On a nice open problem

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Definition1. Let n and k be two natural numbers such that $n \leq k$, and let $A := \{1, 2, ..., k\}$. Denote with A^n all subsets of A with n elements, namely $A^n := \{E \subseteq A| \mid E \mid= n\}$. Let $X \subseteq A^n$ be an arbitrary subset of A^n. The pair $G := (A, X)$ is called n-uniform hypergraph. We say that A is the set of nodes and X is the set of edges of G.
Example: $n := 3 \Rightarrow A = \{1, 2, 3, 4, 5\}$ and
$A^3 = \{(1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 3, 4), (1, 3, 5), (1, 4, 5), (2, 3, 4), (2, 3, 5), (2, 4, 5), (3, 4, 5)\}$.
Let $X := \{(1, 3, 4), (2, 3, 5), (2, 4, 5), (3, 4, 5)\}$, then $G := (A, X)$ is an 3-uniform hypergraph.
Definition2. We say that an n-uniform hypergraph $G := (A, X)$ is 2-colourable if there exists a surjective function $f : A \to \{a, b\}$ such that $\forall x \in X$, the restriction of f relative to x, namely $f|_x$ is surjective too.
A nice open problem is the following one: for every $n \geq 2$ determine the non 2-colourable hypergraph with the smallest number of edges. Note that there is no requierement about the number of nodes. This is the nice thing on this problem. In what follows we will give $\forall n \geq 2$ a non 2-colourable n -uniform hypergraph with some analysis on the number of nodes.
Proposition: Let $n \in \mathbb{N}, n \geq 2$ and let $A := \{1, 2, ..., 2n - 1\}$. The hypergraph $G := (A, A^n)$ is non 2-colourable.
Proof: Suppose that G is 2-colourable. It means that there exists a surjective function $f : A \to \{a, b\}$ such that $\forall x \in A^n, f|_x$ is surjective too. We do following notations: $f^{-1}(a) := \{z \in A|f(z) = a\}$ and $f^{-1}(b) := \{z \in A|f(z) = b\}$. Clearly we have

$$|A| = |f^{-1}(a)| + |f^{-1}(b)| \tag{1}$$

Suppose now that

$$|f^{-1}(a)| \leq n - 1 \vee |f^{-1}(b)| \leq n - 1 \tag{2}$$

This would mean, by (1), that $|A| \leq 2n - 2$ which is certainly false since $|A| = 2n - 1$. It follows that relation (2) is false. Therefore

$$|f^{-1}(a)| \geq n \wedge |f^{-1}(b)| \geq n \tag{3}$$

Consider the case $|f^{-1}(a)| \geq n$. The other one is the same story. Since A^n contains all subsets of A with n elements and since $f^{-1}(a)$ is subset of A, A^n will also contain all subsets with n elements of $f^{-1}(a)$. This means that $\exists x \in A^n$ such that $f|_x = a$, so $f|_x$ is not surjective. But this is contradiction with our initial supposition, therefore the proposition is true.
We proved above that $\forall n \geq 2, G := (A, A^n)$ is a non 2-colourable hypergraph. The number of nodes of G is $2n - 1$ and the number of edges is $\frac{(2n-1)!}{n!(n-1)!}$. We can easily see that with number of nodes less than $2n - 1$ it is impossible to construct a non 2-colourable n-uniform hypergraph and it also can be proved (not so easily)that with $2n$ nodes the "smallest" non 2-colourable n-uniform hypergraph has also $\frac{(2n-1)!}{n!(n-1)!}$ edges.