

Hilbert metrics and Minkowski norms

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Abstract

It is shown that the Hilbert geometry (D, h_D) associated to a bounded convex domain $D \subset \mathbb{E}^n$ is isometric to a normed vector space $(V, \|\cdot\|)$ if and only if D is an open n -simplex. One further result on the asymptotic geometry of Hilbert's metric is obtained with corollaries for the behavior of geodesics. Finally we prove that every geodesic ray in a Hilbert geometry converges to a point of the boundary.

1 Introduction

Busemann wrote on page 105 in [B] that “Plane Minkowskian geometry arises from the Euclidean through replacing the ellipse as unit circle by a convex curve. In a somewhat similar way a geometry discovered by Hilbert arises from Klein's Model of hyperbolic geometry through replacing the ellipse as absolute locus by a convex curve.” In this note we treat the question of when a Hilbert geometry is isometric to a Minkowski space (here meaning a normed finite dimensional real vector space).

We recall the definition of Hilbert's metric. Let \mathbb{E}^n denote the n -dimensional Euclidean space. For the Euclidean distance of $x, y \in \mathbb{E}^n$ we write $|xy|$, for the straight line segment between x and y we write $[x, y]$ and $L(x, y)$ denotes the whole straight line through x and y .

Given a bounded convex domain $D \subset \mathbb{E}^n$ with boundary $\partial D \subset \mathbb{E}^n$, the

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Hilbert metric $h_D : D \times D \longrightarrow \mathbb{R}_0^+$ is defined as follows: For $x, y \in D$ one defines

$$h_D(x, y) := \begin{cases} \log \frac{|y\bar{x}| \cdot |x\bar{y}|}{|x\bar{x}| \cdot |y\bar{y}|} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

where $\bar{x} \in L(x, y) \cap \partial D$ with $|\bar{x}x| < |\bar{x}y|$ and $\bar{y} \in L(x, y) \cap \partial D$ with $|\bar{y}x| > |\bar{y}y|$. The expression $\frac{|y\bar{x}| \cdot |x\bar{y}|}{|x\bar{x}| \cdot |y\bar{y}|}$ is called the cross ratio of the four collinear ordered points \bar{x}, x, y, \bar{y} and is invariant under projective transformations. For the basic properties of the distance h_D see [B] and [dlH]. We shall refer to (D, h_D) as a *Hilbert geometry*. All vector spaces are over the reals.

The following fact is known, see the pages 22-23 in Nussbaum's book [Nu] as well as the pages 110-111 and 113 in de la Harpe's paper [dlH]:

Theorem 1 *Let $D \subset \mathbb{R}^n$ be the interior of the standard n -simplex. Then (D, h_D) is isometric to an n -dimensional normed vector space.*

Recall that a *polytope* is the convex hull of finite number of points and an *n -simplex* in \mathbb{R}^n is the convex hull of $n + 1$ points in general position (hence it has nonempty interior).

In Section 2 of this paper we show that the converse of Theorem 1 is also true, so that we have:

Theorem 2 *A Hilbert geometry (D, h_D) is isometric to a normed vector space if and only if D is the interior of a simplex.*

In particular, a Hilbert geometry is never a Hilbert space.

In Section 3 we obtain an estimate of the asymptotic geometry of Hilbert's metric (see Proposition 2), which will be used to prove:

Theorem 3 *Let (D, h_D) be a Hilbert geometry. Then*

- (i) *every geodesic ray in (D, h_D) has to converge to a point in ∂D .*
- (ii) *every complete geodesic in (D, h_D) has precisely two accumulation points on ∂D .*

Theorem 3 in the case D is the interior of a simplex was already proved by V. Metz with entirely different techniques ([Me]).

2 The proof of Theorem 2

As is explained both in [B] and [dlH], the straight line segments in D are geodesics in (D, h_D) , but in general there can exist geodesics different from such Hilbert straight line segments. Indeed, two points $x, y \in D$ can be joined by no other geodesic than their connecting Hilbert straight line segment, if and only if there do not exist two coplanar but not collinear straight line segments $l_1, l_2 \subset \partial D$ through \bar{x} and \bar{y} such that \bar{x} and \bar{y} are not boundary points of l_1 and l_2 .

The following notation is by now standard:

$$(x|y)_{p_0} := \frac{1}{2} \left[h_D(x, p_0) + h_D(y, p_0) - h_D(x, y) \right],$$

where x, y , and p_0 are three points in the metric space. We recall the following simple but useful fact:

Theorem 4 ([KN]) *Let D be a bounded convex domain. Let $\{x_n\}, \{z_n\}$ be two sequences of points in D . Assume that $x_n \rightarrow \bar{x} \in \partial D$, $z_n \rightarrow \bar{z} \in \partial D$ and $[\bar{x}, \bar{z}] \not\subset \partial D$. Then, for any fixed p_0 , there is a constant $K = K(p_0, \bar{x}, \bar{z})$ such that*

$$\limsup_{n \rightarrow \infty} (x_n|z_n)_{p_0} \leq K.$$

We will need:

Lemma 1 *Let $D \subset \mathbb{R}^m$ be a bounded convex domain such that (D, h_D) is isometric to an m -dimensional normed vector space $(V, \|\cdot\|)$ and such that there exist $\bar{x}_i \in \partial D$, $i = 1, \dots, n$ with $[\bar{x}_i, \bar{x}_j] \not\subset \partial D$ for $i \neq j$. Then there exist n points v_1, \dots, v_n on the unit sphere of $(V, \|\cdot\|)$ such that*

$$\|v_i - v_j\| = 2 \quad \forall i, j = 1, \dots, n, \quad i \neq j. \quad (1)$$

Proof : Let $\varphi : D \rightarrow V$ be an isometry with $\varphi(p_0) = 0$. Let further $\gamma_i : [0, \infty) \rightarrow D$ be the arc length parameterized Hilbert straight line geodesic connecting p_0 to \bar{x}_i . In view of Theorem 4 we find $\tilde{K} > 0$ and $k_0 > 0$ such that

$$h_D(\gamma_i(k), \gamma_j(k)) \geq 2k - 2\tilde{K} \quad \forall i, j = 1, \dots, n, \quad i \neq j,$$

for all $k \geq k_0$. Given $N > 0$, let $k_N := \max\{k_0, 2\tilde{K}N\}$. For each i let

$$v_i^N = \frac{1}{k_N} \varphi(\gamma_i(k_N))$$

which is a point on the unit sphere in $(V, \|\cdot\|)$. Then we have that

$$\|v_i^N - v_j^N\| = \frac{1}{k_N} h_D(\gamma_i(k_N), \gamma_j(k_N)) \geq \frac{1}{k_N} (2k_N - 2\tilde{K}) \geq 2 - \frac{1}{N}.$$

Since V is finite dimensional, the unit sphere is compact and hence we can find a subsequence $N_k \rightarrow \infty$ and v_1, \dots, v_n such that $v_i^{N_k} \rightarrow v_i$ for every i . These limit points clearly satisfy (1). ■

Proposition 1 *Suppose that a Hilbert metric space (D, h_D) is isometric to a normed space. Then D is the interior of a polytope.*

Proof : Suppose that \overline{D} is not the convex hull of a finite number of points. Then one can find an infinite number of points satisfying the hypothesis of Lemma 1. To see that assume we can only find a finite number of such points. Take a maximal such set of points of the boundary. Note that none of these points belongs to the boundary of two different faces, because otherwise we could replace this point by two interior points of these faces, which contradicts the maximality of the chosen set. Now take the union of the closed faces containing points in our maximal set (a priori such a face might just be a point itself). If this union is not all of the boundary, then we add a point outside this union to our chosen set, hence again contradicting maximality. We have thus showed that \overline{D} is the convex hull of a finite number of points.

Therefore if \overline{D} is not a polytope but isometric to a normed vector space, then by Lemma 1 and a diagonal process we can extract an infinite sequence of v_i of mutual distance 2. This clearly contradicts the compactness of the unit sphere in $(V, \|\cdot\|)$. ■

With Proposition 1 at hand, we are ready to provide the proof of Theorem 2:

Proof : The “if part” follows from Theorem 1. For the “only if part” we know by Proposition 1 that \overline{D} must be a polytope. Suppose that \overline{D} is not an n -simplex. Then we find three points $v_1, v_2, e \in \partial D$ such that v_1 and v_2 are vertices of ∂D , the Hilbert straight line geodesics $[e, v_1]$ and $[e, v_2]$ lie entirely in D and the intersection ∂C of the affine plane Σ through e, v_1, v_2 is a polytope such that e is not a vertex point of ∂C . We also write $C := \Sigma \cap D$. Note that $h_C = h_D|_C$.

Under the isometry $\varphi : (D, h_D) \rightarrow (V, \|\cdot\|)$ the Hilbert straight line geodesics $[e, v_1]$ and $[e, v_2]$ have to be mapped to two straight line geodesics $l_1 := \varphi([e, v_1])$ and $l_2 := \varphi([e, v_2])$ in $(V, \|\cdot\|)$ because of the uniqueness of

these geodesics. We will get a contradiction to the assumption that ∂D is not an n -simplex by showing that (i) $l_1 \parallel l_2$ and (ii) $l_1 \not\parallel l_2$:

(i) Let $e_1, e_2 \notin \{v_1, v_2\}$ be the vertices of ∂C with $e \in [e_1, e_2]$ such that e_i and v_i lie in the same connected component of $\bar{C} \setminus [v_1, e]$. Let further $\tilde{e}_i \neq e_j$ be the vertex point of ∂C next to e_i , $i = 1, 2$, $i \neq j$, consider a straight line segment s parallel to $[e_1, e_2]$ with endpoints \tilde{x}_1 and \tilde{x}_2 on $[e_1, \tilde{e}_1]$ and $[e_2, \tilde{e}_2]$, respectively. By x_1 and x_2 we denote the intersection of s with $[e, v_1]$ and $[e, v_2]$, respectively.

All we need to show is that there exists $c \in \mathbb{R}^+$ such that (a) for all $y_1 \in [e, x_1]$ there exists $y_2 \in [e, x_2]$ satisfying $h_D(y_1, y_2) \leq c$ and (b) for all $y_2 \in [e, x_2]$ there exists $y_1 \in [e, x_1]$ satisfying $h_D(y_1, y_2) \leq c$, as this obviously implies $l_1 \parallel l_2$.

Without loss of generality we only prove (a). In order to do that, we set $\tilde{x}_1 := s \cap [e_1, v_1]$ and $\tilde{x}_2 := s \cap [e_2, v_2]$. Given $y_1 \in [e, x_1]$ we denote by r the straight line segment through y_1 in D parallel to $[e_1, e_2]$ and set $y_2 := r \cap [e, v_2]$. Then

$$h_D(y_1, y_2) \leq \log \left[1 + \frac{|x_1 x_2|}{|x_1 \tilde{x}_1|} + \frac{|x_1 x_2|}{|x_2 \tilde{x}_2|} + \frac{|x_1 x_2|^2}{|x_1 \tilde{x}_1| |x_2 \tilde{x}_2|} \right] =: c,$$

where $\tilde{x}_i := s \cap [e, v_i]$.

(ii) Let $\tilde{v}_i \neq v_j$ be the vertex of ∂C such that $[\tilde{v}_i, v_i] \subset \partial C$, $i, j = 1, 2$, $i \neq j$. Given $z_i \in C$ we denote by $\tilde{z}_i \in \partial C$ the unique point on ∂C such that $z_i \in [\tilde{z}_i, v_j]$, $i, j = 1, 2$, $i \neq j$. Let $u_i \in [v_i, e]$ be such that $[u_1, u_2] \parallel [v_1, v_2]$ and $L(u_1, u_2) \cap \partial C \subset [v_1 \tilde{v}_1] \cup [v_2, \tilde{v}_2]$, $i = 1, 2$. Let further $\gamma_i : (-\infty, \infty) \rightarrow D$ be the arc length parameterization of $[e, v_i]$ with $[\gamma_i(-\infty)] = e$ and $[\gamma_i(\infty)] = v_i$, $i = 1, 2$.

For $t_0 \in \mathbb{R}$ we write $w_1 := \gamma_1(t_0)$ and choose $s_0 \in \mathbb{R}$ such that

$$h_D(w_1, \gamma_2(s_0)) = \inf_{s \in \mathbb{R}} h_D(w_1, \gamma_2(s)).$$

Given \tilde{c} arbitrary large we can choose $z_i \in tr(\gamma_i)$ such that $z_i \in [v_i, u_i]$, $\tilde{z}_i \in [v_i, \tilde{v}_i]$ and $|z_i \tilde{z}_i| < e^{-\frac{\tilde{c}}{2}} |u_1 u_2|$, $i = 1, 2$.

Now there exists $t \in \mathbb{R}$ such that for $w_1 := [\gamma_1(t_0 + t), v_2] \cap [v_1, \tilde{v}_1]$ and $w_2 := [\gamma_2(s_0 + t), v_1] \cap [v_2, \tilde{v}_2]$ it holds

$$\max \left\{ |\gamma_i(t_i + t) v_i|, |\gamma_i(t_i + t) w_i| \right\} < |z_i \tilde{z}_i| \quad i = 1, 2,$$

where $t_1 := t_0$ and $t_2 := s_0$. For such a t it is easy to see that

$$h_D(\gamma_1(t_0 + t), \gamma_2(s_0 + t)) \geq \tilde{c},$$

which contradicts $l_1 \parallel l_2$. ■

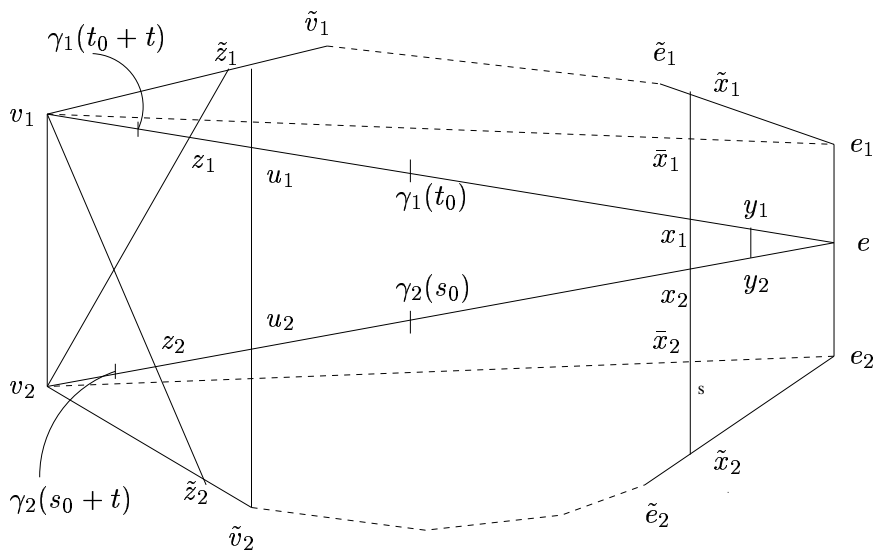


Figure 1: The figure illustrates the situation in the proof of Theorem 2.

3 Some asymptotic geometry

We will need:

Lemma 2 *Let D be an open bounded convex domain in \mathbb{E}^n and $p_0 \in D$. Then there exists $C = C(D, p_0) > 0$ such that the following holds: Let $z \in \partial D$, $z' \in [p_0, z] \cap D$ and $x, y \in \partial D$ such that $z' \in [x, y]$, then*

$$|zz'| \leq C|z'x|. \tag{2}$$

Proof : Without loss of generality assume $x \neq z \neq y$ and let $\delta > 0$ be such that the Euclidean ball $B(p_0, 3\delta)$ of radius 3δ around p_0 is contained in D . Let further Σ denote the intersection of D with the affine plane spanned by x, y and z , and set $\tilde{B} := B(p_0, \delta) \cap D$.

Let γ be the angle between the two straight lines T_1 and T_2 tangent to B with $T_1 \cap T_2 = \{p_0\}$. Since D is bounded, there exists some $\gamma_0 > 0$ such that for all $z \in D$ it holds $\gamma \geq \gamma_0$.

If $|z'x| \geq \delta$, then inequality 2 holds for $C := \frac{\text{diam} D}{\delta}$. Let now $|z'x| < \delta$. Let $\alpha := \angle([xz], [xz'])$ and $\beta := \angle([zz'], [zx])$. Then by the sine law we find

$$|zz'| \leq \frac{\sin \alpha}{\sin \beta} |z'x|.$$

Since $|z'x| < \delta$, we also have

$$\frac{\gamma_0}{2} \leq \beta \leq \pi - \frac{\gamma_0}{2}$$

and the claim follows. ■

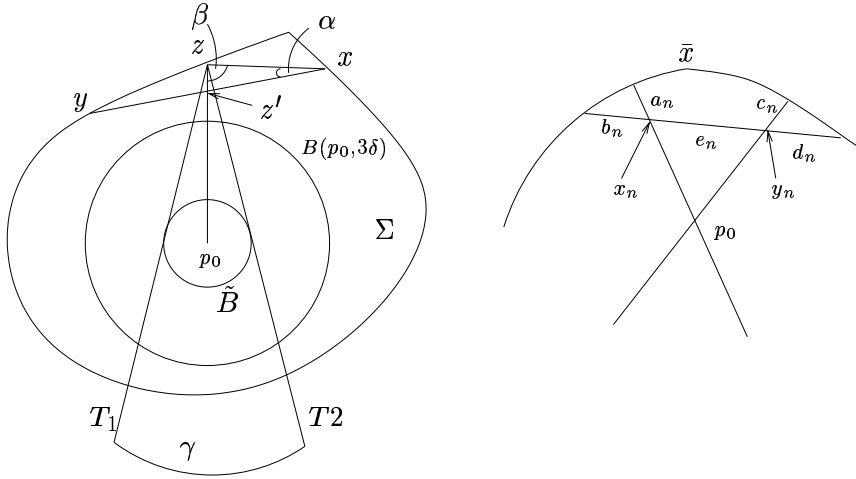


Figure 2: The figure on the left hand side illustrates the situation in the proof of Lemma 2, while the figure on the right hand side explains the notation in the proof of Proposition 2.

The following estimate complements Theorem 4 above:

Proposition 2 *Let $\{x_n\}$ and $\{y_n\}$ be two sequences of points in D both converging to $\bar{x} \in \partial D$. Then for any fixed p_0*

$$(x_n|y_n)_{p_0} \rightarrow \infty.$$

Proof : See Figure 2 for the notation. We can assume that $d(x_n, y_n) \rightarrow \infty$ (because for any subsequence for which x_n and y_n stay bounded the conclusion for that subsequence is immediate). Therefore at least one of b_n or d_n tend to 0.

By compactness of \bar{D} it is clear that there is a constant $C_1 > 0$ such that

$$\begin{aligned} |d(p_0, x_n) - \log \frac{1}{a_n}| &< C_1 \\ |d(p_0, y_n) - \log \frac{1}{c_n}| &< C_1, \end{aligned}$$

for all n .

In view of Lemma 2 there is a constant $C_2 > 0$ such that

$$a_n \leq C_2 b_n \quad \text{and} \quad c_n \leq C_2 d_n$$

Therefore, for some constant C_3 ,

$$\begin{aligned} 2(x_n|y_n) &= d(x_n, p_0) + d(y_n, p_0) - d(x_n, y_n) \\ &\geq -2C_1 + \log \frac{1}{a_n} + \log \frac{1}{c_n} - d(x_n, y_n) \\ &\geq -C_3 + \log \frac{1}{b_n} + \log \frac{1}{d_n} - \log \frac{e_n + b_n}{b_n} \frac{e_n + d_n}{d_n} \\ &= -C_3 - \log(e_n + b_n)(e_n + d_n) \rightarrow \infty, \end{aligned}$$

since e_n , and at least one of b_n and d_n tend to 0. ■

As an immediate application of Theorem 4 we derive the

Lemma 3 *All boundary accumulation points of a geodesic ray must belong to one closed face.*

Proof : Let $\gamma : [0, \infty) \rightarrow D$. Consider two sequences $\gamma(t_i)$ and $\gamma(s_i)$ converging to the boundary. Then

$$(\gamma(t_i), \gamma(s_i))_{\gamma(0)} = \frac{1}{2}(t_i + s_i - |t_i - s_i|) \geq \frac{1}{2} \min\{t_i, s_i\},$$

which tends to infinity as $i \rightarrow \infty$. The proposition now follows from Theorem 4. ■

For two points $x, y \in D$ we denote by $\xi_{xy}, \xi_{yx} \in \partial D$ those points satisfying $x, y \in [\xi_{xy}, \xi_{yx}]$ with $|\xi_{xy}x| < |\xi_{xy}y|$ and $|\xi_{yx}y| < |\xi_{yx}x|$.

The following lemma is well known and simply follows from the fact, that the straight line Hilbert geodesic between two points $x, y \in D$ is the image of the unique geodesic segment connecting the two points if and only if there do not exist two coplanar but not collinear straight line segments $l_1, l_2 \subset \partial D$ through ξ_{xy} and ξ_{yx} such that ξ_{xy} and ξ_{yx} are not boundary points of l_1 and l_2 .

Lemma 4 *Let (D, h_D) be a Hilbert geometry and $x, y, z \in D$ such that $h_D(x, y) + h_D(y, z) = h_D(x, z)$ and $x \notin [y, z]$. Then it holds $[\xi_{yx}, \xi_{zy}] \subset \partial D$ and $[\xi_{xy}, \xi_{yz}] \subset \partial D$.*

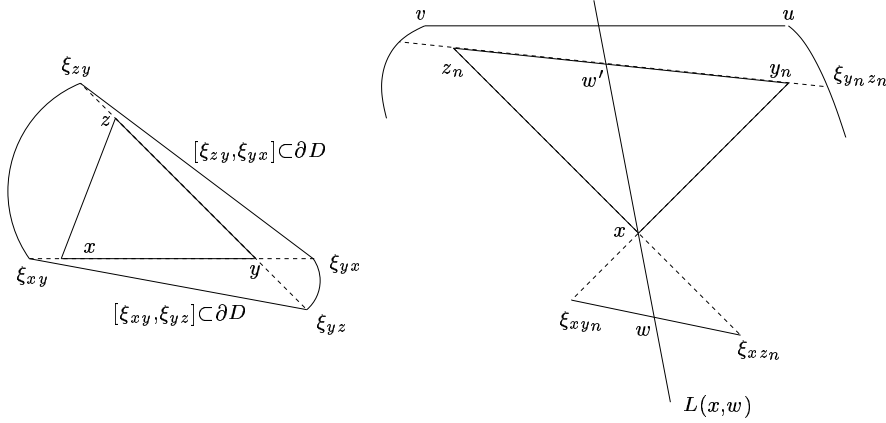


Figure 3: The figure on the left hand side visualizes the situation in Lemma 4, while the figure on the right hand side shows the setup of the proof of Theorem 3.

With Proposition 2 and Lemmata 3 and 4 at hand, we are finally ready to provide the

Proof of Theorem 3: (i) Assume to the contrary of the claim of Theorem 3 that there exist $u, v \in \partial D$, $u \neq v$, such that both, u and v are accumulation points of a geodesic ray γ in (D, h_D) with $x := \gamma(0)$. From Lemma 3 we know that u and v must lie in a common face F of ∂D . Thus we have $[u, v] \subset \partial D$. Under this assumption we prove the following two claims, which cannot simultaneously hold. By that we contradict our assumption!

Claim 1: With the notation introduced above it holds $[\xi_{xu}, \xi_{xv}] \subset \partial D$.

Claim 2: There exist $\eta_{uv} \in l := ((L(u, v) \cap \partial D) \setminus [u, v]) \cup \{u, v\}$ with $|u\eta_{uv}| < |v\eta_{uv}|$ and $\eta_{vu} \in l$ with $|v\eta_{vu}| < |u\eta_{vu}|$ such that

$$[\xi_{xu}, \eta_{uv}], [\xi_{xv}, \eta_{vu}] \subset \partial D. \quad (3)$$

In the following let $\{y_n\}_{n \in \mathbb{N}}, \{z_n\}_{n \in \mathbb{N}}$ be sequences in D converging to u and v , respectively, such that

- (1) $y_n, z_n \in \text{im}\{\gamma\} \quad \forall n \in \mathbb{N}$,
- (2) $h_D(x, y_{n+1}) > h_D(x, z_n) > h_D(x, y_n) \quad \forall n \in \mathbb{N}$,

Proof of Claim 1: We set $E_n := \text{span}\{x, y_n, z_n\}$ for all $n \in \mathbb{N}$ and assume without loss of generality that for all $n \in \mathbb{N}$ this is a nondegenerated, 2-dimensional plane. Note that the Hilbert geometry on $E_n \cap D$ isometrically embeds into that of D .

Fix $n \in \mathbb{N}$ and suppose that $[\xi_{xy_n}, \xi_{xz_n}] \not\subset \partial D$. Then there exists $w \in [\xi_{xy_n}, \xi_{xz_n}]$ such that all straight line segments on $L(x, w) \cap D$ are the images of the unique geodesic segments in $(E_n \cap D, h_{E_n \cap D})$ connecting their endpoints. This clearly contradicts the fact that for $w' := L(x, w) \cap [y_n, z_n]$ the union $[x, y_n] \cup [y_n, w']$ is the image of another geodesic segment in $(E_n \cap D, h_{E_n \cap D})$ connecting x to w' . Thus we have $[\xi_{xy_n}, \xi_{xz_n}] \subset \partial D$ for all $n \in \mathbb{N}$ and by $\xi_{xy_n} \xrightarrow{n \rightarrow \infty} \xi_{xu}$, $\xi_{xz_n} \xrightarrow{n \rightarrow \infty} \xi_{xv}$ and the continuity of ∂D also $[\xi_{xu}, \xi_{xv}] \subset \partial D$.

Proof of Claim 2: Let E_n , $n \in \mathbb{N}$, be as above and $\tilde{E}_n := \text{span}\{x, z_n, y_{n+1}\}$ for all $n \in \mathbb{N}$. As above for E_n we assume without loss of generality that \tilde{E}_n also is a nondegenerate, 2-dimensional plane. Now consider the sequences $\{\xi_{y_n z_n}\}_{n \in \mathbb{N}}$ and $\{\xi_{z_n y_{n+1}}\}_{n \in \mathbb{N}}$. By construction we can pass over to appropriate subsequences that converge to points in l . Let η_{uv} and η_{vu} , respectively, denote their limits. Now it follows by the choices of the sequences $\{y_n\}_{n \in \mathbb{N}}$ and $\{z_n\}_{n \in \mathbb{N}}$ as well as from Lemma 4 that $[\xi_{xy_n}, \xi_{y_n z_n}] \subset \partial D$ and $[\xi_{xy_n}, \xi_{z_n y_{n+1}}] \subset \partial D$ for all $n \in \mathbb{N}$. Due to $\xi_{xy_n} \xrightarrow{n \rightarrow \infty} \xi_{xu}$, $\xi_{xz_n} \xrightarrow{n \rightarrow \infty} \xi_{xv}$, $\xi_{y_n z_n} \xrightarrow{n \rightarrow \infty} \eta_{uv}$, $\xi_{z_n y_{n+1}} \xrightarrow{n \rightarrow \infty} \eta_{vu}$ and the continuity of ∂D we finally obtain the inclusions (3) and hence the validity of Claim 2.

(ii) Suppose that the geodesic line Γ accumulates only at $\xi \in \partial D$, so $\Gamma(t) \rightarrow \xi$ as $t \rightarrow \pm\infty$. Then, on the one hand, $(\Gamma(n)|\Gamma(-n))_{\Gamma(0)} \rightarrow \infty$ by Proposition 2, but, on the other hand, by virtue of being a geodesic, $(\Gamma(n)|\Gamma(-n))_{\Gamma(0)} = 0$. This is a contradiction. ■

Moreover, one can say that if both accumulation points of a complete geodesic belong to a single closed face, then both have to lie on the boundary of the face. Such geodesic lines do exist, the simplest example is the triangle. Here fix one point p in the interior and connect it through straight line segments to two of the vertices. The union of these segments indeed is the image of a geodesic in the associated Hilbert geometry.

References

- [B] H. Busemann, *The Geometry of Geodesics*, Academic Press Inc. (New York), 1955
- [dlH] P. de la Harpe, *On Hilbert's Metric for Simplices*, Geometric Group Theory, Vol.1, (Sussex, 1991), Cambridge Univ. Press, 1993, 97-119,
- [KN] A. Karlsson & G.A. Noskov, *The Hilbert metric and Gromov hyperbolicity*, L'Enseignement Mathématique, t.48 (2002), 73-89

- [Me] V. Metz, Personal communication.
- [Nu] R.D. Nussbaum, *Hilbert's projective metric and iterated nonlinear maps*, Memoirs of the American Mathematical Society Vol. 75 No. 391, Providence, RI, 1988