ON THE DEFINITION OF BOLIC SPACES

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Abstract. We discuss notions of bolic metric spaces introduced by Kasparov and Skandalis. We make some observations which clarify the involved metric axioms and explain all the known basic examples.

1. Introduction

Bolic metric spaces were introduced by Kasparov and Skandalis in [KS94] and [KS00] in relation with conjectures of Baum-Connes and Novikov. This class of metric spaces, which includes Gromov hyperbolic geodesic spaces and non-positively curved spaces, was also considered by V. Lafforgue in [La01]. Further works with some discussion on bolicity include [Tu99], [MY01] and [BCM01].

In view of the importance of the above mentioned works and the potential usefulness of these notions in geometric group theory more generally, it seems worthwhile to try to clarify and examplify the various metric space axioms involved. This is the purpose of this note.

We show that the condition B2 implies B1, hence this latter condition is superfluous in the definition of a δ-bolic space (for all terminology see the next section). In view of this, it is immediate that every CAT(0)-space is δ-bolic for any $\delta > 0$ (for complete spaces the converse holds as well).

We introduce δ-uniformly convex and δ-uniformly smooth metric spaces generalizing standard notions in Banach space theory. We prove that uniform smoothness implies B1 and that uniform convexity implies B2′. This shows that L^p-spaces for $1 < p < \infty$ are weakly δ-bolic for every $\delta > 0$. Furthermore, spaces which satisfy B2 (e.g. δ-hyperbolic geodesic spaces) are both δ-uniformly smooth and δ-uniformly convex. This suggests an alternative and probably not significantly more stringent definition of weak bolicity obtained by replacing δ-B1 by δ-uniform smoothness and δ-B2′ by $(1, \delta)$-uniform convexity.

2. Definitions

Here follow two collections of metric space axioms. Examples are given in the last section of this paper.

Bolic and weakly bolic spaces. We recall the definition of Kasparov and Skandalis:

Definition. Let δ be a positive number. A metric space (X, d) is called δ*-bolic* if the following two conditions hold:

- δ-B1

$$\forall\, r>0, \exists\, R=R(r,\delta)>0 \text{ such that } \forall\, x_1,x_2,y_1,y_2\in X, \text{ if}$$
$$d(x_1,x_2)+d(y_1,y_2)\le r,$$
$$d(x_1,y_1)+d(x_2,y_2)\ge R,$$
$$\text{then } d(x_1,y_1)+d(x_2,y_2)\le d(x_1,y_2)+d(x_2,y_1)+2\delta.$$

- δ-B2

$$\exists\, m: X\times X\longrightarrow X \text{ such that } \forall\, x,y,z\in X,$$
$$2d(m(x,y),z)\le\sqrt{2d(x,z)^2+2d(y,z)^2-d(x,y)^2}+4\delta.$$

The conditions that Kasparov and Skandalis in fact need for their work on the Novikov conjecture are the following:

Definition. Let δ be a positive number. A metric space (X,d) is called *weakly δ-bolic* if the following two conditions hold:

- δ-B1
- δ-B2$'$
  (1) there exists a δ-middle point map $m: X\times X\longrightarrow X$, that is, for every $x,y\in X$,
  $$|2d(x,m(x,y))-d(x,y)|\le 2\delta,$$
  $$|2d(y,m(x,y))-d(x,y)|\le 2\delta,$$
  (2) for every $x,y,z\in X$,
  $$d(m(x,y),z)\le\max(d(x,z),d(y,z))+2\delta,$$
  (3) for every $p\in\mathbb{R}_+$, there exists $N(p)\in\mathbb{R}_+$ such that for every $N\ge N(p)$, if
  $$d(x,z)\le N,\ d(y,z)\le N \text{ and } d(x,y)>N,$$
  then
  $$d(m(x,y),z)<N-p.$$

By a (*weakly*) *bolic space* we mean a metric space which is (weakly) δ-bolic for some $\delta>0$.

In Lafforgue's work on the Baum-Connes conjecture it is of importance for the bolic spaces to satisfy δ-B1 for every $\delta>0$. This is sometimes refered to as *strong bolicity.*

Uniform smoothness and convexity. The following are possible definitions of uniform smoothness and uniform convexity for arbitrary metric spaces generalizing the corresponding concepts in Banach space theory to be found for example in [Di75].

For a $\delta\ge 0$ we shall call a *δ-midpoint map* a symmetric map $m: X\times X\to X$ such that for every $x,y\in X$,

$$|2d(x,m(x,y))-d(x,y)|\le 2\delta,$$
$$|2d(y,m(x,y))-d(x,y)|\le 2\delta.$$

For example, geodesic spaces have 0-midpoint maps.

Definitions.

(1) A metric space (X,d) is δ-*uniformly smooth*, if there exists a δ-midpoint map m and for every $\varepsilon>0$ there exists $\eta=\eta(\varepsilon)>0$ such that for every $x,y,z\in X$ satisfying $\frac{d(y,z)}{d(x,z)}\leq\eta$ the following inequality holds

$$d(m(x,y),z)+\frac{1}{2}d(x,y)\leq d(x,z)+\varepsilon d(y,z)+2\delta.$$

(2) A metric space (X,d) is δ-*uniformly convex*, if there exists a δ-midpoint map m and for every $\varepsilon>0$ there exists $\rho=\rho(\varepsilon)>0$ such that for any x,y,z in X satisfying $d(x,y)\geq\varepsilon\max\{d(x,z),d(y,z)\}$ the following inequality holds

$$d(m(x,y),z)\leq(1-\rho)\max\{d(x,z),d(y,z)\}+2\delta.$$

We call a space *uniformly convex (smooth)* if the corresponding condition holds with $\delta=0$. We call a metric space $(1,\delta)$-*uniformly convex* if the condition of uniform convexity is only required to hold for every $\varepsilon\geq1$.

These conditions hold for L^p-spaces, $1<p<\infty$, see section 4.

As is indicated in the next section, B1 is a midpoint-free version of uniform smoothness. Compare this with the standard definitions of a δ-hyperbolic space: one definition is formulated in terms of geodesics (thin triangles) while the other, which is recalled in section 4, is more general in that it is "midpoint-free".

Any subset A of a metric space (X,d) is a metric space by restriction of the metric. For example, when a group G acts freely on X, one gets a metric on G by identifying it with an orbit Gx. If (X,d) satisfies a certain condition, then (A,d) will automatically satisfy the midpoint-free version of that same condition.

3. Relations between the axioms

Uniform smoothness implies B1. We will show:

Proposition 1. *If a metric space is δ-uniformly smooth, then it satisfies $(2\delta+\kappa)$-B1 for any $\kappa>0$.*

Proof. Pick $\kappa>0$ and $r>0$. Set $\varepsilon=\frac{\kappa}{r}$ and $R=\frac{r}{\eta(\varepsilon)}$. Let $x,y,z\in X$ be such that $d(x,z)\geq R$ and $d(y,z)\leq r$. Then $\frac{d(y,z)}{d(x,z)}\leq\frac{r}{R}=\eta(\varepsilon)$ so that by uniform smoothness

$$\begin{aligned} d(m(x,y),z) &\leq d(x,z)-\frac{1}{2}d(x,y)+\varepsilon d(y,z)+2\delta \\ &\leq d(x,z)-\frac{1}{2}d(x,y)+\kappa+2\delta \end{aligned}$$

In view of Lemma 2 below we are done. □

The following lemma gives a criterium (with midpoints) slightly stronger than δ-B1, but which is easier to handle.

Lemma 2. *Let (X,d) be a metric space, $m:X\times X\to X$ a map, and $\delta>0$. If for every $r'>0$ there exists $R'=R'(r',\delta)>0$ such that for every*

$x, y, z \in X$ satisfying

$$d(x,z) \geq R', \qquad d(y,z) \leq r'$$

the following inequality holds

$$d(m(x,y),z) \leq d(x,z) - \frac{1}{2}d(x,y) + \delta,$$

then (X,d) satisfies δ-B1.

Proof. Pick $r > 0$ and set $R = R(r,\delta) = 2R'(r,\delta) + 2r$. Let $x_1, x_2, y_1, y_2 \in X$ be such that they satisfy the hypothesis of B1. Then clearly $d(x_1,x_2)$, $d(y_1,y_2) \leq r$ and at least one of $d(x_1,y_1)$ and $d(x_2,y_2)$ is greater or equal to $R/2$. Therefore by the triangle inequality

$$d(x_i,y_j) \geq \frac{R}{2} - r = R'(r,\delta)$$

for any $i,j \in \{1,2\}$. Therefore y_2, x_2, x_1 and x_2, y_2, y_1 satisfy the conditions of the assumed criterium. Now it remains to compute

$$\begin{aligned} d(x_1,y_1) &\leq d(x_1, m(x_2,y_2)) + d(m(x_2,y_2),y_1) \\ &\leq d(y_2,x_1) - \frac{1}{2}d(x_2,y_2) + \delta + d(x_2,y_1) - \frac{1}{2}d(x_2,y_2) + \delta. \end{aligned}$$

Hence (X,d) satisfies δ-B1. □

B2 implies B1. The following shows that one can drop the B1-condition in the definition of δ-bolic spaces.

Proposition 3. *If a metric space is δ-B2 for some $\delta > 0$, then it is δ-uniformly smooth.*

Proof. Given $\varepsilon > 0$, pick $\eta = 2\varepsilon$. Let $x, y, z \in X$ be such that $d(y,z) \leq \eta d(x,z)$. Then using δ-B2 and the inequalities

$$\begin{aligned} \sqrt{1+t} &\leq 1 + \frac{1}{2}t, \\ 2t^2 - u^2 &\leq (2t-u)^2, \end{aligned}$$

for $t \geq 0$, we obtain

$$\begin{aligned} 2d(m(x,y),z) &\leq \sqrt{2d(x,z)^2 + 2d(y,z)^2 - d(x,y)^2} + 4\delta \\ &\leq 2d(x,z)\sqrt{1 + \Big(\frac{d(y,z)}{d(x,z)}\Big)^2} - d(x,y) + 4\delta \\ &\leq 2d(x,z) + \frac{d(y,z)^2}{d(x,z)} - d(x,y) + 4\delta \\ &\leq 2(d(x,z) - \frac{1}{2}d(x,y) + \varepsilon d(y,z) + 2\delta), \end{aligned}$$

as required. □

From Propositions 1 and 3 we hence get:

Corollary 4. *If a metric space is δ-B2 for some $\delta > 0$ then it is $(2\delta+\kappa)$-B1 for any $\kappa > 0$, hence it is 3δ-bolic.*

Uniform convexity implies B2$'$.

Proposition 5. *If a metric space is δ-uniformly convex (or more generally, $(1, \delta)$-uniformly convex), then δ-B2$'$ holds.*

Proof. We only need to check condition (3). Let $p \in \mathbb{R}_+$. Take $N(p) > (p + 2\delta)/\rho(1)$, where $\rho(1)$ corresponds to $\varepsilon = 1$ in the uniform convexity definition, and $N \geq N(p)$. Suppose $d(x, y) \leq N$, $d(y, z) \leq N$ and $d(x, y) > N$. Then we get

$$\begin{aligned} d(m(x,y), z) &\leq (1 - \rho(1)) \max\{d(x, z), d(y, z)\} + 2\delta \\ &\leq N - N(p)\rho(1) + 2\delta < N - p. \end{aligned}$$

□

B2 implies B2$'$. We now show that, as expected (see [KS94]), bolic spaces are weakly bolic.

Proposition 6. *If a metric space satisfies δ-B2, then it is δ-uniformly convex.*

Proof. The map m in δ-B2 is actually a δ-midpoint map. Indeed, for every $x, y \in X$,

$$\begin{aligned} 2d(x, m(x, y)) &\leq \sqrt{2d(y, x)^2 - d(x, y)^2} + 4\delta \\ &= d(x, y) + 4\delta. \end{aligned}$$

Similarly, $2d(y, m(x, y)) \leq d(x, y) + 4\delta$, so that

$$d(x, y) - 2d(x, m(x, y)) \leq 2d(y, m) - d(x, y) \leq 4\delta,$$

and hence

$$|2d(x, m(x, y)) - d(x, y)| \leq 2\delta.$$

Let $R = \max\{d(x, z), d(y, z)\}$, from B2 we now get

$$\begin{aligned} d(m(x, y), z) &\leq \frac{1}{2}\sqrt{R^2 - d(x, y)^2} + 2\delta \\ &= R\sqrt{1 - \left(\frac{d(x, y)}{2R}\right)^2} + 2\delta \\ &= R\sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2} + 2\delta. \end{aligned}$$

This shows the condition (2) and the δ-uniform convexity which implies condition (3) in view of the previous proposition. □

Propositions 5 and 6 imply:

Corollary 7. *A δ-bolic metric space is weakly δ-bolic.*

B1 versus B2′. Note that there exist metric spaces which satisfy B2′ but not B1. Indeed, define in $\mathbb{R}^2$ a norm which is given by its unit sphere: a regular octagon. Proposition 5 (or a remark in [KS00]) implies that this space satisfies B2′, but on the other hand this space does not satisfy B1 as follows similarly as for L^1, see Section 4.

Conversely, there also exist metric spaces which satisfy B1 but not B2′: the induced metric space obtained by mapping $\mathbb{Z}$ into the real line with its usual distance by sending n to (for example) n^2, yields a discrete metric space without a δ-midpoint map and which clearly satisfies B1. (Note moreover that this metric space is δ-hyperbolic).

4. Examples

Non-positive curvature. Let us recall the beautiful characterization of non-positive curvature due to Bruhat and Tits (see e.g. [L99]): a complete metric space (X, d) is CAT(0) if and only if for any x, y there is a point m for which

$$d(x,y)^2 + 4d(m,z)^2 \leq 2d(x,z)^2 + 2d(y,z)^2$$

holds for every z in X. With this in mind it is now clear that all the conditions in Section 2 are satisfied for such spaces.

Corollary 8. *A metric space is δ-B2 for every $\delta > 0$ if and only if it is δ-bolic for every $\delta > 0$, which is equivalent to being* CAT(0) *if the space is complete.*

Proof. It remains to show that δ-B2 for every $\delta > 0$ implies CAT(0). Suppose a complete metric space (X, d) is δ-bolic $\forall\, \delta > 0$. We get a sequence of δ-midpoint maps which converge to a limit m, since writing $m_\delta := m_\delta(x, y)$, $m_{\delta'} := m_{\delta'}(x, y)$, we obtain

$$\begin{aligned} 2d(m_{\delta'}, m_\delta) &\leq \sqrt{2d(x,m_\delta)^2 + 2d(y,m_\delta)^2 - d(x,y)^2} + 4\delta' \\ &\leq \sqrt{2\big(\frac{1}{2}d(x,y)+\delta\big)^2 + 2\big(\frac{1}{2}d(x,y)+\delta\big)^2 - d(x,y)^2} + 4\delta' \\ &= \sqrt{4\delta\big(d(x,y)+\delta\big)} + 4\delta', \end{aligned}$$

which is arbitrarily small for δ, δ' small enough. The limit map so obtained is a midpoint map and makes the inequality of B2 hold with $\delta = 0$, so that X is CAT(0). □

In [KS94] and [KS00] it was shown that every non-positively curved simply connected Riemanian manifold and every Euclidean building is δ-bolic for every $\delta > 0$.

Normed vector spaces. For normed vector spaces the concepts of uniform convexity and smoothness coincide with the standard ones. These two properties are dual to each other in the sense that a normed space is uniformly convex (respectively smooth) if and only if its dual space is uniformly smooth (respectively convex). Every L^p-space for $1 < p < \infty$ is uniformly convex, due to Clarkson [Cl36], and hence by duality also uniformly smooth, see [Di75].

The proof of the following proposition provides a simple argument showing that a Banach space is CAT(0) if and only if it is a Hilbert space.

Proposition 9. *If a normed vector space satisfies δ-B2 for some $\delta > 0$, then it is a pre-Hilbert space.*

Proof. First note that δ can be set to 0. Indeed, if three points $x, y, 0$ satisfy

$$2d(m(x,y),0) \geq \sqrt{2d(x,0)^2 + 2d(y,0)^2 - d(x,y)^2} + \varepsilon^2,$$

then by multiplying the inequality by a number t such that $t\varepsilon$ is much larger than δ and using the normed linear structure we see that the δ-B2 inequality would fail.

We now have for any vectors x and y using B2 twice:

$$\begin{aligned} ||x+y||^2 + ||x-y||^2 &\leq 2\,||x||^2 + 2\,||y||^2 \\ &\leq 2\,||x||^2 + 2\left(2\left\|0 - \frac{x+y}{2}\right\|^2 + 2\left\|y - \frac{x+y}{2}\right\|^2\right. \\ &\quad \left. -4\left\|\frac{y}{2} - \frac{x+y}{2}\right\|^2\right) \\ &= ||x+y||^2 + ||x-y||^2, \end{aligned}$$

which shows that we in fact have equality everywhere. Hence the parallelogram law holds which was to be proved. □

Proposition 10. *The L^p-spaces are δ-bolic for every $\delta > 0$ if and only if $p = 2$. They are weakly δ-bolic for every $\delta > 0$ if and only if $1 < p < \infty$. (We exclude the 1-dimensional spaces $L^p(\{x\})$.)*

Proof. In view of Propositions 1, 3, 5 and 9, it remains to check that L^1 and L^∞ are not δ-B1. For $L^1(\{x,y\})$, given $r > 0$ consider the four points $x_1 = (0,0)$, $x_2 = (0, r/2)$, $y_1 = (R, r/2)$ and $y_2 = (R, 0)$. Then we get on one hand

$$d(x_1, y_1) + d(x_2, y_2) = R + r,$$

and on the other hand

$$d(x_1, y_2) + d(x_2, y_1) = R,$$

and it is clear that no matter which δ and R we pick, for $r > 2\delta$, δ-B1 fails. One can argue similarly for L^∞. □

Remarks.

- It can be shown that for Banach spaces, uniform smoothness is equivalent to δ-B1 for every $\delta > 0$, and δ-B2$'$ is equivalent to $(1,0)$-uniform convexity.
- It is proven in [KS00] that a finite dimensional normed space whose dual's unit ball is strictly convex satisfies δ-B1 for every $\delta > 0$. This is a particular case of Proposition 5 since (for normed spaces) the notions of uniform smoothness and uniform convexity are dual to each other.
- The following can be found in [KS00]. Let E be a finite dimensional normed space. If there are no segments of length 1 in the unit sphere of E, then E satisfies δ-B2$'$ for every $\delta > 0$. This is essentially contained in Proposition 5 above.

Hyperbolic spaces. A metric space (X, d) is called *δ-hyperbolic* if for every x, y, z, w in X

$$d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\} + 2\delta.$$

This is equivalent to the usual definition by expanding the terms of the Gromov products.

Proposition 11. *A δ-hyperbolic space with δ-midpoints is $3\delta/2$-uniformly smooth, $3\delta/2$-uniformly convex, and $3\delta/2$-bolic.*

Proof. We reproduce the proof in [KS00]. Applying the defining inequality to $w = m(x, y)$ we get

$$d(m(x, y), z) \leq -\frac{1}{2}d(x, y) + \max\{d(x, z), d(y, z)\} + 3\delta.$$

The $3\delta/2$-uniform smoothness is now clear: η can be taken to be 1 for any ε, since then $d(y, z) \leq d(x, z)$. (Uniform convexity is also clear from this inequaility.) The condition δ-B1 is immediate from the definition.

Now, if s, t, u are nonnegative numbers such that $|t - u| \leq s$, we have

$$(2t - u)^2 + u^2 = 2t^2 + 2(t - u)^2 \leq 2t^2 + 2s^2.$$

Setting $s = \min\{d(x, z), d(y, z)\}$, $t = \max\{d(x, z), d(y, z)\}$ and $u = d(x, y)$, we find

$$2\max\{d(x, z), d(y, z)\} - d(x, y) \leq (2d(x, z)^2 + 2d(y, z)^2 - d(x, y)^2)^{1/2},$$

which combined with the first inequality completes the proof of B2. □

The central issue in [MY01] is to find a left-invariant metric on a hyperbolic group which is quasi-isometric to the word metric and for which not only δ-B2 holds for some δ, but also δ-B1 for every $\delta > 0$.

Products. Following [KS94] and [KS00] we note that the class of bolic spaces is closed under metric products:

Proposition 12. *The product of two bolic spaces endowed with the distance such that $d((x, y), (x', y'))^2 = d(x, x')^2 + d(y, y')^2$ is bolic.*

Proof. Let $x_1, y_1, z_1 \in X_1$ and $x_2, y_2, z_2 \in X_2$. Put

$$A_i = \sqrt{2d(x_i, z_i)^2 + 2d(y_i, z_i)^2 - d(x_i, y_i)^2}.$$

We have

$$\begin{aligned} 4d(m_1(x_1, y_1), z_1)^2 + 4d(m_2(x_2, y_2), z_2)^2 &\leq (A_1 + 4\delta)^2 + (A_2 + 4\delta)^2 \\ &\leq ((A_1^2 + A_2^2)^{1/2} + 4\sqrt{2}\delta)^2, \end{aligned}$$

and the condition B2 (hence also B1) follows. (The last inequality can be verified straightforwardly for any positive numbers by expanding the squares, simplify and square.) □

For a more systematic investigation of what properties of metric spaces are preserved under various products, we refer to [BFS02].

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