Relation between Engel Groups and Nilpotent Groups

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Abstract

One knows that every nilpotent group of class n is an n-Engel group, but the converse is fulse in general. In this paper we study the converse of the problem and impose conditions on the groups to get some partial answer in the case of 4-Engel groups.

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§ 1. **Introduction**

A group G is an Engel group if for each ordered pair (x, y) of elements in G there is a positive integer $n(x, y)$ such that

$$[x, [, ..., [x, [x, y]]]] = 1 \qquad (n \;\; copies \;\; of \;\; x). \tag{1}$$

The bracketing is done from the right. But since

$$[[[[y, x], x], ...,], x] = [x^{-1}, [, ..., [x^{-1}, [x^{-1}, y]]]]^{x^n}$$

it does not make difference we use bracketing from the right or left in the definition.

The origin of Engel groups lies in the theory of Lie algebras. As an example, one of the basic classical results for Engel Lie algebras is Engel's Theorem. It states that every finite dimensional Engel Lie algebra over a field is nilpotent. In 1936 Zorn [17] proved a corresponding theorem for Engel groups.

Zorn's Theorem A finite Engel group is nilpotent.

So for finite groups the Engel condition is equivalent to nilpotency, but it is much weaker in general. It is clear from the definition that every locally nilpotent group is an Engel group. But the converse does not hold in general since E. S. Golod [3] has constructed finitely generated Engel groups, which are not nilpotent. For some special classes of groups we do have that the Engel condition is equivalent to local nilpotency. K. W. Gruenberg [4] showed that this is true for soluble groups and in [2], R. Baer shows that this also holds for groups with maximal condition.

Now suppose in (1), the non-negative integer $n = n(x, y)$ can be chosen independent of x and y. We then say that G is an n-Engel group. Now turn back to Engel Lie algebras. We have the following two results of E. I. Zelmanov.

Theorem Z1 [14] Every n-Engel Lie algebra over a field K of characteristic zero is nilpotent.

Theorem Z2 [15] An n-Engel Lie algebra over an arbitrary field is locally nilpotent.

Now consider the corresponding statements for Engel groups, one can raise the following questions:

Question1 Is every torsion free n-Engel group nilpotent?

Question2 Is every n-Engel group locally nilpotent?

The answers to both of the above questions are known to be positive for $n \leq 3$, but the questions remain open for $n \geq 4$. This is of course obvious for $n = 1$, since 1-Engel groups are exactly the abelian groups. In 1942, F. W. Levi [11] solved the problem for $n = 2$. In fact he proved that a group G is a 2-Engel group if and only if the normal closure x^G of an arbitrary element x in G is abelian. Furthemore we have that every 2-Engel group is nilpotent of class at most 3. For 3-Engel groups the problem is much harder. In this case H. Heineken [8] in 1961 gave a positive answer to both questions. He proved that every 3-Engel group G is nilpotent of class at most 4 if G has no elements of order 2 or 5. There are 3-Engel 2-groups and 5-groups which are not nilpotent. In fact there is a 3-Engel 5-group that is not soluble [1] but N. D. Gupta [5] has shown that 3-Engel 2-groups are soluble.

In 1972, L. C. Kappe and W. P. Kappe [10] showed that the following statements are equivalent:

(i) G is a 3-Engel group;

(ii) x^G is a 2-Engel group for all $x \in G$;

(iii) x^G is nilpotent of class at most 2, for all $x \in G$. It follows from property (iii) that a 3-Engel group with r generators has nilpotency class at most $2r$. We do not have a corresponding characterization for 4-Engel groups. N. D. Gupta and F. Levin [7] have constructed a 4-Engel group with an element x such that the nilpotency class of x^G is greater than 3. Finally, to answer the question 1, G. Traustason (see [13]) showed that all 4-Engel groups are locally nilpotent if torsion-free 4-Engel groups are nilpotent and 4-Engel p-groups are locally finite for every prime p. Also to answer the question 2, E. I. Zelmanov (see [15]) proved that a torsion-free locally nilpotent n-Engel group is nilpotent.

In the next section, we give some partial answer to these questions.

§ 2. Relation between 4-Engel groups and Nilpotent groups

In this section we investigate conditions for which some subgroups of 4-Engel groups are nilpotent groups. Then we shall give some partial answer to questions 1 and 2.

The following lemma can be proved easily.

Lemma 2.1 Let G be a torsion-free n-Engel group and $x, y \in G$. If $[x^s, y] = 1$, for some positive integer $s \geq 1$, then $[x, y] = 1$.

Proof. Let $i \geq 1$ be the minimum positive integer such that $[y, x, ..., x] = [y,_i x] = 1 \quad (i \ copies \ of \ x)$, and assume $i > 1$.Then by assumption we have $[[y,_{i-2}x], x^s] = 1$. Therefore $[[y,_{i-2}x], x]^s = 1$, which implies that $[y,_{i-1}x] = 1$, since G is torsion-free. This gives a contradiction to the minimality of i .

The following technical lemmas useful for further investigation.

Lemma 2.2 Let G be a torsion-free n-Engel group, and $x, y \in G$. If $< x^s, y >$ is nilpotent group of class k, say. Then $< x, y >$ is also nilpotent of class k, for every positive integer s.

Proof. By Lemma 2.1, it follows that

$$Z(< x^s, y >) = Z(< x, y >) \cap < x^s, y > .$$

Assume that $k > 1$, and argue by induction on k, and assume the result holds for $k - 1$. The factor group $\dfrac{< x, y >}{Z(< x, y >)}$ is still a torsion-free n-Engel group, and

$$\frac{< x^s, y > Z(< x, y >)}{Z(< x, y >)} \cong \frac{< x^s, y >}{Z(< x^s, y >)}$$

is nilpotent of class $k - 1$. Hence by induction, $\dfrac{< x, y >}{Z(< x, y >)}$ is nilpotent of class $k - 1$, and so $< x, y >$ is nilpotent of class k.

Lemma 2.3 Let G be a 4-Engel group, $x, y \in G$, and $a = x^{-1}yx$. Then $< a, a^y >$ is nilpotent of class at most 2.

Proof. Clearly the indentity $[x^{-1}, y, y, y, y] = 1$, gives $[[y, x]^{x^{-1}}, y, y, y] = 1$, and $[y, x, y^x, y^x, y^x] = 1$.

Clearly $1 = [y^{-1}y^x, y^x, y^x, y^x] = [y^{-1}, y^x, y^x, y^x]^{y^x}$ and $[y^{-1}, a, a, a] = 1$, i.e. $[[a, y]^{y^{-1}}, a, a] = 1$, and $1 = [a, y, a^y, a^y] = [a^{-1}a^y, a^y, a^y]$. Hence $[a^{-1}, a^y, a^y] = 1$. Arguing with y^{-1} instead of y and a^{-1} instead of a, it implies that $[a, (a^{-1})^{y^{-1}}]$ commutes with $a^{y^{-1}}$, i.e. $[a^y, a^{-1}]$ commutes with a. Hence $[a^y, a^{-1}] = [a, a^y]^{a^{-1}}$ commutes with a and a^y and so $[a, a^y] \in Z(< a, a^y >)$, i.e. $< a, a^y >$ is nilpotent of class at most 2.

By Lemmas 2.1, 2.2 and 2.3 we are able to prove the following proposition.

Proposition 2.4 Let G be a torsion-free 4-Engel group, and a, y are conjugate elements of G. Then the subgroups

$$< a, a^y >, < y, y^a >, < y, y^{a^3} >, < a, a^{y^3} >, < a, a^{y^2} >$$

are all nilpotent of class at most 2.

Proof. The nilpotence properties of $< a, a^y >$ and $< y, y^a >$ follow by Lemma 2.3. Similarly $< y^3, (y^3)^{a^3} >$ is nilpotent of class at most 2. Lemma 2.2, follows that $< y, y^{a^3} >$ is nilpotent of class at most 2. One may argue for other subgroups, in the same way.

Now, we are able to prove the main Theorem of this paper, which provides some partial answer to the questions of section 1, under some condition.

Theorem 2.5 Let G be a torsion-free 4-Engel group, and a, y are conjugate elements of G. Then $< a, y >$ is nilpotent of class at most 4.

Proof. By the assumption a, y are conjugate elements of G. So by Proposition

2.4, $< [a^3, y], y >$ is nilpotent of class at most 2. But

$$[a^3, y] = [a^2, y]^a[a, y] = [a, y]^{a^2}[a, y]^a[a, y] = [a, y][a, y, a]^2[a, y, a][a, y],$$

since by Proposition 2.4 $< [a, y], a >$ is nilpotent of class at most 2.

Hence

$$[a^3, y] = [a, y]^3[a, y]^{-3}([a, y]^a)^3 = ([a, y]^a)^3.$$

Therefore $< ([a, y]^a)^3, y >$ is nilpotent of class at most 2, and $< [a, y]^a, y >$ is nilpotent of class at most 2, by Lemma 2.2.

It follows that $[a, y, y^{a^{-1}}]$ commutes with $y^{a^{-1}}$, i.e.

$$[[a, y], y[y, a^{-1}]] = [[a, y], [y, a^{-1}]][a, y, y]^{[y,a^{-1}]}$$

commutes with y^{-1}.

But

$$[[a, y], [y, a^{-1}]] = [[a, y], [a, y]^{a^{-1}}] = [[a, y], [a, y][a, y, a^{-1}]] = 1$$

since $< [a, y], a >$ is nilpotent of class at most 2. Hence $[a, y, y]^{[y,a^{-1}]}$ commutes with $y^{a^{-1}}$, and $[a, y, y]^{y^{-1}a}$ commutes with y.

Also $[a, y, y, y^{-1}] = 1$, since $< y, y^a >$ is nilpotent of class at most 2 and we have that $[a, y, y]^{y^{-1}a} = [a, y, y]^a$ commutes with y, and $[a, y, y, a]$ commutes with y. But $[a, y, y, a] = [a, y, y]^{-1}[a, y, y]^a$ also commutes with y^a , since $[a, y, y]$ does and $[a, y, y, y] = 1$. Therefore $[a, y, y, a]$ commutes with y and $[a, y]$.

From $< [a, y^2], a >$ nilpotent of class at most 2, by Proposition 2.4, we also get that $[[a, y]^2[a, y, y], a]$ commutes with a. Hence $[[a, y, y][a, y]^2, a] = [a, y, y, a]^{[a,y]^2}[[a, y]^2, a]$

commutes with a, from which $[a, y, y, a]$ commutes with a, since $[[a, y]^2, a] = [a, y, a]^2$ commutes with a, because $< [a, y], a >$ is nilpotent of class at most 2. Hence $[a, y, y, a] \in Z(< a, y >)$. From $[a, y, y, y] = 1$ we get that $[a, y, y] \in Z_2(< a, y >)$. Arguing similarly $[y, a, a] \in Z_2(< a, y >)$. Hence $[y, a] \in Z_3(< a, y >)$ and $< a, y >$ is nilpotent of class at most 4, as required.

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