Teodor J. Stepien
Pedagogical Academy , Cracow
Poland

PROOF OF WEAK CONSISTENCY OF PEANO'S ARITHMETIC SYSTEM

Abstract At first we introduce a definition of weak consistency of deductive systems by the well-known interpretation . Hence we can prove that **Peano's Arithmetic System** is consistent in traditional sense .

1. Terminology

Let C, N, A, K, E denote connectives of implication , negation , disjunction , conjuction and equivalence respectively . $P_n^k(t_1, ..., t_k)$ are atomic formulas . By $At_0 = \{p_1^1, p_2^1, ..., p_1^2, p_2^2, ..., p_1^k, p_2^k, ...\}$ we denote the set of all propositional variables . Hence S_0 is the set of all well-formed formulas , those are built in the usual manner from propositional variables by means of logical connectives . Sp denotes the set of all well-formed formulas of Peano's Arithmetic System . $vf(\alpha)(\alpha \in Sp)$ denotes the set of all free variables occuring α . Hence $\overline{Sp} = \{\alpha \in Sp : vf(\alpha) = \emptyset\}$. R_{Sp} denotes the set of all rules over Sp (see [1]) . For any $X \subseteq Sp, R \subseteq R_{Sp}, Cn(R, X)$ is the smallest subset of Sp containing X and closed under the rules of R . The couple $\langle R, X \rangle$ is called a system, whenever $X \subseteq Sp$ and $R \subseteq R_{Sp}$. r_o denotes modus ponens and r_+ denotes the generalization rule. $R_{o+} = \{r_o, r_+\}$. We use $\Rightarrow, \Leftrightarrow, \wedge, \vee, \forall, \exists$ as metalogical symbols . L_2 and A_r denote the set of all logical axioms and the set of all specific axioms in **Peano's Arithmetic System** respectively . Hence $\langle R_{o+}, L_2 \cup A_r \rangle$ is **Peano's Arithmetic System .**

We define the function $i : Sp \to S_0$ as follows :

(a) $i(P_n^k(t_1, ..., t_k)) = p_n^k(p_n^k \in At_0)$,
(b) $i(N\alpha) = Ni(\alpha)$,
(c) $i(F\alpha\beta) = i(\alpha)Fi(\beta)$,
(d) $i(\wedge x\alpha) = i(\vee x\alpha) = i(\alpha)$,

where $\alpha, \beta \in Sp$ and $F \in \{C, A, K, E\}$.

Definition 1. $\langle R, X\rangle \in Cns^T \Leftrightarrow$
$\Leftrightarrow (\forall\alpha \in \overline{Sp})[\alpha \notin Cn(R, X) \vee N\alpha \notin Cn(R, X)]$.

At last $\mathcal{M}_0$ denotes here the fixed logical matrix, where 0 is the fixed nondistinguished element of the matrix $\mathcal{M}_0$ (for details see [2]) . Thus,

Definition 2. $\langle R_{o+}, L_2 \cup X\rangle \in Cns^w \Leftrightarrow$
$\Leftrightarrow (\forall\alpha \in Sp)(\forall\beta \in X)[\alpha \in Cn(R_{o+}, L_2 \cup \{\beta\}) \Rightarrow$
$\Rightarrow (\forall v : At_0 \rightarrow | \mathcal{M}_0 |)\ h^v(i(C\beta\alpha)) \neq 0]$,

where $X \subseteq Sp$, $\beta = K...K\alpha_1...\alpha_k$ and $\alpha_1, ..., \alpha_k \in \overline{Sp}$.

2. THE MAIN RESULT

LEMMA . $\langle R_{o+}, L_2 \cup A_r\rangle \in Cns^w \Rightarrow \langle R_{o+}, L_2 \cup A_r\rangle \in Cns^T$.

Proof . Elementary and Combinatorical .

THEOREM . $\langle R_{o+}, L_2 \cup A_r\rangle \in Cns^T$.

Proof by LEMMA . (Elementary and Combinatorical , cf. [3]) .

References :

[1] . Pogorzelski , W . , The classical calculus of quantifiers , PWN , Warszawa 1981 .
[2] . Pogorzelski , W. and Wojtylak , P . , Elements of the theory of completeness in Propositional Logic , Silesian University (1982) .
[3] . Von Neuman, J. , Die formalistische Grundlegung der Mathematik „ Erkenntis" t2 (1931) .

1991 Mathematics Subject Classification : 03B10, 03B30

Key words : a weak consistency, the traditional consistency