Zeros and extreme values of the Riemann zeta-function on the critical line

Rasa Šleževičienė (Šiauliai, Lithuania),
Jörn Steuding (Frankfurt, Germany)

The value-distribution of the Riemann zeta-function $\zeta(s)$ is of special interest in number theory. The famous yet unproved Riemann hypothesis claims that all nontrivial (non-real) zeros lie on the critical line $\operatorname{Re} s = 1/2$. Moreover, it is conjectured that all or at least almost all zeros are simple. It is known that more than 2/5-th of the zeros lie on the critical line and are simple.

Denote by t_n the distinct positive ordinates of zeros of $\zeta(s)$ on the critical line in ascending order. We prove

$$\sum_{n\leq N} |\zeta'(1/2+it_n)| \ll T(\log T)^{9/4},$$

and

$$\sum_{n\leq N} \max_{t_n<t<t_{n+1}} |\zeta(1/2+it)| \ll T(\log T)^{5/4},$$

where N is the number of ordinates t_n below T; note that $N \asymp N(T)$ where $N(T) \sim \frac{1}{2\pi} T \log T$ counts the number of all nontrivial zeros of $\zeta(s)$ with imaginary part in $(0,T]$. These estimates support special cases of conjectures on the vertical zero-distribution of $\zeta(s)$ (Montgomery's pair corelation conjecture, predicitions from random matrix theory). [The proof of the first estimate is based on an idea of Garaev; the authors learned that in the meantime also Garaev found this estimate.]