Modeling and numerical simulation of causal nonlinear systems

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The constructive approximation of non-linear operators is particularly relevant to the study of dynamical systems where a proper description of non-linear behaviour is essential to the development of efficient algorithms for real time processing [1–3]. We consider the mathematical basis for a suitable description.

To construct a mathematical model of a *realistic* dynamical system it is necessary to formalise definitions of such crucial physical properties as *causality*, *finite memory* and *stationarity*. The philosophy of realistic systems has been considered by many authors including Russell, Paley and Wiener, Foures and Segal, Falb and Freedman, Willems, Gohberg and Sandberg and Xu.

We propose a generic topological structure to describe *realistic* non-linear systems and extend the methods of Torokhti and Howlett to prove *stable* approximation theorems for numerical simulation of these systems. We define a class of $\mathcal{R}$-operators and prove that an $\mathcal{R}$-continuous operator F can be approximated by an $\mathcal{R}$-continuous operator S constructed from an algebra of elementary functions by a finite arithmetic process. The approximation is *stable* to small disturbances.

Our main result is a wide generalisation of the Stone-Weierstrass theorem. Theorems of this type were extended to operators on topological vector spaces by Prenter and Bruno. A Stone-Weierstrass theory for approximation of continuous functions by superpositions of a sigmoidal function was given by Cybenko. Daugavet considered non-linear operator approximation by generalised causal operators.

We provide a substantial extension of this work and show that our definition of the $\mathcal{R}$-continuous operator includes the accepted notions of causality and other fundamental realistic properties as special cases. Several key results on operator approximation also follow from particular applications of our main theorems. We also show how certain specific approximation problems can be formulated and solved for $\mathcal{R}$-operators.

References

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