k-CURVATURE HOMOGENEOUS PSEUDO-RIEMANNIAN MANIFOLDS WHICH ARE NOT LOCALLY HOMOGENEOUS

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Let R be the Riemann curvature tensor of a pseudo-Riemannian manifold (M, g_M) of signature (p, q) on a smooth manifold M of dimension $m := p + q$. We say that (M, g_M) is *k-curvature homogeneous* if given any two points $P, Q \in M$, there exists an isomorphism $\phi_{P,Q}$ from T_PM to T_QM so that

$$\phi^* g_Q = g_P,\ \phi^* R_Q = R_P,\ ...,\ \phi^* \nabla^k R_Q = \nabla^k R_P\,.$$

This means that the metric, curvature tensor, and covariant derivatives up to order k of the curvature tensor "look the same" at each point. Takagi [4] was the first to exhibit 0-curvature homogeneous Riemannian manifolds which were not locally homogeneous; his examples were non compact. Compact examples were first exhibited by Ferus, Karcher, and Münzer [2]; many other examples have been found subsequently. In the Lorentzian setting, 1-curvature homogeneous manifolds which are not locally homogeneous were constructed by Bueken and Vanhecke [1]. There were, however, no known examples of pseudo-Riemannian manifolds which were k-curvature homogeneous but not locally homogeneous for $k \geq 2$. In this note, we exhibit k-curvature homogeneous manifolds for arbitrary k which are of neutral signature and which are not locally homogeneous [3].

Let $k = p + 2 \geq 2$ be given. Let $(x, y, z_0, ..., z_p, \bar{x}, \bar{y}, \bar{z}_0, ..., \bar{z}_p)$ be coordinates on $\mathbb{R}^{2p+6}$. Let $f = f(y) \in C^\infty(\mathbb{R})$. Let $g_{2p+6,f}$ be the pseudo-Riemannian manifold of balanced (i.e. neutral) signature $(p + 3, p + 3)$ on $\mathbb{R}^{2p+6}$ where:

$$\begin{aligned}
&F_{2p+6,f}(y, \vec{z}) := f(y) + yz_0 + y^2 z_1 + ... + y^{p+1} z_p,\\
&g_{2p+6,f}(\partial_{z_i}, \partial_{\bar{z}_j}) = \delta_{ij}, \quad g_{2p+6,f}(\partial_x, \partial_{\bar{x}}) = 1,\\
&g_{2p+6,f}(\partial_y, \partial_{\bar{y}}) = 1, \quad \text{and} \quad g_{2p+6,f}(\partial_x, \partial_x) = -2F_{2p+6,f}(y, \vec{z})\,.
\end{aligned}$$

Theorem: *Assume $f^{(p+3)} > 0$ and $f^{(p+4)} > 0$. Let $\mathcal{M} := (\mathbb{R}^{2p+6}, g_{2p+6,f})$.*

(1) *All geodesics in $\mathcal{M}$ extend for infinite time.*
(2) $\exp_P : T_P\mathbb{R}^{2p+6} \to \mathbb{R}^{2p+6}$ *is a diffeomorphism for any $P \in \mathbb{R}^{2p+6}$.*
(3) *$\mathcal{M}$ is $p + 2$-curvature homogeneous.*
(4) *For generic f, $\mathcal{M}$ is not $p + 3$-affine curvature homogeneous.*
(5) *$\mathcal{M}$ is Ricci flat, nilpotent Osserman, and nilpotent Ivanov-Petrova.*

References

[1] P. Bueken and L. Vanhecke, Examples of curvature homogeneous Lorentz metrics, *Classical Quantum Gravity* **14** (1997), L93–L96.
[2] D. Ferus, H. Karcher, H. Münzner, Cliffordalgebren und neue isoparametrische Hyperflächen, *Math. Z.* **177** (1981), 479–502.
[3] P. Gilkey and S. Nikčević, Complete k-curvature homogeneous pseudo-Riemannian manifolds (preprint).
[4] H. Takagi, On curvature homogeneity of Riemannian manifolds, *Tôhoku Math. J.* **26** (1974), 581–585.