The Nonlinear Ergodic Theorems In Banach Space

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Abstract. In this paper, one of the our main results is Theorem 1 which is a generalization of a result in [1], and among many other results, we study nonlinear ergodic properties for an amenable semigroup of nonexpansive mappings of type (γ) in a Banach space in our version (see Theorem 2, for example). In this paper we replace the compactness condition in the Atsushiba and Takahashi theorem [1], by some imposed conditions on nonexpansive mappings and extend the theorem. Moreover we prove the existence of an ergodic retraction as in [2] in a Banach space with such imposed conditions on nonexpansive mappings.

Key words: *Nonlinear ergodic theorem, fixed point, nonexpansive mapping, of type (γ), strong convergence, amenable semigroup*

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The main results

Let E be a real Banach space and let C be a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$. We denote by $F_\varepsilon(T)$ the ε-approximate fixed points of T, i.e. $F_\varepsilon(T) = \{x \in C : \|x - Tx\| \leq \varepsilon\}$. If C is bounded, then $F_\varepsilon(T) \neq \phi$ for each $\varepsilon > 0$. We denote by $\mathrm{dis}(x, A)$ the distance from x to A for each x in E. Let $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots \supseteq A$ be a decreasing chain of nonempty closed subsets of E. Then the notation $A_n \xrightarrow{dis} A$ means that, if (x_n) is a sequence with $x_n \in A_n$ for each $n \geq 1$, then $\lim\limits_n dis(x_n, A) = 0$. If C is compact, then $\overline{co}F_{\frac{1}{n}}(T) \xrightarrow{dis} F(T)$ (see[3]).

Theorem 1

Let C be a nonempty closed convex subset of a Banach space E and $T : C \to C$ be a nonexpansive mapping of type (γ) such that $F(T) \neq \phi$ and $\overline{co}F_{\frac{1}{n}}(T) \xrightarrow{dis} F(T)$ and let $x \in C$. If $F(T)$ is compact, then $\frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x$ converges strongly to a fixed point of T uniformly in $h \geq 0$.

Theorem 2

Let C be a nonempty weakly compact convex subset of a Banach space E, $\varphi = \{T_t : t \in S\}$ be a nonexpansive semigroup on C of type (γ) with the condition $\overline{co}F_{\frac{1}{n}}(T_t) \xrightarrow{dis} F(T_t)$, for every $t \in S$, and X be an invariant subspace of $B(S)$ such such that $1 \in X$ and the mapping $t \mapsto < T_t x, x^* >$ is an element of X for each $x \in C$ and $x^* \in E^*$, and μ be a mean on X. If X is amenable, then $F(\varphi) \neq \phi$ and there exists a nonexpansive retraction P form C onto $F(\varphi)$, such that $PT_t = T_t P = P$ for each $t \in S$, and $Px \in \overline{co}\{T_t x : t \in S\}$ for each $x \in C$, and if the norm of E is Frechet differentiable, then P is unique.

Furthermore, if $\{\mu_\alpha\}$ is an asymptotically invariant net of means on X, then for each $x \in C$, $\{T_{\mu_\alpha} x\}$ converges weakly to Px.

References

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