Margo Kondratieva and Sergey Sadov

Large wavenumber asymptotics of the symbol of the Dirichlet-to-Neumann operator in an exterior problem

Consider the exterior Dirichlet problem for the 2D Helmholtz equation outside a bounded domain Ω with smooth boundary Γ

$$u_{xx} + u_{yy} + k^2 u = 0 \quad \text{in} \quad \mathbf{R}^2 \setminus \bar{\Omega}, \qquad u|_\Gamma = f. \tag{1}$$

If $k^2 > 0$, then Sommerfeld's radiation condition is imposed to uniquely determine the solution. For $\operatorname{Im} k^2 > 0$, there exists a unique L_2 solution.

We study the Dirichlet-to-Neumann operator $\mathcal{N} : \, f \to \partial_n u\,|_\Gamma$. It depends on k. Using parametrization of the boundary by the normalized arclength $s \bmod 2\pi$, we treat $\mathcal{N}(k)$ as a pseudodifferential operator of order 1 on the unit circle. Let $\sigma(s, n;\, k)$, $n \in \mathbf{Z}$, be its discrete symbol.

In case of Γ being a circle, problem (1) has an explicit solution in terms of Hankel functions. The uniform asymptotic behaviour of the symbol $\sigma(s, n;\, k)$ is as follows

$$\sigma(s, n;\, k) \;\sim\; i\sqrt{k^2 - n^2}, \qquad k, n \to \infty, \quad k/n \to \text{const.} \tag{2}$$

The branch of the root satisfies $\operatorname{Re}\sqrt{} \geq 0$, $\operatorname{Im}\sqrt{} \geq 0$.

Asymptotics (2) is valid for an arbitrary boundary Γ, if k tends to infinity along a ray in the open upper complex half-plane. A similar asymptotics holds for the interior problem. Proof is based on ellipticity with parameter of the operator $\Delta + k^2$.

Conjecture. *Asymptotics* (2) *holds for real* k *in the exterior problem with any smooth* Γ.

The conjecture is backed by numerical experiments, including non-convex domains. It is consistent (at the physical level of rigor) with Kirchhoff approximation. Formula (2), unlike Kirchhoff's, isn't sensitive to the presence of flattened boundary regions.

This work is a part of research aimed at a robust numerical algorithm for diffraction problems in mid-high frequency range.

Reference: M. Kondratieva, S. Sadov, Symbol of the Dirichlet-to-Neumann operator in 2D diffraction problems with large wavenumber, Proc. "Days on Diffraction-03", St. Petersburg University, 2003, 88–98, arXiv/physics/0310048.