symbols. This means that we interprete the twisted expression for w as an infinite twisted expression using the generators $s_i = (i, i+1)$, $i \in \mathbb{Z}$, and throw away all generators with negative index or with index at least m. In particular, $\phi_m(w)$ depends on the choice of the reduced twisted expression for w in general. We make the following observations which follow directly from our definition of the projection ϕ_m:

- The affine involution $w \in \tilde{I}_n$ is boolean if and only if its projection $\phi_m(w)$ is boolean for all m.
- We have pointwise convergence of $\phi_m(w)$ to w on the set of positive integers, i.e. $(\phi_m(w))(l) \to w(l)$ for $m \to \infty$ for all $l > 0$.
- The sequence $w(1) \ldots w(l)$ contains 4321, 45312 or 456123 for some $l \in \mathbb{N}$ if and only if $\phi_m(w)$ contains one of those patterns for large m.

We conclude that w is boolean if and only if $w(1)w(2) \ldots w(l)$ avoids the patterns 4321, 45312 and 456123 for all $l \in \mathbb{N}$. This means, that we have to check if an infinite sequence avoids those three patterns. Fortunately, this can be done by looking at a finite subsequence only (for n fixed).

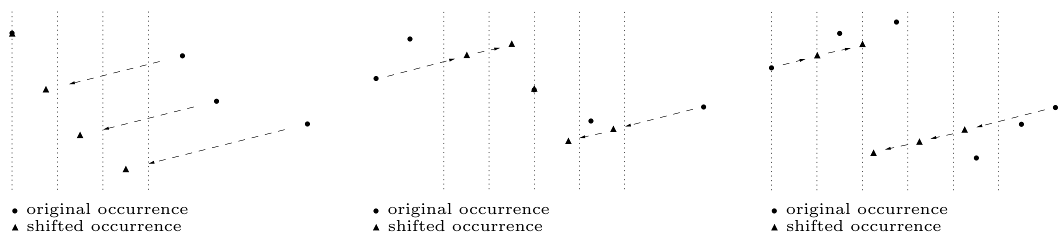
Lemma 5.27. *Let $w \in \tilde{S}_n$ and $n \ge 3$. Then, $w(1)w(2) \ldots w(k)$ avoids the patterns 4321, 45312 and 456123 for arbitrary large $k \in \mathbb{N}$ if and only if $w(1)w(2) \ldots w(6n)$ avoids them.*

Proof. "$\Rightarrow$" is trivial.
"$\Leftarrow$". Assume that $w(1) \ldots w(k)$ contains 4321 for some $k \in \mathbb{N}$. We show that we can choose an occurrence of 4321 in the first $4n$ entries. If the positions of $\langle 4 \rangle$ and $\langle 3 \rangle$ differ

by more than n then we can choose another occurrence of $\langle 321\rangle$ which is a multiple of n positions to the left of the original occurrence such that $\langle 4\rangle$ and $\langle 3\rangle$ lie "close enough". In the same way we can ensure that $\langle 2\rangle$ and $\langle 1\rangle$ are at most $2n$ respectively $3n$ positions to the right of $\langle 4\rangle$. Figure 4(a) shows the part of the diagram representation of w corresponding to the original and the "shifted" occurrence. We can finally choose the occurrence of 4321 such that $\langle 4\rangle$ is in position at most n. Then $w(1)\dots w(4n)$ contains 4321.

Similar methods are successful if $w(1)\dots w(k)$ contains 45312 or 456123. In Figure 4(b) and 4(c) we show how to choose the "shifted" occurrence of both patterns if the positions of the first and last pattern entry are too far from each other. Thus, the occurrence of 45312 and 456123 can be chosen such that $\langle 4\rangle$ is in position at most n and $\langle 2\rangle$ respectively $\langle 3\rangle$ in position at most $5n$ respectively $6n$.

□



(a) $\langle 4321\rangle$ (b) $\langle 45312\rangle$ (c) $\langle 456123\rangle$

Figure 5.4: Shifting occurrences in $w \in \tilde{S}_n$

We showed that we can dedice if an affine involution $w \in \tilde{I}_n$ is boolean by looking at the sequence $w(1)w(2)\dots w(6n)$.

Proposition 5.28. *The affine involution $w \in \tilde{I}_n$ with $n \geq 2$ is boolean if and only if $w(1)w(2)\dots w(6n)$ avoids the patterns* 4321, 45312 *and* 456123. □

6 Enumeration

Let P be a poset. We denote by $b(P)$ and $b_{\max}(P)$ the number of boolean elements in P and the number of maximal boolean elements in P, respectively.

$$\begin{aligned} b(P) &:= |\{x \in P : x \text{ is boolean}\}| \\ b_{\max}(P) &:= |\{x \in P : x \text{ is maximal boolean}\}| \end{aligned}$$

Furthermore, for $k \in \mathbb{N}$ and $f_1, \ldots, f_k : P \to \mathbb{N}$ given, let

$$b^{f_1,\ldots,f_k}(P, i_1, \ldots, i_k) := |\{x \in P : x \text{ is boolean}, f_1(x) = i_1, \ldots, f_k(x) = i_k\}|$$

be the number of boolean elements with respect to the statistics $f_1, \ldots, f_k$.

We will derive formulas for $b(P)$ with $P \in \{I_n, I_n^B, I_n^D, \tilde{I}_n\}$ and see, that $b_{\max}(P)$ plays a significant role. For I_n we can give some more detailed statistics.

6.1 Boolean involutions in $\mathbf{I}_n$

The reader is by now well-acquainted with the combinatorial structure of I_n. It should not be a big surprise that this knowledge enables us to compute the number of boolean elements of I_n in total and with respect to the statistics ρ, exc and inv.

Proposition 6.1. *The number of boolean involutions in I_n can be computed recursively by $b(I_1) = 1$, $b(I_2) = 2$, $b(I_3) = 4$ and*

$$b(I_n) = 2b(I_{n-1}) + b(I_{n-2}) - b(I_{n-3})$$

for all $n \geq 4$. In particular, the sequence $b(I_n)$ can be found in [19, Sequence A052534].

Proof. All involutions in I_n are boolean for $n = 1, 2, 3$. Thus, $b(I_1) = |I_1| = 1$, $b(I_2) = |I_2| = 2$ and $b(I_3) = |I_3| = 4$.

We will count the number of boolean involutions in I_n depending on the entries at positions $n, n-1, n-2$. For that, let

$$\begin{aligned} q(n) &:= |\{w \in I_n : w \text{ boolean}, w(n) = n\}|, \\ r(n) &:= |\{w \in I_n : w \text{ boolean}, w(n) = n-1\}|, \\ s(n) &:= |\{w \in I_n : w \text{ boolean}, w(n) \neq n,\ w(n-1) = n-1\}|, \\ t(n) &:= |\{w \in I_n : w \text{ boolean}, w(n) = n-2,\ w(n-1) \neq n-1\}|. \end{aligned}$$

Because of the characterization of boolean elements in I_n, we have $b(I_n) = q(n) + r(n) + s(n) + t(n)$. Obviously the number of boolean involutions $w \in I_n$ with $w(n) = n$ is equal to the number of boolean involutions $w' \in \mathrm{I}_{n-1}$, i.e. $q(n) = b(I_{n-1})$. Using similar arguments, we deduce

$$\begin{aligned} r(n) &= b(I_{n-2}) \\ s(n) &= b(I_{n-1}) - q(n-1) \\ t(n) &= s(n-1) \end{aligned}$$

for $n \geq 4$ and compute

$$b(I_n) = 2b(I_{n-1}) + b(I_{n-2}) - b(I_{n-3}).$$

□

n	1	2	3	4	5	6	7	8	9	10
$b(I_n)$	1	2	4	9	20	45	101	227	510	1146

Table 6.1: Values of $b(I_n)$ for $n \leq 10$

The concept of generating functions is explained in [21]. We note that the method of computing the generating function of a sequence from its recursion formula used there can be applied to sequences with higher dimensional indices as well. From the recursion formula and the values for $n = 1, 2, 3$ it is straightforward to compute the generating function of $b(I_n)$ as follows.

Corollary 6.2. *The generating function of $b(I_n)$ is*

$$F(x) = \sum_{n \geq 1} b(I_n) x^n = \frac{x(1 - x^2)}{1 - 2x - x^2 + x^3}$$

□

We found a recursive formula for the number of boolean involutions in I_n. The corresponding sequence is referred to as the number of Motzkin paths with certain properties in [19]. We will present a bijection between those paths and the boolean involutions in I_n. This yields another proof of our formula.

A *lattice path* from v_1 to v_n is a sequence $L = (v_1, \dots, v_n)$ where $v_i \in \mathbb{N}^2$ and $v_{i+1} - v_i$ is in a certain set of allowed steps (for example $\{(1,0), (0,-1)\}$ in [21, Section 2.7]).

Definition 6.3. *A* Motzkin path *of length n is a lattice path from $(0,0)$ to $(n,0)$ with steps $(1,0)$, $(1,1)$, and $(1,-1)$ that never goes below the x-axis. Letting l, u, and d represent the steps $(1,0)$, $(1,1)$, and $(1,-1)$ respectively, we encode such a path with a word over $\{l, u, d\}$ (where the k-th letter of the word corresponds to the k-th step in the path). The set of Motzkin paths of length n is denoted by M_n.*

Let $M_n^r \subseteq M_n$ denote the set of Motzkin paths of length n with $(1,0)$ steps occurring only on level at most 1 and which never go higher than level 2. We call a path in M_n^r a *restricted Motzkin path* of length n.

Proposition 6.4. *It holds that $b(I_n) = |M_n^r|$ for all $n \in \mathbb{N}$.*

Proof. We establish a bijection between the boolean involutions in I_n and the restricted Motzkin paths of length n. Consider the mapping $\phi : I_n \to M_n$ which maps an involution

w to the Motzkin path $\phi(w)$ with l,u and d as k-th letter if $w(k)$ is a fixed point, an excedance or a deficiency, respectively.

For every $w \in I_n$, $\phi(w)$ is a lattice path by definition. It goes from $(0,0)$ to $(n,0)$, because w has the same number of excedances and deficiencies in w, and it obviously doen not go below the x-axis. Thus, $\phi(w)$ is a Motzkin path for all $w \in I_n$ and ϕ is well-defined.

Assume that the k-th step of $\phi(w)$ is $(1,0)$ and on level p (i.e. it goes from $(k-1,p)$ to (k,p)). Then there are exactly p elements l in $[k-1]$ such that $w(l) > k$. If $p > 1$ there are $l_1, l_2 \in [k-1]$ such that $w(l_1) > k$ and $w(l_2) > k$. Assuming $l_1 < l_2$, we have $l_2 < k < w(l_1)$. Thus, if $\phi(w)$ is a path with a $(1,0)$ step on level 2 or higher, then w is not boolean. Similarly, it follows that if $\phi(w)$ goes to a level > 2, then w is not boolean. Therefore every boolean involution is mapped to a restricted Motzkin path and

$$\phi(\{w \in I_n : w \text{ is boolean}\}) \subseteq M_n^r.$$

For showing the reverse inclusion, fix a restricted Motzkin path. We construct an involution $w \in I_n$ such that $\phi(w)$ is exactly this path. For $k \in [n]$ define $w(k) = k$ if the k-th letter of the corresponding Motzkin word is l. If the k-th letter is u or d and it is the m-th occurrence of u or d, respectively, then define $w(k) = p$ where p is such that the p-th letter in the word is the m-th occurrence of d or u, respectively. This obviously defines a unique involution in I_n. Observe that the given restrictions on the Motzkin path ensure that the constructed involution is boolean. This proves $\phi(I_n) = M_n^r$.

Observe that by Theorem 5.11 a boolean involution is uniquely determined by its sets of excedances and deficiencies. Thus, ϕ yields a bijection between the boolean elements of I_n and M_n^r. □

Furthermore, we can compute the rank $\rho(w)$ of a boolean $w \in I_n$ from the restricted Motzkin path $\phi(w)$.

Proposition 6.5. *Let $w \in I_n$ be boolean and let $\phi(w)$ be the corresponding restricted Motzkin path as defined above. Let $a(w)$ denote the area of the region lying below $\phi(w)$ and between the lines $y = 0$ and $y = 1$. Then it holds that $\rho(w) = a(w)$.*

Proof. Because $w \in I_n$ is boolean, no reduced twisted expression for w contains any letter more than once. We conclude $\rho(w) = \{k \in [n-1] : s_k = \underline{s}_k \leq w\}$.

On the other hand, we note that $a(w) = (n-1) - |\{k \in [n-1] : (k,0) \in \phi(w)\}|$. From the definition of $\phi(w)$ we conclude that $(k,0) \in \phi(w)$ if and only there is no $i \in [n]$ with $i \leq k$ and $w(i) > k$. This is the case if and only if $s_k \not\leq w$. Thus, $a(w) = (n-1) - |\{k \in [n-1] : s_k \not\leq w\}| = |\{k \in [n-1] : s_k \leq w\}| = \rho(w)$. □

Example 6.6. Consider the boolean involution $4261573598 \in I_9$ with graph representation as in Figure 1(a). Then $\phi(w)$ is shown in Figure 1(b) and the gray shaded area determines $\rho(w) = a(w) = 7$.

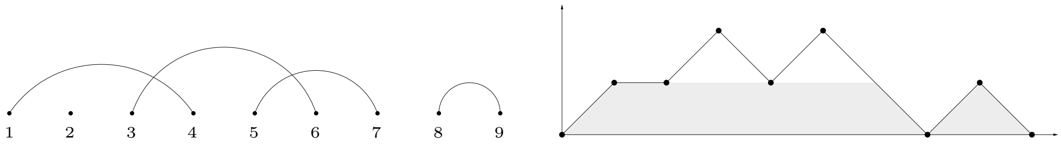


Figure 6.1: A boolean involution and the corresponding restricted Motzkin path

In a completely analogous way to Proposition 6.1 it is possible to count boolean involutions in I_n with respect to the most natural statistics ρ, inv and exc. We will state those results without their proofs, as all of them build on the same strategy as the proof of Proposition 6.1 and use knowledge about how shrinking or deleting cycles changes the rank or number of inversions and excedances.

With respect to the rank ρ we get the recursion

$$b^\rho(I_n, k) = b^\rho(I_{n-1}, k) + b^\rho(I_{n-1}, k-1) + b^\rho(I_{n-2}, k-2) - b^\rho(I_{n-3}, k-2) \quad (6.1)$$

for $n \geq 4$ and $k \geq 2$. The values of b^ρ for $n < 4$ or $k < 2$ can be found in table 6.2. The generating function of $b^\rho(I_n, k)$ is

$$F^\rho(x, y) = \sum_{n\geq 1}\sum_{k\geq 0} b^\rho(I_n, k)\, x^n y^k = \frac{x(1-x^2y^2)}{(1-x^2y^2)(1-x) - xy}.$$

$b^\rho(I_n, k)$	$k = 0$	$k = 1$	$k \geq 2$
$n = 1$	1	0	0
$n = 2$	1	1	0
$n = 3$	1	2	1
$n \geq 4$	1	$n-1$	equation (6.1)

Table 6.2: Distribution of $b^\rho(I_n, k)$ for small n and k

$b^{\text{inv,exc}}(I_n, k, l)$	$k = 0$ $l = 0$	$k = 1$ $l = 1$	$k = 2$ $l = 2$	$k = 3$ $l = 1$
$n = 1$	1	0	0	0
$n = 2$	1	1	0	0
$n = 3$	1	2	0	1
$n \geq 4$	1	$n-1$	$\frac{1}{2}(n^2-5n+6)$	equation (6.2)

Table 6.3: Distribution of $b^{\text{inv,exc}}(I_n, k, l)$ for small n, k and l

The recursion with respect to the number of inversions and excedances is

$$\begin{aligned}b^{\mathrm{inv,exc}}(I_n,k,l) = b^{\mathrm{inv,exc}}(I_{n-1},k,l) + b^{\mathrm{inv,exc}}(I_{n-2},k-1,l-1)\\ + b^{\mathrm{inv,exc}}(I_{n-1},k-2,l) - b^{\mathrm{inv,exc}}(I_{n-2},k-2,l)\\ + b^{\mathrm{inv,exc}}(I_{n-2},k-3,l-1) - b^{\mathrm{inv,exc}}(I_{n-3},k-3,l-1) \quad (6.2)\end{aligned}$$

for $n \geq 4$, $k \geq 3$ and $l \geq 1$. Table 6.3 contains the non-zero values of $b^{\mathrm{inv,exc}}$ for smaller n, k or l. We compute the corresponding generating function as

$$\begin{aligned}F^{\mathrm{inv,exc}}(x,y,z) &= \sum_{n\geq 1}\sum_{k\geq 0}\sum_{l\geq 0} b^{\mathrm{inv,exc}}(I_n,k,l)x^n y^k z^l\\ &= \frac{x^2yz + x - x^2y^2 - x^3y^3z}{1 - x - x^2yz - xy^2 + x^2y^2 - x^2y^3z + x^3y^3z}.\end{aligned}$$

Actually, we could have deduced $F^{I,\rho}$ from $F^{I,\mathrm{inv,exc}}$ and the formula for the rank function of I_n saying $\rho(w) = \frac{1}{2}(\mathrm{inv}(w) + \mathrm{exc}(w))$. Indeed, we have

$$F^{\rho}(x,y) = F^{\mathrm{inv,exc}}(x,\sqrt{y},\sqrt{y})$$

and by definition

$$F(x) = F^{\rho}(x,1) = F^{\mathrm{inv,exc}}(x,1,1).$$

6.2 Counting reduced twisted expressions

We want to develop a more general method to compute the number of boolean involutions of a Coxeter group W. Remember that W_T denotes the standard parabolic subgroup of the Coxeter group W generated by $T \subseteq S$. In particular, $W_S = W$.

Proposition 6.7. *Let (W,S) be a Coxeter system. Then*

$$b(\mathcal{I}(W,id)) = \sum_{T\subseteq S} b_{\max}((\mathcal{I}(W_T,id)))$$

and

$$b^{\rho}(\mathcal{I}(W,id),k) = \sum_{T\subseteq S,|T|=k} b_{\max}((\mathcal{I}(W_T,id))).$$

Proof. Every boolean involution $w \in \mathcal{I}(W,\mathrm{id})$ determines the set $T \subseteq S$ of generators which is used in the reduced twisted expressions for w. We have $w \in \mathcal{I}(W_T,\mathrm{id})$ and that w is a maximal boolean involution in $\mathcal{I}(W_T,\mathrm{id})$. Thus, every boolean involution is a maximal boolean involution in some parabolic subgroup of W. Of course, the converse is also true. Furthermore, w is of rank k in $\mathcal{I}(W,\mathrm{id})$ if and only if it is maximal in some $\mathcal{I}(W_T,\mathrm{id})$ with $|T| = k$. □

In Proposition 6.1 and equation (6.1) we deduced a recursive formula for the number of (maximal) boolean elements in I_n using the combinatorial structure of the involved permutations. However, we do not have such a nice combinatorial description as for I_n for the more complex cases of signed and affine involutions. Instead, we need to work with reduced twisted expressions to derive formulas for the number of boolean involutions.

We will apply Proposition 6.7 to the Coxeter groups S_n, S_n^B, S_n^D and $\tilde{S}_n$. Therefore we have to find the number of maximal boolean involutions of those groups and their standard parabolic subgroups.

It is easy to see from the Coxeter graph of S_n that all standard parabolic subgroups of S_n are products of smaller symmetric groups. Similarly, the standard parabolic subgroups of S_n^B and S_n^D are products of smaller groups of type A, B or D. Thus, it suffices to know the number of maximal boolean elements of I_m, I_m^B and I_m^D for arbitrary $m \in \mathbb{N}$ and this is what we compute first.

Let F_k denote the k-th Fibonacci number defined by $F_1 = F_2 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for all $k \geq 3$. It will prove useful later to let $F_k = 0$ for all $k \leq 0$.

Proposition 6.8. *The numbers of maximal boolean elements of I_n, I_n^B and I_n^D are as follows.*

(i) $b_{\max}(I_n) = F_{n-1}$ *for all* $n \geq 2$ *and* $b_{\max}(I_1) = 1$,

(ii) $b_{\max}(I_n^B) = F_{n+1}$ *for all* $n \geq 1$,

(iii) $b_{\max}(I_n^D) = 2F_{n-1} + F_{n-3}$ *for all* $n \geq 3$ *and* $b_{\max}(I_1^D) = b_{\max}(I_2^D) = 1$.

Proof. (i). We will count maximal boolean involutions by counting certain reduced twisted expressions. Every involution has a lexicographically first reduced twisted expression, where we consider the set of generators $s_1, \ldots, s_{n-1}$ as lexicographically ordered. We will see that the lexicographically first reduced twisted expressions of maximal boolean involutions have quite a special form.

Let $w \in I_n$ be maximal boolean and let $\underline{s}_{i_1} \cdots \underline{s}_{i_{n-1}}$ be a reduced and lexicographically first expression for w. Thus, we have $i_k \leq i_{k+1} + 1$ for all $k \in [n-2]$, because otherwise we would get a lexicographically smaller expression by exchanging the positions of s_{i_k} and $s_{i_{k+1}}$. (This would be possible because both would commute in that case.) Assume that we have $i_k > i_{k+1}$ for some $k \in [n-2]$ such that s_{i_k} and $s_{i_{k+1}}$ commute with $\underline{s}_{i_1} \cdots \underline{s}_{i_{k-1}}$. Then we have

$$\begin{aligned}
\underline{s}_{i_1} \cdots \underline{s}_{i_{k+1}} &= s_{i_{k+1}} \underline{s}_{i_1} \cdots \underline{s}_{i_{k-1}} s_{i_k} s_{i_{k+1}} \\
&= s_{i_k} \underline{s}_{i_1} \cdots \underline{s}_{i_{k-1}} s_{i_{k+1}} s_{i_k} \\
&= \underline{s}_{i_1} \cdots \underline{s}_{i_{k-1}} \underline{s}_{i_{k+1}} \underline{s}_{i_k}
\end{aligned}$$

This is a contradiction to our assumption because we constructed a lexicographically smaller reduced expression. From the last observation we can deduce, that there is no decreasing subsequence $i_k > i_{k+1} > i_{k+2}$ of length 3 in $\underline{s}_{i_1} \dots \underline{s}_{i_{n-1}}$ and that $i_1 = 1$. The reader may observe that the reduced expressions which we constructed in the proof of Proposition 5.8 are exactly the lexicographically first ones.

Consider the lexicographically first reduced twisted expressions for maximal boolean involutions in I_n with $n \geq 4$ which have $\underline{s}_{n-1}$ as last letter and observe that they are in bijection with corresponding expressions for elements of I_{n-1} by deleting the last letter. Similarly, we have a bijection between the expressions for elements in I_n which have $\underline{s}_{n-1}\underline{s}_{n-2}$ as last two letters (in this order) and those of elements in I_{n-2}. But there are no more possibilities for the last letters because of our characterization of lexicographically minimal expressions. Thus, $b_{\max}(I_n) = b_{\max}(I_{n-1}) + b_{\max}(I_{n-2})$ for $n \geq 4$. It is easy to check that $b_{\max}(I_1) = b_{\max}(I_2) = b_{\max}(I_3) = 1$.

(ii). Analogous to *(i)* with the exception that the given recursion already holds for $n \geq 3$. The initial values are $b_{\max}(I_1^B) = 1 = F_2$ and $b_{\max}(I_2^B) = 2 = F_3$.

(iii). Let $n \geq 3$. We count the number of maximal boolean $w \in I_n^D$ having a reduced twisted expression ending with $\underline{s}_0'$ first. This is the case if and only if $w\underline{s}_0'$ is a maximal boolean element of I_n and we can conclude that there are exactly $b_{\max}(I_n)$ of those $w \in I_n^D$. If $w \in I_n^D$ is maximal boolean and does not have a reduced twisted expression ending with $\underline{s}_0'$, it will have a reduced twisted expression starting with $\underline{s}_0'\underline{s}_1$ or $\underline{s}_0'\underline{s}_3\underline{s}_2\underline{s}_1$. The first case is counted by $b_{\max}(I_{n-2}^B)$ and the latter one by $b_{\max}(I_{n-4}^B)$ as we can interpret the words without the first letters from above as maximal boolean elements of I_{n-2}^B or I_{n-4}^B after shifting all indices by -2 and -4 respectively. We conclude

$$b_{\max}(I_n^D) = b_{\max}(I_n) + b_{\max}(I_{n-2}^B) + b_{\max}(I_{n-4}^B) = F_{n-1} + F_{n-1} + F_{n-3}$$

for all $n \geq 3$. Of course, it holds that $b_{\max}(I_1^D) = b_{\max}(I_2^D) = 1$. □

We can now compute the number of boolean elements of I_n, I_n^B and I_n^D using Proposition 6.7. It makes the expressions much nicer if we let $b(I_0) = 1$ so we do this here although we did not take $n = 0$ into account in the previous computations of $b(I_n)$.

Proposition 6.9. *The numbers of boolean elements of I_n, I_n^B and I_n^D are*

(i) $b(I_n) = b(I_{n-1}) + \sum_{k=2}^{n} F_{k-1} b(I_{n-k})$ *for all* $n \geq 1$,

(ii) $b(I_n^B) = \sum_{k=0}^{n} F_{k+1} b(I_{n-k})$ *for all* $n \geq 1$,

(iii) $b(I_n^D) = 2b(I_n) - b(I_{n-1}) + b(I_{n-2}) + \sum_{k=3}^{n}(2F_{k-1} + F_{k-3}) b(I_{n-k})$ *for all* $n \geq 2$.

Proof. *(i)*. Let $(I_n)_T = \mathcal{I}((S_n)_T, \mathrm{id})$ and $i_T := \min(\{i : s_i \notin T\} \cup \{n\})$. Then

$$\begin{aligned} b(I_n) &= \sum_{T \subseteq S} b_{\max}((I_n)_T) \\ &= \sum_{i=1}^{n} \sum_{T \subseteq S, i_T = i} b_{\max}((I_n)_T). \end{aligned}$$

If s_i is the first generator missing, then every element is a product of a maximal element of the subgroup generated by the first $i-1$ generators and an element of the subgroup generated by the last $n-i$ generators. Hence,

$$\begin{aligned}
b(I_n) &= \sum_{i=1}^{n} b_{\max}(I_i) \sum_{T \subseteq \{s_{i+1},\dots,s_{n-1}\}} b_{\max}((I_n)_T) \\
&= \sum_{i=1}^{n} b_{\max}(I_i) b(I_{n-i}) \\
&= b(I_{n-1}) + \sum_{i=2}^{n} F_{i-1} b(I_{n-i}).
\end{aligned}$$

(ii). Follows in the same way as (i).

(iii). Let $n \geq 2$. With i_T as defined in (i), we compute

$$\begin{aligned}
b(I_n^D) &= \sum_{T \subseteq S} b_{\max}((I_n^D)_T) \\
&= \sum_{T \subseteq S \setminus \{s_0'\}} b_{\max}((I_n^D)_T) + \sum_{T \subseteq S \setminus \{s_1\}} b_{\max}((I_n^D)_T) \\
&\quad - \sum_{T \subseteq S \setminus \{s_0', s_1\}} b_{\max}((I_n^D)_T) \\
&\quad + \sum_{k=2}^{n} \sum_{\{s_0', s_1\} \subseteq T \subseteq S, i_T = k} b_{\max}((I_n^D)_T) \\
&= b(I_n) + b(I_n) - b(I_{n-1}) + \sum_{k=2}^{n} b_{\max}(I_k^D) b(I_{n-k}) \\
&= 2b(I_n) - b(I_{n-1}) + b(I_{n-2}) + \sum_{k=3}^{n} (2F_{k-1} + F_{k-3}) b(I_{n-k}).
\end{aligned}$$

□

We have seen that we can compute $b(I_n^B)$ and $b(I_n^D)$ from the numbers $b(I_m)$ for $m = 1, \dots, n$. However, this is not very useful as long as we do not have an explicit formula for $b(I_m)$ which is easy to compute. Instead it turns out that $b(I_n^B)$ and $b(I_n^D)$ fulfill the same recursion as $b(I_n)$ which we deduced in Proposition 6.1. This is not just accidental. Recall that for a graph $G = (V, E)$ and a vertex $v \in V$, the set of neighbors of v is denoted by $N(v)$ following [5].

Proposition 6.10. *Let W be a Coxeter group with Coxeter graph G and set of generators $S = \{s_1, \dots, s_n\}$. Let $s_n, s_{n-1}, s_{n-2} \in S = V(G)$ be such that $N(s_n) = \{s_{n-1}\}$ and $N(s_{n-1}) = \{s_n, s_{n-2}\}$. Then G is as in Figure 6.2. Write $W_i = W_{S \setminus \{s_{i+1},\dots,s_n\}}$ for the* *standard parabolic subgroup generated by all generators except $s_{i+1}, \dots, s_n$ and $f(i) = b(\mathcal{I}(W_i, id))$ for the number of boolean involutions in W_i. Then it holds that*

$$f(n) = 2f(n-1) + f(n-2) - f(n-3).$$

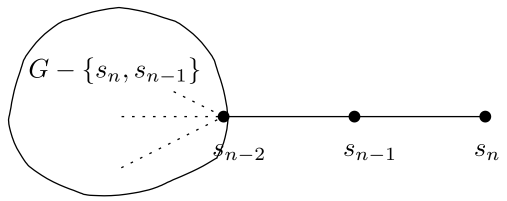


Figure 6.2: Coxeter graph of W_n

Proof. Let $w \in \mathcal{I}(W_n, \mathrm{id})$ be boolean. If $s_n \not\leq w$ then w is a boolean involution in W_{n-1}. There are exactly $f(n-1)$ such w. Otherwise, consider the lexicographically first reduced twisted expression for w. If this expression ends with $\underline{s}_n$ then $w\underline{s}_n \in \mathcal{I}(W_{n-1}, \mathrm{id})$. The converse is also true. Thus, those w are counted by $f(n-1)$, too. If the expression does not end with $\underline{s}_n$, it ends with $\underline{s}_n\underline{s}_{n-1}$. This is the case if and only if $w\underline{s}_{n-1}\underline{s}_n \in \mathcal{I}(W_{n-2}, \mathrm{id}) \backslash \mathcal{I}(W_{n-3}, \mathrm{id})$. There are $f(n-2) - f(n-3)$ such w. It now follows directly that

$$f(n) = 2f(n-1) + f(n-2) - f(n-3).$$

□

Corollary 6.11. *It holds that*

$$b(I_n^B) = 2b(I_{n-1}^B) + b(I_{n-2}^B) - b(I_{n-3}^B)$$

for all $n \geq 4$ and that

$$b(I_n^D) = 2b(I_{n-1}^D) + b(I_{n-2}^D) - b(I_{n-3}^D)$$

for all $n \geq 5$. □

Obviously the previous proposition does not help for $\tilde{I}_n$. The situation is a little different here because the graph of $\tilde{S}_n$ is not a tree but a cycle. Nevertheless, the method of counting reduced expressions still works.

Proposition 6.12. *The number of maximal boolean elements in $\tilde{I}_n$ is given by*

$$b_{\max}(\tilde{I}_n) = F_{n-1} + F_{n+1} - 1$$

for all $n \geq 3$.

Proof. Let $n \geq 3$ and let

$$k = \max\{i \in [n] : w(i) > i+1, w(i+1) < i\}.$$

If there is a reduced twisted expression for w ending with $\underline{\tilde{s}}_l$, then $w(l) > l+1$ and $w(l+1) < l$ because w is maximal boolean. This means that k is well-defined and $1 \leq k \leq n$.

If $k \geq 2$ then the lexicographically first reduced twisted expression for w is given by $\underline{\tilde{s}}_{i_1} \dots \underline{\tilde{s}}_{i_{k-1}} \underline{\tilde{s}}_n \dots \underline{\tilde{s}}_k$ where $v = \underline{s}_{i_1} \dots \underline{s}_{i_{k-1}}$ is the lexicographically first reduced twisted expression for a maximal boolean involution $v \in I_k$.

Conversely, if $k = 1$ then $\underline{\tilde{s}}_{i_1} \dots \underline{\tilde{s}}_{i_{n-1}} \underline{\tilde{s}}_1$ is the lexicographically first reduced twisted expression for w where $v = \underline{s}_{i_1} \dots \underline{s}_{i_{n-1}}$ with $\{i_1, \dots, i_{n-1}\} = \{2, \dots, n\}$ is the lexicographically first twisted reduced expression for a maximal boolean involution $v \in I_{\{2,\dots,n+1\}}$.

We conclude

$$\begin{aligned} b_{\max}(\tilde{I}_n) &= \sum_{k=2}^{n} b_{\max}(I_k) + b_{\max}(I_n) = \sum_{k=2}^{n} F_{k-1} + F_{n-1} \\ &= F_{n-1} + \sum_{k=1}^{n-1} F_k = F_{n-1} + F_{n+1} - 1 \end{aligned}$$

where the last step follows from the identity $\sum_{i=1}^{m} F_i = F_{m+2} - 1$. □

Corollary 6.13. *The number of boolean elements of $\tilde{I}_n$ is given by*

$$b(\tilde{I}_n) = F_{n-1} + F_{n+1} - 1 + \sum_{k=0}^{n-1} \frac{n}{n-k} b^{\rho}(I_n, k)$$

for all $n \geq 3$.

Proof. Let $T \subset S$ and $s_i \notin T$. Then $\tilde{I}_n^T := \mathcal{I}((\tilde{S}_n)_T, \mathrm{id})$ is isomorphic to a subposet of $I_{\{i+1,\dots,i+n+1\}} \cong I_n$ and every maximal boolean element $w \in \tilde{I}_n^T$ can be identified with a boolean element of I_n. If $|T| = k$ then s_i can be chosen in $n-k$ different ways. Applying an idea used in [21, Lemma 2.3.4] we conclude

$$(n-k) \sum_{T \subseteq S, |T|=k} b_{\max}(\tilde{I}_n^T) = n b^{\rho}(I_n, k)$$

for all $k = 0, \dots, n-1$ and thus

$$b(\tilde{I}_n) = b_{\max}(\tilde{I}_n) + \sum_{k=0}^{n-1} \sum_{T \subset S, |T|=k} b_{\max}(\tilde{I}_n^T) = F_{n-1} + F_{n+1} - 1 + \sum_{k=0}^{n-1} \frac{n}{n-k} b^{\rho}(I_n, k).$$

□

7 Conclusion and perspectives

We showed that the property of having a boolean lower order ideal in the partially ordered set of involutions induced by the Bruhat order can be completely characterized by pattern avoidance for Coxeter groups of type A, B and D. The main theorem has been proved using the combinatorial properties of the symmetric group. It implies the result for the signed permutations. The even signed permutations have been treated a little differently. We can even give some characterization for type $\tilde{A}$.

A similar question arises for twisted involutions. There are exactly two graph automorphisms of S_n and the non-trivial one is given by $\theta_0 : S_n \to S_n$ with $\theta_0(w) = w_0 w w_0$ for all $w \in W$. There is also exactly one non-trivial order-preserving automorphism of S_n^D induced by the graph automorphism θ_1 mapping s_0' to s_1 and having all other vertices as fixed points. In both cases, we can ask if the boolean twisted involutions can be characterized via pattern avoidance.

Problem 7.1. Can the boolean twisted involutions of $\mathcal{I}(S^n, \theta_0)$ or $\mathcal{I}(S_n^D, \theta_1)$ be characterized in terms of pattern avoidance?

There is an order-reversing bijection between I_n and $\mathcal{I}(S_n, \theta_0)$ given by $\varphi : I_n \to \mathcal{I}(S_n, \theta_0)$ with $\varphi(w) = w_0 w$ for all $w \in I_n$. Thus, the boolean elements of $\mathcal{I}(S_n, \theta_0)$ are in bijection with the elements of I_n having an upper order ideal isomorphic to some boolean lattice. Furthermore, we can see that $w \in I_n$ avoids a pattern $p \in S_k$ if and only if $\varphi(w) \in \mathcal{I}(S_n, \theta_0)$ avoids the pattern $w_{0,k}p$ where $w_{0,k}$ denotes the maximal element of S_k. We can thus characterize the boolean elements of $\mathcal{I}(S_n, \theta_0)$ in terms of pattern avoidance if and only if we can express the property of having an upper order ideal $[w, w_0]$ by pattern avoidance for all $w \in I_n$. Computer experiments have verified the following conjecture for $n \leq 10$.

Conjecture 7.2. *The involution $w \in I_n$ has an upper order ideal $[w, w_0]$ isomorphic to some boolean lattice if and only if it avoids all of the patterns below.*

1234	1243	1324	2134
14523	21354	34125	
214365	215634	216543	
321654	341265	351624	
426153	432165	456123	
5276143	5471263	65872143	

In section 4.2 we presented results, that characterize the elements with rank-symmetric lower order ideal in terms of pattern avoidance for Coxeter groups of type A, B and D. This leads directly to the corresponding question for involutions.

Problem 7.3. Can the property of having a rank-symmetric lower order ideal be characterized in terms of pattern avoidance for the (twisted) involutions in I_n, I_n^B or I_n^D?

It would not be less interesting to find a characterization in terms of reduced twisted expressions valid for any Coxeter group W and involutive group automorphism θ.

In section 6.1 we found a recursive formula for the number of boolean involutions in I_n. From the recursion it is easy to deduce an explicit formula which unfortunately involves irrational complex numbers raised to high powers and thus is not really useful to compute $b(I_n)$ for large n. In section 6.2 we deduced formulas for the number of boolean involutions in I_n^B and I_n^D using the numbers $b(I_n)$. Thus, it would be very interesting to find an explicit combinatorial formula for $b(I_n)$, because it would immediately yield explicit formulas for $b(I_n^B)$ and $b(I_n^D)$, as well.

Problem 7.4. Find an explicit combinatorial formula for $b(I_n)$.

Although the method of counting reduced twisted expressions that we used in section 6.2 can be applied for any Coxeter group, the necessary argumentation will depend on the structure of the corresponding Coxeter graph in every single case. It would be helpful to find a general formula for the number of (maximal) boolean involutions at least for certain classes of Coxeter graphs such as trees or anything related. Using Proposition 6.10 we can compute the number of boolean involutions for all Coxeter groups whose Coxeter graph is a tree if we can do it for all trees which do not have leaves adjacent to a vertex of degree 2. This includes for example the following graph.

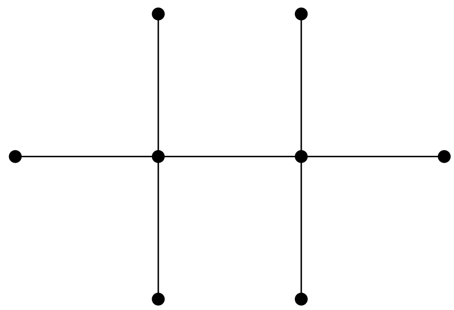


Figure 7.1: A tree with no leaf adjacent to a vertex of degree 2

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Theses

- A twisted involution $w \in \mathcal{I}(W, \theta)$ is boolean if and only if for every reduced twisted expression $\underline{s}_1 \dots \underline{s}_k$ for w and every $i, j \in [k]$ with $i \neq j$ it holds that $\underline{s}_i \neq \underline{s}_j$.
- An involution $w \in \mathcal{I}(W, \mathrm{id})$ is boolean if and only if there is a twisted expression for w without repeated letters.
- An involution $w \in I_n$ is boolean if and only if w avoids the patterns 4321, 45312 and 456123.
- If an involution $w \in I_n$ contains an induced p-pattern and $p \in I_k$ is 2143-avoiding then there is a reduced twisted expression for w which begins with a shift of a reduced twisted expression for p.
- A signed involution $w \in I_n^B$ is boolean if and only if it avoids all of the following signed patterns.

4321	45312	456123	$\underline{12}$	$1\underline{32}$	$\underline{321}$
$21\underline{3}$	$42\underline{3}1$	$4\underline{32}1$	$3\underline{4}1\underline{2}$	$\underline{4}53\underline{1}2$	$45\underline{3}12$
$\underline{4}32\underline{1}$	$5\underline{4}3\underline{2}1$	$\underline{4}56\underline{1}23$	$5\underline{4}6\underline{2}13$		

- An even signed involution $w \in I_n^D$ is boolean if and only if it avoids all signed patterns below.

4321	45312	456123	$1\underline{23}$	$21\underline{3}$	$31\underline{2}$
$32\underline{1}$	$\underline{123}$	$\underline{132}$	$\underline{213}$	$\underline{231}$	$\underline{312}$
$12\underline{43}$	$\underline{3}4\underline{21}$	$4\underline{23}1$	$4\underline{3}12$	$15\underline{43}2$	$1\underline{5432}$
$\underline{5}2\underline{431}$	$\underline{54}3\underline{21}$	$\underline{4}56\underline{1}23$	$6\underline{5432}1$	$\underline{654321}$	

- The number of boolean elements in P_n, fulfills the recursion

$$b(P_n) = 2b(P_{n-1}) + b(P_{n-2}) - b(P_{n-3})$$

  for all $n \geq n_P$ for some $n_P \in \mathbb{N}$ if $P \in \{I, I^B, I^D\}$.
- The generating function of the number of boolean involutions in I_n with respect to the rank is

$$\sum_{n \geq 1} \sum_{k \geq 0} b^\rho(I_n, k)\, x^n y^k = \frac{x(1 - x^2y^2)}{(1 - x^2y^2)(1 - x) - xy} \; .$$

- There is a bijection between the boolean involutions $w \in I_n$ and certain restricted Motzkin paths.
- The number of maximal boolean involutions of S_n, S_n^B and S_n^D respectively can be computed from the Fibonacci numbers.