On a Two-Temperature Problem for the Klein-Gordon Equation

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Consider the Klein-Gordon equation, with constant or variable coefficients in $\mathbb{R}^n$, $n \geq 2$,

$$\left\{ \begin{array}{l} \ddot{u}(x,t) = \sum_{j=1}^{n} (\partial_j - iA_j(x))^2 u(x,t) - m^2 u(x,t), \quad x \in \mathbb{R}^n, \\ u|_{t=0} = u_0(x), \quad \dot{u}|_{t=0} = v_0(x). \end{array} \right. \tag{1}$$

Here $m > 0$, $(A_1(x), \ldots, A_n(x))$ a vector potential of a magnetic field; we assume that functions $A_j(x)$ vanish outside a bounded domain. The solution $u(x,t)$ is considered as a complex-valued function. Denote $Y(t) \equiv (u(\cdot,t), \dot{u}(\cdot,t))$, $Y_0 \equiv (u_0, v_0)$. The initial date Y_0 is a random element of a complex functional space $\mathcal{H}$ consisting of states with a finite local energy, $\|Y\|_R^2 = \int_{|x|<R} (|u(x)|^2 + |\nabla u(x)|^2 + |v(x)|^2) dx < \infty$, $\forall R > 0$. Given $t \in \mathbb{R}$, denote by μ_t the probability measure that gives the distribution of $Y(t)$, the random solution to (1). We study the asymptotics of μ_t as $t \to \pm\infty$.

We identify $\mathbb{C} \equiv \mathbb{R}^2$ and denote by $\otimes$ tensor product of real vectors. We assume that the initial correlation matrices $Q_0^{ij}(x,y) := E\Big(Y_0^i(x) \otimes Y_0^j(y)\Big)$ $x, y \in \mathbb{R}^n$, $i,j = 0,1$, have the form

$$Q_0^{ij}(x,y) = \left\{ \begin{array}{l} q_+^{ij}(x-y), \quad x_n, y_n > a, \\ q_-^{ij}(x-y), \quad x_n, y_n < -a. \end{array} \right.$$

Here $q_\pm^{ij}(x-y)$ are the correlation matrices of some translation-invariant measures $\mu_\pm$ with zero mean value in $\mathcal{H}$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $a > 0$. The measure μ_0 is not translation-invariant if $q_-^{ij} \neq q_+^{ij}$.

We also assume that the measure μ_0 has zero mean and the initial mean 'energy' density is uniformly bounded: $E[|u_0(x)|^2 + |\nabla u_0(x)|^2 + |v_0(x)|^2] = \mathrm{tr}\Big(Q_0^{00}(x,x) + [\nabla_x \cdot \nabla_y Q_0^{00}(x,y)]|_{y=x} + Q_0^{11}(x,x)\Big) < \infty$, a.a. $x \in \mathbb{R}^n$. Finally, we assume that measure μ_0 satisfies a mixing condition of a Rosenblatt- or Ibragimov-Linnik type, which means that $Y_0(x)$ and $Y_0(y)$ are asymptotically independent as $|x-y| \to \infty$.

Our main result is the (weak) convergence $\mu_t \rightharpoondown \mu_\infty$, $t \to \infty$, to an equilibrium measure μ_∞, which is a translation-invariant Gaussian measure on $\mathcal{H}$. A similar convergence holds for $t \to -\infty$. We construct generic examples of the initial measures μ_0 satisfying all assumptions imposed. An explicit formulas are then given for the correlation matrices of measure μ_∞.

The proof is based on the Bernstein 'room-corridor' argument and oscillatory integrals estimates. The application to the case of the Gibbs measures $\mu_\pm = g_\pm$ with two different temperatures $T_\pm$ is given. Limiting mean energy current density *formally* is $-\infty \cdot (0, \ldots, 0, T_+ - T_-)$ for the Gibbs measures, and it is finite and equals $-C(0, \ldots, 0, T_+ - T_-)$ with $C > 0$ for a smoothed solution.

Bibliography

[1] T.V. Dudnikova, A.I. Komech, E.A. Kopylova, Yu.M. Suhov, On convergence to equilibrium distribution, I. The Klein-Gordon equation with mixing, *Com. Math. Phys.* **225** (2002), no.1, 1-32.

[2] T.V. Dudnikova, A.I. Komech, H. Spohn, On a two-temperature problem for wave equation, *Markov Processes and Related Fields* **8** (2002), 43-80.

[3] T. Dudnikova, A. Komech, N.J. Mauser, On two-temperature for harmonic crystals, *J. Stat. Phys.* **114** (2004), no.3/4, 1035-1083.

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