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Title of poster

State Symmetries in Matrices and Vectors on Finite State Spaces.

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Abstract
Let $C = \{1, 2, \ldots, N\}$ be a finite set of natural numbers (the state space). Let $\rho \in \mathcal{S}_N$ be a permutation of the elements of C, and let $\mathbb{V} = \mathbb{R}^N$ be the vector space of real column vectors and $\mathbb{M}$ be the vector space of real $N \times N$ matrices.

A state symmetry of a vector or a matrix is defined as a permutation ρ for which holds $v^\rho = v$, where $v^\rho = (v_i^\rho) := \left(v_{\rho(i)}\right)$ (similarely $M^\rho = M$ with $M^\rho = \left(m_{i\,j}^\rho\right) := \left(m_{\rho(i)\,\rho(j)}\right)$), so that ρ is acting on $\mathbb{V}$ and $\mathbb{M}$. If $v^\rho = v$ (or $M^\rho = M$) for a permutation $\rho \neq id$, v (or M) is called state symmetric. Hence, $\mathcal{S}_v := \{\rho | v^\rho = v\}$ and $\mathcal{S}_M := \{\rho | M^\rho = M\}$ are isotropy subgroups.

In addition, $\mathbb{V}_\rho := \{v \in \mathbb{V} | v^\rho = v\}$ and $\mathbb{M}_\rho := \{M \in \mathbb{M} | M^\rho = M\}$ are vector spaces. The dimension of $\mathbb{V}_\rho$ is the number of disjoint cycles of ρ, and a canonical basis of this vector space can be derived directly. For M_ρ the dimension is the sum of all $GCD(n_1, n_2)$ where the sum is taken over all pairs of cycles c_1 and c_2 in ρ and n_r is the length of c_r.

The set $G = \mathbb{M}_\rho \cap GL(N)$ of invertible matrices with state symmetry ρ is a group. Also this group acts on $\mathbb{V}_\rho$ and $\mathbb{M}_\rho$.

If $v^{(1)} = M\, v^{(0)}$ and $v^{(2)} = M\, v^{(1)}$, then the existance of a common state symmetry of the vectors $v^{(0)}$ and $v^{(1)}$ is generally not sufficient for the existance of a state symmetry of M, neither for the existance of a state symmetry of $v^{(2)}$.

However, the following hypothesis has been raised for vector sequences $u = (v^{(i)})_{i=0,\ldots,N}$: Let $\mathbb{W}_u := \{M | v^{(i+1)} = M\, v^{(i)}$ for $i = 0, \ldots, N-1\}$ be the set of "generators" of u. If ρ is a proper state symmetry ρ of all vectors of u, then $|\mathbb{W}_u| \geq 2$ and there is at least one matrix $\hat{M} \in \mathbb{W}_u$ which is state symmetric. In this case, all elements of the infinite sequence $u = (v^{(i)})_i$ are state symmetric.

Applications to simplify special Markow chains or structures of graphs will be presented.