THE PURE PART OF THE IDEALS IN C(X)

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Abstract

Let C(X) be the ring of all continuous real valued functions defined on a completely regular T_1-space. For each ideal I in C(X) let mI be the pure part of the ideal I.

In this article we show that $mI = O^{\theta(I)}$, where $\theta(I) = \bigcap_{f \in I} cl_{\beta X} Z(f)$.

The pure part of many ideals in C(X) is calculated.

We found that $mC_K(X)$, the pure part of the ideal of functions with compact support, is finitely generated if and only if βX-$\theta(C_K(X))$ is compact, $mC_K(X)$ is countably generated if and only if βX-$\theta(C_K(X))$ is Lindelőff and $mC_K(X)$ is generated by a star finite set if and only if βX-$\theta(C_K(X))$ is paracompact. Similar results are obtained for the pure part of the ideal $C_\Psi(X)$, the ideal of functions with pseudocompact support.

1. INTRODUCTION

Let X be a completely regular T_1-space, βX the Stone-Čech compactification of X and υX the Hewitt realcompactification of X. Let C(X) be the ring of all continuous real valued functions defined on X. For each $f \in C(X)$, let $Z(f) = \{x \in X: f(x) = 0\}$, coz f = X-Z(f), the support of $f = S_X(f) = cl_X coz(f)$, $S_{\upsilon X}(f^\upsilon) = cl_{\upsilon X}(\upsilon X\text{-}Z(f^\upsilon))$, where f^υ is the extension of f to υX, $S_{\beta X}(f^\beta) = cl_{\beta X}(\beta X\text{-}Z(f^\beta))$, where f^β is the continuous extension to βX for the function

$$f^*(x) = \begin{cases} 1 & f(x) \geq 1 \\ f(x) & -1 \leq f(x) \leq 1 \\ -1 & f(x) \leq -1 \end{cases}$$

If I is an ideal in C(X), then coz $I = \bigcup_{f \in I}$ coz f.

For each subset $A \subseteq \beta X$, let $M^A = \{f \in C(X): A \subseteq cl_{\beta X} Z(f)\}$ and $O^A = \{f \in C(X): A \subseteq Int_{\beta X}\, cl_{\beta X} Z(f)\} = \{f \in C(X): A \subseteq Int_{\beta X} Z(f^\beta)\}$.

An ideal I of C(X) is called a **pure ideal** if for each $f \in I$, there exists $g \in I$ such that $f = fg$. It is clear that in this case $g = 1$ on $\mathrm{S_X}(f)$.

For any undefined terms here the reader may consult [12].

Purity attracted the attention of a lot of people working in ring and module theories. A large class of commutative rings can be classified through the pure ideals of the ring. The pure ideals in C(X) were completely characterized in [3].

Arbitrary sum and finite intersection of pure ideals is a pure ideal, see [7]. One might ask whether intersection of arbitrary family of pure ideals is pure. In fact the answer is not true in general. The following example was given in [3]. Let $\mathbb{R}$ be the space of reals. The ideal $I = \mathrm{O}^{(0,1)} = \bigcap_{x\in(0,1)} \mathrm{O}^{\mathrm{x}}$ in $\mathrm{C}(\mathbb{R})$ is not pure, since the function $f(x) = \begin{cases} 0 & x \le 1 \\ x-1 & x > 1 \end{cases}$

belongs to I, while there is no $g \in I$ such that $f = fg$, since in this case $\mathrm{S}_{\mathbb{R}}(f) = [1, \infty) \subseteq \mathrm{coz}\, g$ which implies that $(0,1) \nsubseteq \mathrm{Int_X}(\mathrm{Z}(g))$.

Recall that a space X is called **basically disconnected** if for each $f \in$C(X), $\mathrm{S_X}(f)$ is open in X. We will show that intersection of arbitrary family of pure ideals in C(X) is pure if and only if X is basically disconnected, but first we'll need some preliminaries.

Proposition 1.1 (Al-Ezeh [3]). An Ideal I of C(X) is pure if and only if $I = \mathrm{O^A}$ for some closed subset $\mathrm{A} \subset \beta\mathrm{X}$.

Proposition 1.2 (Abu Osba [1]**).** For each $\mathrm{A} \subseteq \beta\mathrm{X}$. The ideal $\mathrm{O^A}$ is pure if and only if $\mathrm{O^A} = \mathrm{O}^{\mathrm{cl}_{\beta X}\mathrm{A}}$.

Theorem 1.3. Intersection of any family of pure ideals in C(X) is pure if and only if $\mathrm{O^A}$ is pure for each $\mathrm{A} \subseteq \beta\mathrm{X}$.

Proof. Suppose the condition is satisfied. Then for each $\mathrm{A} \subset \beta\mathrm{X}$, $\mathrm{O^A} = \bigcap_{x\in\mathrm{A}} \mathrm{O^x}$ is an intersection of pure ideals and so it is pure.

Conversely, Let $I = \bigcap_{\alpha\in\Delta} I_\alpha$, where I_α is pure for each $\alpha \in \Delta$. Then it follows by Proposition 1.1 that $I_\alpha = \mathrm{O}^{\mathrm{A}_\alpha}$ for some closed subset $\mathrm{A}_\alpha \subset \beta\mathrm{X}$.

So $I = \bigcap_{\alpha\in\Delta} \mathrm{O}^{\mathrm{A}_\alpha} = \mathrm{O}^{(\bigcup_{\alpha\in\Delta} \mathrm{A}_\alpha)}$ is pure.□

Theorem 1.4. The following statements are equivalent:
(1) The space X is basically disconnected.
(2) The ideal $\mathrm{O^A}$ is pure for each $\mathrm{A} \subset \beta\mathrm{X}$.
(3) For each $f \in$ C(X), the ideal $\mathrm{O}^{Int_{\beta\mathrm{X}}\mathrm{Z}(f^\beta)}$ is pure.

Proof**.** (1) ⇒ (2): Suppose that X is basically disconnected, then so is βX, see [12]. Let $\mathrm{A} \subset \beta\mathrm{X}$, then $\mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}} \subseteq \mathrm{O^A}$.

Let $f \in \mathrm{O}^{\mathrm{A}}$, then $\mathrm{A} \subset \mathrm{Int}_{\beta\mathrm{X}}\mathrm{Z}(f^{\beta})$. But $\mathrm{Int}_{\beta\mathrm{X}}\mathrm{Z}(f^{\beta})$ is closed in βX since βX - $\mathrm{Int}_{\beta\mathrm{X}}\mathrm{Z}(f^{\beta}) = \mathrm{cl}_{\beta\mathrm{X}}(\beta\mathrm{X}$ - $\mathrm{Z}(f^{\beta})) = \mathrm{S}_{\beta\mathrm{X}}(f^{\beta})$ is open in βX.

Hence $\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A} \subset \mathrm{Int}_{\beta\mathrm{X}}\mathrm{Z}(f^{\beta})$, and $f \in \mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}}$.

Thus $\mathrm{O}^{\mathrm{A}} = \mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}}$ is pure by Proposition 1.2.

(2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (1): Let $f \in$ C(X). Then $f \in \mathrm{O}^{\mathrm{Int}_{\beta\mathrm{X}}\mathrm{Z}(f^{\beta})} = \mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{Int}_{\beta\mathrm{X}}\mathrm{Z}(f^{\beta})}$. So $\mathrm{cl}_{\beta\mathrm{X}}\mathrm{Int}_{\beta\mathrm{X}}\mathrm{Z}(f^{\beta}) \subseteq \mathrm{Int}_{\beta\mathrm{X}}\mathrm{Z}(f^{\beta}) \subseteq \mathrm{cl}_{\beta\mathrm{X}}\mathrm{Int}_{\beta\mathrm{X}}\mathrm{Z}(f^{\beta})$. Hence $\mathrm{S}_{\beta\mathrm{X}}(f^{\beta}) = \beta\mathrm{X}$ - $\mathrm{Int}_{\beta\mathrm{X}}\mathrm{Z}(f^{\beta})$ is clopen in βX.

Now, coz f = coz f^{*}= (βX - Z(f^{β})) $\cap$ X.

$\mathrm{S}_{\mathrm{X}}(f) = \mathrm{cl}_{\mathrm{X}}((\beta X - Z(f^{\beta})) \cap \mathrm{X}) = \mathrm{cl}_{\beta\mathrm{X}}((\beta\mathrm{X}$ - $\mathrm{Z}(f^{\beta})) \cap \mathrm{X}) \cap \mathrm{X}$.

$= \mathrm{cl}_{\beta\mathrm{X}}((\beta\mathrm{X}$ - $\mathrm{Z}(f^{\beta})) \cap \mathrm{cl}_{\beta\mathrm{X}}\mathrm{X}) \cap \mathrm{X}$, since βX - Z(f^{β}) is open in βX.

$= \mathrm{cl}_{\beta\mathrm{X}}((\beta\mathrm{X}$ - $\mathrm{Z}(f^{\beta})) \cap \beta\mathrm{X}) \cap \mathrm{X}$.

$= \mathrm{cl}_{\beta\mathrm{X}}((\beta\mathrm{X}$ - $\mathrm{Z}(f^{\beta})) \cap \mathrm{X}$.

$= \mathrm{S}_{\beta\mathrm{X}}(f^{\beta}) \cap \mathrm{X}$.

Thus $\mathrm{S}_{\mathrm{X}}(f)$ is open in X for each $f \in$ C(X), which implies that X is basically disconnected.□

Recall that a ring R is called a **PP-ring** if every principal ideal of R is a projective R-module.

Corollary 1.5. The following statements are equivalent:
(1) The space X is basically disconnected.
(2) The ring C(X) is a PP-ring.
(3) Intersection of arbitrary family of pure ideals in C(X) is pure.

Proof. For the equivalence of (1) and (2) see[9].□

The above corollary shows that in a basically disconnected space X, the pure ideals of C(X) form a complete lattice with $I \vee J = I + J$ and $I \wedge J = I \cap J$.

If the ideal is not pure, then one can study the pure part of the ideal which have some interesting properties and it is a tool for studying the ideal itself.

Definition 1.6. For each ideal I of C(X), let $mI = \{f \in$ C(X): $f \in fI\}$. The ideal mI will be called the pure part of I.

It is clear that, $mI \subseteq I$ and I is pure if and only if $I = mI$. It is shown in [7] that for any ideal I of C(X), mI is a pure ideal.

The pure part of the ideal was studied by many authors, such as in [5], [7], [10], [14] and [15]. Studying the pure part of the ideal is a tool for studying the ideal itself. In fact it is easier to deal with the pure part since it is pure.

In section 2, we characterize the pure part of the ideal I to be the ideal $\mathrm{O}^{\theta(I)}$. In paricular, we show that the pure part of a maximal ideal M^{x} is the ideal O^{x} and $m\mathrm{M}^{\mathrm{A}} = \mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}}$.

In section 3, we find the pure part for some important well known ideals in C(X), such as $C_K(X)$, the ideal of functions with compact support, the ideal $C_\Psi(X)$, the ideal of functions with pseudocompact support, ...etc.

In section 4, we study the relations between the generators of the ideal $mC_K(X)$ and the topological properties of the set βX - $\theta(C_K(X))$. Similar results are obtained for the generators of $mC_\Psi(X)$ and the topological properties of βX - $\theta(C_\Psi(X))$.

2. THE PURE PART OF THE IDEAL

For each ideal I of C(X), let $\theta(I) = \{x \in \beta X: I \subseteq M^x\}$.

Proposition 2.1(Gillman and Jerison[12]). For each ideal I of C(X), $\theta(I) = \bigcap_{f\in I} cl_{\beta X}Z(f)$.

It is clear that for each ideal I of C(X), $O^{\theta(I)}$ is a pure ideal, since $\theta(I)$ is a closed set in βX.

We give now another characterization for mI using the set $\theta(I)$. This new characterization is some times helpful.

Theorem 2.2. For each ideal I of C(X), $mI = O^{\theta(I)}$.

Proof. Let $f \in O^{\theta(I)}$. Then $\theta(I) \subseteq Int_{\beta X}Z(f^\beta) \subseteq Z(f^\beta)$.

So βX-$Z(f^\beta) \subseteq S_{\beta X}(f^\beta) \subseteq \beta X - \theta(I) = \bigcup_{g\in I} \beta X\text{-}cl_{\beta X}Z(g)$. Compactness of $S_{\beta X}(f^\beta)$ implies that there exists n$\in \mathbb{N}$ such that $S_{\beta X}(f^\beta) \subseteq \bigcup_{i=1}^{n} \beta X\text{-}cl_{\beta X}Z(g_i)$, where $g_i \in I$ for each i.

Hence $S_X(f) = S_{\beta X}(f^\beta) \cap X \subseteq (\bigcup_{i=1}^{n} \beta X\text{-}cl_{\beta X}Z(g_i)) \cap X \subseteq \bigcup_{i=1}^{n} X\text{-}Z(g_i) = \text{coz } g$, where $g = \sum_{i=1}^{n} g_i^2$. It is clear that $g \in I$.

Define $h(x) = \begin{cases} \frac{f}{g}(x) & x \in S_X(f) \\ 0 & \text{otherwise} \end{cases}$

Then $h \in C(X)$ and $f = gh \in I$. So $O^{\theta(I)} \subseteq I$. But $O^{\theta(I)}$ is a pure ideal, since $\theta(I)$ is a closed set in βX, see 1.1 above. So there exists $k \in O^{\theta(I)} \subseteq I$ such that $f = fk \in fI$ which implies that $f \in mI$. Thus $O^{\theta(I)} \subseteq mI$.

Now, if $f \in mI$, then $f = fg$, for some $g \in I$. Then $f^\beta = f^\beta g^\beta$, which implies that $S_{\beta X}(f^\beta) \subseteq \beta X - Z(g^\beta)$.

So $\theta(I) \subseteq cl_{\beta X}Z(g) \subseteq Z(g^\beta) \subseteq \beta X - S_{\beta X}(f^\beta) \subseteq Z(f^\beta)$.

Hence $f \in O^{\theta(I)}$.□

We now characterize the pure part of any maximal ideal in C(X). This characterization would be very useful in the rest of this article. But first we find the pure part for an intersection of maximal ideals.

Theorem 2.3. For each subest A $\subseteq \beta$X, $m\mathrm{M}^{\mathrm{A}} = \mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}}$.

Proof**.** It follows by Proposition 1.1 that $\mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}}$ is pure, so $\mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}} \subseteq m\mathrm{M}^{\mathrm{A}}$.

Let $f \in m\mathrm{M}^{\mathrm{A}}$, then there exists $g \in \mathrm{M}^{\mathrm{A}}$ such that $f = fg$,which implies that $f^{\beta} = f^{\beta}g^{\beta}$.

Hence βX - Z$(f^{\beta}) \subseteq \mathrm{S}_{\beta\mathrm{X}}(f^{\beta}) \subseteq \beta$X - Z$(g^{\beta})$.

So A $\subseteq \mathrm{cl}_{\beta\mathrm{X}}\mathrm{Z}(g) \subseteq \mathrm{Z}(g^{\beta}) \subseteq \beta\mathrm{X}\text{-}\mathrm{S}_{\beta\mathrm{X}}(f^{\beta}) \subseteq \mathrm{Z}(f^{\beta})$, which implies that $\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A} \subseteq \mathrm{Z}(g^{\beta}) \subseteq \mathrm{Int}_{\beta\mathrm{X}}\mathrm{Z}(f^{\beta})$.

Hence $f \in \mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}}$.□

The following corollary is an easy consequence of the above theorem and it characterizes the pure part of a maximal ideal in C(X). Recall that a point x $\in \beta$X is called a **P-point** if $\mathrm{M}^{x} = \mathrm{O}^{x}$.

Corollary 2.4. For each x $\in \beta$X, $m\mathrm{M}^{x} = \mathrm{O}^{x}$, and so M^{x} is pure if and only if x is a P-point.

Recall that a subset A $\subseteq \beta$X is called a **round subset** of βX if $\mathrm{O}^{\mathrm{A}} = \mathrm{M}^{\mathrm{A}}$, see [13].

Corollary 2.5. Let A $\subseteq \beta$X, then M^{A} is pure if and only if $\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}$ is a round subset of βX.

Proof**.** The ideal M^{A} is pure if and only if $\mathrm{M}^{\mathrm{A}} = m\mathrm{M}^{\mathrm{A}}$ if and only if $\mathrm{M}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}} = \mathrm{M}^{\mathrm{A}} = \mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}}$ if and only if $\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}$ is a round subset of βX.□

Theorem 2.6. For each A$\subseteq \beta$X, $m\mathrm{O}^{\mathrm{A}} = \mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}}$.

Proof**.** Since $\mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}}$ is a pure ideal contained in O^{A}, it follows that $\mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}} \subseteq m\mathrm{O}^{\mathrm{A}} \subseteq m\mathrm{M}^{\mathrm{A}} = \mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}}$.□

Corollary 2.7. For each A$\subseteq \beta$X, M^{A} is pure if and only if O^{A} is pure and A is a round subset of βX.

Proof**.** If M^{A} is pure, then $\mathrm{O}^{\mathrm{A}} \subseteq \mathrm{M}^{\mathrm{A}} = m\mathrm{M}^{\mathrm{A}} = \mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}} \subseteq \mathrm{O}^{\mathrm{A}}$. So $\mathrm{O}^{\mathrm{A}} = \mathrm{M}^{\mathrm{A}} = \mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}}$ which implies that A is a round subset of βX and O^{A} is pure. For the converse, we have $\mathrm{O}^{\mathrm{cl}_{\beta\mathrm{X}}\mathrm{A}} = \mathrm{O}^{\mathrm{A}} = \mathrm{M}^{\mathrm{A}}$, and hence the result.□

For each space X we have βX - υX is a round subset of βX, but $\mathrm{M}^{\beta\mathrm{X}-\upsilon\mathrm{X}}$ is not always pure, see [1]. If T is the Tychonoff blank, then $\mathrm{O}^{\mathrm{t}_0}$ is pure but $\mathrm{M}^{\mathrm{t}_0}$ is not, since $\mathrm{M}^{\mathrm{t}_0} \neq \mathrm{O}^{\mathrm{t}_0}$, where $\{\mathrm{t}_0\} = \beta T - T$, see [12]. This shows that the two conditions are both necessary in the above corollary to prove that M^{A} is pure.

3. EXAMPLES

We now find the pure part for some important ideals in C(X). We will use the results obtained in the previous section to characterize some properties of the space X using purity of these ideals.

The following sets and facts are well known:
$C_K(X) = \{ f \in C(X)\colon S_X(f)$ is compact $\} = O^{\beta X\text{-}X}$.
$C_\Psi(X) = \{ f \in C(X)\colon S_X(f)$ is pseudocompact $\} = O^{\beta X\text{-}\upsilon X} = M^{\beta X\text{-}\upsilon X}$.
$C_\rho(X) = \{ f \in C(X)\colon S_X(f)$ is realcompact $\}$.
$I(X) = M^{\beta X\text{-}X}$.
$O^{\upsilon X\text{-}X} = \{ f \in C(X)\colon S_X(f) = \mathrm{cl}_{\upsilon X} S_X(f) \}$.
$M^{\upsilon X\text{-}X} = \{ f \in C(X)\colon \mathrm{coz}(f)$ is realcompact $\}$.

The following lattice of inclusions exists in the general case.

$$\begin{array}{lllll} & & & & C_\Psi(X) \\ & & & \nearrow & \\ C_K(X) & \to & I(X) & & \\ \downarrow & & & \searrow & \\ O^{\upsilon X\text{-}X} & \to & C_\rho(X) & \to & M^{\upsilon X\text{-}X} \end{array}$$

A space X is called $\mu-$**compact (** $\lambda-$**compact)** if $O^{\beta X\text{-}X} = M^{\beta X\text{-}X}(O^{\upsilon X\text{-}X} = M^{\upsilon X\text{-}X})$.

For more informations about these ideals and spaces the reader may consult [6].

For each space X, let $X_L = \{x \in X\colon$ x has a compact neighborhood in X$\}$. Then $X_L = \bigcup_{f \in C_K(X)} \mathrm{coz}\ f = \mathrm{coz}(C_K(X)) = \mathrm{Int}_{\beta X} X$, the space X is locally compact if and only if $X = X_L$, see [2]. We also have βX - $X_L = \beta X - \mathrm{Int}_{\beta X} X = \mathrm{cl}_{\beta X}(\beta X\text{-}X) = \theta(C_K(X))$.

For each space X, let $kX = \{x \in \upsilon X\colon$ x has a compact neighborhood in $\upsilon X\}$. Then $kX = \{x \in \beta X\colon C_\Psi(X) \not\subseteq M^x\} = \bigcup_{f \in C_\Psi(X)} \upsilon X \text{ - } Z(f^\upsilon) = \mathrm{coz}(C_K(\upsilon X)) = \mathrm{Int}_{\beta X}(\upsilon X)$, the space υX is locally compact if and only if $\upsilon X = kX$, see [1]. We also have βX - $kX = \beta X$- $\mathrm{Int}_{\beta X}(\upsilon X) = \mathrm{cl}_{\beta X}(\beta X\text{-}\upsilon X) = \theta(C_\Psi(X))$.

Now, $mC_K(X) = mO^{\beta X\text{-}X} = O^{\mathrm{cl}_{\beta X}(\beta X\text{-}X)} = O^{\beta X\text{-}X_L}$.

$mC_\Psi(X) = mM^{\beta X\text{-}\upsilon X} = O^{\mathrm{cl}_{\beta X}(\beta X\text{-}\upsilon X)} = O^{\beta X\text{-}kX}$ and $C_\Psi(X)$ is pure if and only if βX-kX is round.

$mI(X) = mM^{\beta X\text{-}X} = O^{\mathrm{cl}_{\beta X}(\beta X\text{-}X)} = O^{\beta X\text{-}X_L}$ and I(X) is pure if and only if βX - X_L is round if and only if $C_K(X)$ is pure and X is $\mu-$compact.

$mM^{\upsilon X\text{-}X} = O^{\mathrm{cl}_{\beta X}(\upsilon X\text{-}X)}$ and $M^{\upsilon X\text{-}X}$ is pure if and only if $\mathrm{cl}_{\beta X}(\upsilon X$ - X) is round if and only if $O^{\upsilon X\text{-}X}$ is pure and X is $\lambda-$compact. Note that in this case $C_\rho(X)$ would be an ideal of C(X), see [16].

If $M^{\upsilon X\text{-}X}$ and $C_\Psi(X)$ are pure ideals, then so is I(X), since βX - X = (βX - υX) $\bigcup$ (υX - X) and the union of two round subsets is round, see [13].

4. GENERATORS FOR THE PURE PART OF $C_K(X)$ AND $C_\Psi(X)$

In this section we study the relations between the topological properties of βX-$\theta(C_K(X))$, βX-$\theta(C_\Psi(X))$ and the generators of the ideals $mC_K(X)$ and $mC_\Psi(X)$ respectively.

It was proved in [1] that for each ideal $I \subseteq C_K(X)$, I is pure if and only if coz $I = \bigcup_{f\in I} S_X(f)$. We will use this result quite often later on.

Lemma 4.1. For each ideal I of C(X) coz mI = coz I.

Proof**.** The element $x \in$ coz I if and only if $x \in X$ and $I \nsubseteq M^x$ if and only if $x \in X$ and $mI \nsubseteq M^x$ if and only if $x \in$ coz mI, since $I \subseteq M^x$ if and only if $mI \subseteq M^x$, see [14].□

Corollary 4.2. For each space X, coz($mC_K(X)$) = coz($C_K(X)$) = X_L, and coz ($mC_K(\upsilon X)$) = coz($C_K(\upsilon X)$) = kX.

Recall that $f = f^2$ if and only if $f(x) = \begin{cases} 1 & x \in \text{coz } f \\ 0 & \text{otherwise} \end{cases}$
and in this case coz $f = S_X(f)$, see [4].

Theorem 4.3. The ideal $mC_K(X)$ is finitely generated if and only if X_L is compact.

Proof**.** If $mC_K(X)$ is finitely generated, then it is generated by an idempotent, since it is a pure ideal. So there exists $f \in mC_K(X)$, such that $f = f^2$ and $mC_K(X) = (f)$. Then X_L = coz $f = S_X(f)$ is compact.
Conversely, suppose that X_L is compact.
Define $f(x) = \begin{cases} 1 & x \in X_L \\ 0 & \text{otherwise} \end{cases}$
Then $f \in mC_K(X)$ and $mC_K(X) = (f)$.□

A family of sets $\{ U_\alpha : \alpha \in \Delta \}$ is called **star finite (countable)** if for each $\alpha \in \Delta$, $U_\alpha \cap U_\beta = \phi$ for all but finitely (countably) many $\beta \in \Delta$. A family of functions $\{f_\alpha : \alpha \in \Delta\}$ is called star finite (countable) if the family {coz $f_\alpha : \alpha \in \Delta$} is star finite (countable).

Theorem 4.4. The following statements are equivalent:
(1) The space $X_L$is strongly paracompact.

(2) The space X_L is paracompact.
(3) The ideal $mC_K(X)$ is generated by a star finite family.
(4) The ideal $mC_K(X)$ is generated by a star countable family..

Proof. (1) $\Rightarrow$ (2): Clear.
(2) $\Rightarrow$ (3): Since $mC_K(X)$ is pure, $S_X(f) \subseteq X_L$ for each $f \in mC_K(X)$. Now the result follows from [8, 3.6 and 3.10].
(3) $\Rightarrow$ (4): Clear.
(4) $\Rightarrow$ (1): Suppose that $mC_K(X) = (f_\alpha : \alpha \in \Delta)$, where the family $\{f_\alpha : \alpha \in \Delta\}$ is star countable.
Let $\{U_\beta : \beta \in \Lambda\}$ be any open cover of X_L. For each $\alpha \in \Delta$, $S_X(f_\alpha) \subseteq X_L \subseteq \bigcup\limits_{\beta\in\Lambda} U_\beta$, which implies that coz $f_\alpha \subseteq S_X(f_\alpha) \subseteq \bigcup\limits_{i=1}^{n_\alpha} U_{\beta_i}$.
Thus the family $\{\text{coz } f_\alpha \cap U_{\beta_i} : \alpha \in \Delta,\ i = 1, 2, 3, ..., n_\alpha\}$ is a star countable refinement of $\{U_\beta : \beta \in \Lambda\}$, and therefore X_L is strongly paracompact. □

Theorem 4.5. The ideal $mC_K(X)$ is countably generated if and only if X_L is Lindelőff.

Proof. Suppose that $mC_K(X) = (f_1, f_2, f_3, ...)$. For each $f \in mC_K(X)$, $f = \sum\limits_{i=1}^{n} g_i f_i$, and coz $f \subseteq \bigcup\limits_{i=1}^{n}$ coz f_i.
Hence $\bigcup\limits_{i=1}^{\infty} S_X(f_i) \subseteq \bigcup\limits_{g\in mC_K(X)} S_X(g) = \bigcup\limits_{g\in mC_K(X)}$ coz $g = X_L = \bigcup\limits_{i=1}^{\infty}$ coz $f_i \subseteq \bigcup\limits_{i=1}^{\infty} S_X(f_i)$. Thus X_L is $\sigma-$compact and so it is Lindelőff.
Conversely, assume that X_L is Lindelőff. Then $X_L = \bigcup\limits_{i=1}^{\infty}$ coz f_i, where $f_i \in mC_K(X)$ for each i.
Let $f \in mC_K(X)$, then $S_X(f) \subseteq X_L = \bigcup\limits_{i=1}^{\infty}$ coz f_i. Compctness of $S_X(f)$ implies that there exists n $\in \mathbb{N}$ such that $S_X(f) \subseteq \bigcup\limits_{i=1}^{n}$ coz f_i = coz g, where $g = \sum\limits_{i=1}^{n} f_i^2$.

Define $h(x) = \begin{cases} \frac{f}{g}(x) & x \in S_X(f) \\ 0 & \text{otherwise} \end{cases}$

Then $h \in C(X)$ and $f = hg = h \sum\limits_{i=1}^{n} f_i^2 = \sum\limits_{i=1}^{n} (hf_i) f_i \in (f_1, f_2, f_3, ...)$.
Hence $mC_K(X)$ is countably generated.□

We now turn to the pure part of the ideal $C_\Psi(X)$. The isomorphisim $f \to f^\upsilon$ from $C(X)$ onto $C(\upsilon X)$ maps $C_\Psi(X)$ onto $C_K(\upsilon X)$ and maps $mC_\Psi(X)$ onto $mC_K(\upsilon X)$, see [1]. Then the following results are easily obtained.

Theorem 4.6. The ideal $m\mathrm{C}_\Psi(\mathrm{X})$ is finitely generated if and only if kX is compact.

Theorem 4.7. The following statements are equivalent:
(1) The space kX is strongly paracompact.
(2) The space kX is paracompact.
(3) The ideal $m\mathrm{C}_\Psi(\mathrm{X})$ is generated by a star finite family.
(4) The ideal $m\mathrm{C}_\Psi(\mathrm{X})$ is generated by a star countable family.

Theorem 4.8. The ideal $m\mathrm{C}_\Psi(\mathrm{X})$ is countably generated if and only if kX is Lindelőff.

REFERENCES

[1] E. Abu Osba, Purity of the Ideal of Continuous Functions with Pseudocompact Support, *Internat. J. Math. Math. Sci.*, **29:7** (2002), 381-388.

[2] E. Abu Osba and H. Al-Ezeh, Purity of the Ideal of Continuous Functions with Compact Support, *Math. Jor. Okayama Univ.*, **41** (1999), 111-120.

[3] H. Al-Ezeh, Pure Ideals in Reduced Gelfand Rings with Unity, *Arch. Math.*, **35** (1989), 266-269.

[4] H. Al-Ezeh, M. A. Natsheh and D. Husein, Some Properties of the Ring of Continuous Functions, *Arch. Math.*, **51** (1988), 60-64.

[5] F. Azarpanah, O. A. S. Karamzadeh and A. Rezai Aliabad, On Ideals Consisting Entirely of Zero Divisors, *Communications in Algebra,* **28(2)** (2000), 1061-1073.

[6] I. Blum and S. Swaminthan, A Note on Intersection of Ideals in Rings of Continuous Functions, *Colloquia Mathematica Societatis János Bolyai,* **23** Topology, Budapest (Hungary), (1978), 155-162.

[7] F. Borceux and G. Van Den Bossche, Algebra in a Localic Topos with Applications to Ring Theory, *LNM* **1038** Berlin-Heidelberg-New York, (1983).

[8] J. G. Brookshear, Projective Ideals in Rings of Continuous Functions, *Pacif. Jor. Math.*, **71(2)** (1977), 313-333.

[9] J. G. Brookshear, On Projective Prime Ideals in C(X), *Proc. Amer. Math. Soc.*, **69** (1978), 203-204.

[10] G. DeMarco, Projectivity of Pure Ideals, *Rend. Sem. Mat. Univ. Padova*, **68** (1983), 289-304.

[11] R. Engelking, General Topology, *Sigma Series in Pure Math., **6** Heldermann Verlag Berlin, (*1989).

[12] L. Gillman and M. Jerison, Rings of Continuous Functions, *Graduate Texts in Math.*, **43** Berlin-Heidelberg-New York, (1976)

[13] M. Mandelker, Round z-Filters and Round Subsets of βX, *Israel J.Math.,* **7** (1969), 1-8.

[14] D. Rudd, On Isomorphisms Between Ideals in Rings of Continuous Functions, *Trans. Amer. Math. Soc.,* **159** (1971), 335-353.

[15] D. Rudd, P-ideals and F-ideals in Rings of Continuous Functions, *Fund. Math.,* **88(1)** (1975), 53-59.

[16] J. Schommer, Functions with Realcompact Support, *Kyungpook Math. J.,* **35** (1995), 93-103.