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Some theorems on the multistage iterations and solvability of linear Cauchy problem

We are interested in conditions sufficient for the unique solvability of the n-dimensional initial value problem of the form

$$u'(t) = (lu)(t) + f(t), \quad t \in [a,b]; \qquad u(\tau) = c \tag{1}$$

where $-\infty < a < b < +\infty$, $u : [a,b] \to \mathbb{R}^n$, $n \in \mathbb{N}$. Here, $c \in \mathbb{R}^n$, $f \in L([a,b],\mathbb{R}^n)$, $\tau \in [a,b]$, and $l = (l_k)_{k=1}^n : C([a,b],\mathbb{R}^n) \to L([a,b],\mathbb{R}^n)$ is a linear operator which, for some fixed $\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \subset \{-1,1\}^n$, is $(\sigma_1, \sigma_2, \ldots, \sigma_n; \tau)$-positive in the sense that

$$\sigma_k (l_k u)(t) \operatorname{sign}(t-\tau) \geq 0 \quad \text{for } 1 \leq k \leq n \text{ and a.e. } t \in [a,b],$$

whenever the continuous function $u = (u_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$ satisfies the condition $\sigma_k u_k(t) \geq 0, \quad 1 \leq k \leq n, \ t \in [a,b]$.

Theorem. *Let l in (1) be $(\sigma_1, \sigma_2, \ldots, \sigma_n; \tau)$-positive. Assume that there exist some $\rho \in (1,+\infty)$, $m, r \in \mathbb{N}$, $m \geq r \geq 1$, $\{\beta_k \mid k = 1,2,\ldots,m\} \subset [0,\infty)$, $\beta_0 \in (0,\infty)$, $\{\alpha_i \mid i = 1,2,\ldots,r\} \subset [0,\infty)$ and absolutely continuous vector-functions $y_0 = (y_{j0})_{j=1}^n, y_1 = (y_{j1})_{j=1}^n, \ldots, y_{r-1} = (y_{jr-1})_{j=1}^n$, such that*

$$y_{jk}(\tau) = 0; \quad \sigma_j y_{jk}(t) > 0 \text{ for } t \in [a,b] \setminus \{\tau\}, \ 1 \leq j \leq n, \ k = 0,1,\ldots,r-1$$

and

$$\sigma_j \sum_{k=0}^{r-1} \beta_k y_{jk}(t) > 0 \quad \text{for all} \quad t \in [a,b] \setminus \tau$$

and, furthermore, the following condition is satisfied:

$$\sigma_j \Big[\sum_{k=0}^{r-1} \beta_k y'_{jk}(t) + \sum_{k=0}^{m-1} \Big(\sum_{\nu \in T_{r,m}(k)} \beta_{\nu+k} \alpha_\nu - \rho \beta_k \Big) (l_j y_k)(t) - \\ - \rho \beta^m (l_j y_m)(t) \Big] \operatorname{sign}(t-\tau) \geq 0 \tag{2}$$

for all j, $1 \leq j \leq n$, and almost every $t \in [a,b]$, where, by definition,

$$y_k(t) := \sum_{i=1}^r \alpha_i \int_\tau^t (l y_{k-i})(s) ds, \qquad t \in [a,b], \quad k \geq r$$

and $T_{r,m}(k) := \{\nu \in \mathbb{N} \mid \nu \leq r \leq \nu + k \leq m\}$.

Then the Cauchy problem (1) has a unique solution $u(\cdot)$ for arbitrary $c \in \mathbb{R}^n$ and $f \in L([a,b],\mathbb{R}^n)$.

It can be shown that condition (2) with $\rho = 1$ may not be sufficient for the unique solvability of problem (1) and, thus, the strict inequality $\rho > 1$ is essential in the theorem above.