Nonexpanding Maps, Busemann Functions, and Multiplicative Ergodic Theory

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Abstract We describe some results on the dynamics of nonexpanding maps of metric spaces.

First, considering nonexpanding maps of proper metric spaces, we explain generalizations of some results of Beardon, which extends the Wolff–Denjoy theorem in complex analysis.

Second, we consider certain cocycles, or 'random products', of nonexpanding maps of nonpositively curved spaces. In a joint work with Margulis, we obtained that almost every trajectory lies on sublinear distance from a geodesic ray. In the special case where the metric space is the symmetric space of positive definite, symmetric matrices and the cocycles are isometries, this statement is equivalent to Oseledec's multiplicative ergodic theorem.

Further consequences concerning random ergodic theorems, cocycles of bounded linear operators, random walks and Poisson boundaries, are briefly discussed.

1 Introduction

Let (Y, d) be a metric space. A *nonexpanding map* (or a *semicontraction*) is a map $\phi : Y \to Y$ such that

$$d(\phi(x), \phi(y)) \leq d(x, y)$$

for any $x, y \in Y$. In particular, any isometry is a nonexpanding map.

Examples of nonexpanding maps include holomorphic maps of Kobayashi-hyperbolic complex spaces (e. g. Teichmüller spaces), affine maps of convex domains with the Hilbert metric, and invertible linear operators acting on the corresponding symmetric space. Note that all these examples are generalizations of the hyperbolic disk in the complex plane.

Here we will gain some understanding of nonexpanding maps ϕ that have unbounded orbits by studying the iterates $\phi^n(y)$. The Busemann functions of the space will be a central notion, generalizing the picture given in the Wolff–Denjoy theorem in complex dynamics. The novelty here is the simplicity of the arguments and the generality of the statements; only properness of the metric space is assumed, which is a considerably more general situation than what was previously treated.

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We will also explain a multiplicative ergodic theorem obtained, in a joint article of Margulis and the author, for integrable cocycles of nonexpanding maps $u(n,x) = w(x)w(Lx)...w(L^{n-1}x)$ of nonpositively curved spaces Y. This theorem contains (the convergence part of) the ergodic theorems of von Neumann ($Y = L^2(\Omega)$), Birkhoff ($Y = \mathbb{R}$) and, more generally, of Oseledec ($Y = GL_N(\mathbb{R})/O_N(\mathbb{R})$). It can also be viewed as a random mean ergodic theorem. Furthermore, it has consequences for certain Markov processes and bounded harmonic functions on groups. The proof of the theorem relies on a modification of the proof of Kingman's subadditive ergodic theorem, and some geometric arguments, see [KaMa 99].

The last section of the paper outlines a strategy for the study of random products of elements in C^*-algebras.

2 Busemann Functions

Let (Y,d) be a metric space and let $C(Y)$ denote the space of continuous functions on Y equipped with the topology of uniform convergence on bounded subsets. Fixing a point y, the space Y is continuously injected into $C(Y)$ by

$$\Phi : z \mapsto d(z,\cdot) - d(z,y)\,.$$

A metric space is called ***proper*** if every closed ball is compact. If Y is a proper metric space, then the Arzelá–Ascoli theorem implies that the closure of the image $\overline{\Phi(Y)}$ is compact. The points on the boundary $\partial Y := \overline{\Phi(Y)} \setminus \Phi(Y)$ are called ***Busemann*** (or ***horo) functions***, here denoted by b_γ, and we say that $y_n \to \gamma \in \partial Y$ as $n \to \infty$ if

$$b_\gamma(\cdot) = \lim_{n\to\infty} d(y_n,\cdot) - d(y_n,y)\,.$$

For $C \in \mathbb{R}$, the sublevel set

$$\{z : b_\gamma(z) \le C\}$$

is called a ***horoball*** (***centered at*** γ). We refer to [BGS 85] for details.

Remark 2.1. At first this compactification looks quite abstract, but it follows from the triangle inequality that we always have a map

$$\{\text{geodesic rays at } y\} \to \partial Y\,.$$

If Y is a Hadamard space, i. e. a complete metric space satisfying the semi-parallelogram law (the CAT(0)-condition, 'nonpositive curvature'), then this map is an isomorphism, see [B 95]. In other words, the boundary ∂Y and closure $\overline{\Phi(Y)}$ described above are homeomorphic to the standard ray boundary and closure. Perhaps this property can be called 'reflexivity', motivated by Banach space theory. For a Hilbert space, the linear functionals of norm 1 (Busemann functions) correspond via the inner product to unit vectors (geodesic rays).

Remark 2.2. In view of the results below it seems interesting to investigate the Busemann functions and the asymptotic geometry of various Kobayashi hyperbolic complex spaces Y, and also how ∂Y compares with some extrinsic boundary. For example it was announced in [BaBo 99] that the intrinsic metrics on a bounded, strictly pseudo-convex domains with C^2-boundary are Gromov hyperbolic and the extrinsic boundary coincide with the intrinsic Gromov boundary.

3 Subadditivity

The proofs of the statements about nonexpanding maps below will use the distances between orbit points. Fundamental for this is the well known observation of subadditivity, namely for two nonexpanding maps ϕ, ψ we have

$$d(\phi(\psi(y)), y) \leq d(\phi(\psi(y)), \phi(y)) + d(\phi(y), y) \leq d(\psi(y), y) + d(\phi(y), y)\,.$$

So the statements in this section will be applied to the distances from the origin y to the n-th point of an orbit, e. g. $a_n = d(y, \phi^n(y))$.

It is an elementary fact that subadditivity $a_{n+m} \leq a_n + a_m$ for all $m, n \geq 1$, implies that

$$A := \lim_{n\to\infty} \frac{1}{n} a_n = \inf_{m>0} \frac{1}{m} a_m\,.$$

The following observation is trivial:

Observation 1 *Let a_n be a sequence of real numbers which is unbounded above. Then there are infinitely many n such that*

$$a_m < a_n$$

for all $m < n$.

When this is applied to $a_n - (A - \varepsilon)n$ we get:

Observation 2 *Let a_n be a sequence of real numbers and assume that $A := \limsup a_n/n$ is finite. Then for any $\varepsilon > 0$ there are infinitely many n such that*

$$a_n - a_{n-k} \geq (A - \varepsilon)k$$

for all k, $1 \leq k \leq n$.

Now we describe an ergodic theoretic generalization of the second observation.

Let (X, μ) be a measure space with $\mu(X) = 1$, and let $L : X \to X$ be an ergodic measure preserving transformation. A *subadditive cocycle* is a function $a : \mathbb{N}\times X \to \mathbb{R}$ such that

$$a(n + m, x) \leq a(n, L^m x) + a(m, x)$$

for all $n, m \geq 1$ and $x \in X$. It is assumed that a is *integrable*, that is

$$\int_X |a(1,x)| d\mu(x) < \infty.$$

It follows that

$$A := \inf_{m>0} \frac{1}{m} \int_X a(m,x) d\mu(x)$$

is finite. The following lemma can be proved in a few pages by modifying standard methods of subadditive or pointwise ergodic theory:

Lemma 3.1 ([KaMa 99]). *Let E be the set of x with the property that for any $\varepsilon > 0$ there exist an integer K and infinitely many integers n (depending on x and ε) such that*

$$a(n,x) - a(n-k, L^k x) \geq (A - \varepsilon)k$$

for all k, $K \leq k \leq n$. Then $\mu(E) = 1$.

With one additional argument, one can from Lemma 3.1 now deduce the subadditive ergodic theorem of Kingman :

Theorem 3.2 ([Ki 68]). *For almost every x,*

$$\lim_{n\to\infty} \frac{1}{n} a(n,x) = A\,.$$

Problem 3.3. Fix some sequence $\varepsilon_i \searrow 0$. Let F be the set of x with the property that there exists $n_i = n_i(x) \to \infty$ such that

$$a(n_i,x) - a(n_i - k, L^k x) \geq (A - \varepsilon_j)k$$

for all k, $n_j \leq k \leq n_i$, and all $j < i$.

Note that the set $F \subseteq E$ is L-invariant and thus either $\mu(F) = 0$ or $\mu(F) = 1$. If a is an additive cocycle, then Birkhoff's theorem guarantees that $\mu(F) = 1$. If $X = \{x\}$, then $\mu(F) = 1$ by the second observation above. How generally is it true that $\mu(F) = 1$?

4 Nonexpanding Maps with Unbounded Orbit

If the orbit $\{\phi^n(y)\}_{n\geq 0}$ of a nonexpanding map of a complete metric space is bounded, then one would like to deduce the existence of a fixed point. Here are two sample facts of this type: any contraction of a proper metric space with bounded orbit has a unique fixed point (for a simple proof see e. g. [Be 90]), and any nonexpanding map of a Hadamard space (in fact, only uniform convexity and completeness are needed) with bounded orbit has a fixed point. For more statements about fixed points and relevant counterexamples, see the book [GK 90].

If, on the other hand, any orbit of ϕ is unbounded, then the points of ∂Y, the Busemann functions, and the horoballs will play the role of fixed points and metric balls, respectively. More precisely we have:

Theorem 4.1 ([Ka 99]). *Let ϕ be a nonexpanding map of a proper metric space (Y, d) and let $A \geq 0$ denote the linear rate of escape of orbits. Then any orbit lies inside a (horo)ball. In the unbounded orbit case, there is in fact a point ξ in ∂Y such that for all $k \geq 0$*

$$b_\xi(\phi^k(y)) \leq -Ak\,.$$

In particular, for any z we have

$$\lim_{n\to\infty} -\frac{1}{n} b_\xi(\phi^n(z)) = A\,.$$

To exemplify how the subadditivity observations in the previous sections are used and to illustrate the simplicity of the proof we give the argument proving the first part.

Proof. Let $a_n = d(y, \phi^n(y))$ and assume the orbit is unbounded. By Observation 1 and compactness, we can pick $n_i \to \infty$ such that $a_m < a_{n_i}$ for all $m < n_i$, and

$$\phi^{n_i}(y) \to \xi \in \partial Y\,.$$

Then for any $k \geq 0$,

$$\begin{aligned} b_\xi(\phi^k(y)) &= \lim_{i\to\infty} d(\phi^{n_i}(y), \phi^k(y)) - d(\phi^{n_i}(y), y) \\ &\leq \lim_{i\to\infty} a_{n_i - k} - a_{n_i} \leq 0\,, \end{aligned}$$

which means that every orbit point belongs to the horoball centered at ξ passing through the starting point y. □

With similar argument one deduces the existence of ϕ-invariant domains. For simplicity we will assume the following *projectivity condition* (following the terminology of W. Woess): for any two sequences $\{z_n\}_{n>0}$ and $\{y_n\}_{n>0}$ of points in Y,

$$\text{if } z_n \to \xi \in \partial Y, \text{ and } \sup d(z_n, y_n) < C \text{ for some } C, \text{ then } y_n \to \xi\,.$$

Theorem 4.2 ([Ka 99]). *Let (Y, d) be a proper metric space satisfying the projectivity condition and let $\phi : Y \to Y$ be a nonexpanding map. Either the orbit of ϕ is bounded or there is a point $\xi \in \partial Y$ such that every horoball $\mathcal{H}$ centered at ξ is an invariant set, that is $\phi(\mathcal{H}) \subset \mathcal{H}$.*

In view of the Schwarz–Pick lemma [L 99] and elementary hyperbolic geometry we have the following corollary:

Corollary 4.3 ([W 26a], [D 26], [W 26b]). *Let f be a holomorphic map of the unit disk D into itself. Then either f has a fixed point in D (in which case every ball centered at such a point is an invariant set) or there is a point $\xi \in \partial D$ such that the iterates f^n converge locally uniformly on D to the constant map taking the value ξ and every horoball centered at ξ is an invariant set under f.*

Generalizations of the Wolff–Denjoy theorem, mostly concerning holomorphic maps of more general complex spaces have been obtained in several papers, for example by Heins [He 41], Herve [H 63], Abate [A 89] and Ma [M 91]. For more complete references we refer to the bibliographies of the recent contributions [KKR 99], [Me 00], and the book [Ko 98]. In [Be 90] and [Be 97], Beardon proved essentially the above statements in the special cases of Cartan–Hadamard manifolds and the Hilbert metric on strictly convex domains. The basic method in most, if not all, of these papers is first to compose ϕ by strict contractions (which need to be constructed) and then apply the usual contracting mapping principle to get a unique fixed point for each of these maps that approximates ϕ. Then one uses compactness selecting a subsequence of these fixed points. Instead, it seems both simpler and more powerful to argue directly in terms of the orbit points as illustrated above.

Remark 4.4. Other related statements can be proved in a similar way, see [Ka 99]. For example, if Y is a Gromov hyperbolic space (not necessarily locally compact) and $d(y,\phi^n(y)) \to \infty$ as $n \to \infty$, then the whole orbit converges to a point on the boundary.

Remark 4.5. A metric space may of course lack nontrivial isometries, nevertheless there are quite generally many nonexpanding maps. For example, given a curve $\alpha : \mathbb{R} \to Y$ parametrized by arclength, a Busemann function b_γ, and a constant C, then the map

$$z \mapsto \alpha(b_\gamma(z) + C)$$

is a nonexpanding map $Y \to Y$.

Problem 4.6. It is not known to the author what can be said in general (e. g. for Euclidean spaces and $A = 0$) about the limit set of the orbit at infinity ∂Y (other than that it has to lie in the 'intersection' of the distinguished horoballs and ∂Y).

5 Multiplicative Ergodic Theory

Let (X,μ) be a measure space with $\mu(X) = 1$ and let $L : X \to X$ be an ergodic and measure preserving transformation. Let $w : X \to End(Y,d)$ be a measurable map taking values in the semigroup $End(Y,d)$ of nonexpanding maps (e. g. isometries) of Y. Assume that

$$\int_X d(y, w(x)y)d\mu(x) < \infty \,,$$

and let (note the order)

$$u(n,x) = w(x)w(Lx) \cdots w(L^{n-1}x) \,.$$

The following multiplicative ergodic theorem can be proved using Lemma 3.1 and some geometric arguments:

Theorem 5.1 ([KaMa 99]). *Assume that (Y, d) is a Hadamard space. Then for almost every x there exist $A \geq 0$ and a geodesic ray $\gamma(\cdot, x)$ starting at y such that*

$$\lim_{n\to\infty} \frac{1}{n} d(\gamma(An, x), u(n, x)y) = 0.$$

Here follow some remarks and consequences of this theorem.

Remark 5.2. As usual nothing depends on the point y. If $A > 0$, then the γ_x's are unique and the orbit $u(n, x)y$ converges to this point on the boundary at infinity for a. e. x. Note that the order in which the increments $w(L^k x)$ are multiplied is crucial for the convergence in direction. This order makes $\{u(n, x)y\}_{n\geq 0}$ look like a trajectory of a random walk.

Remark 5.3. In some ways Theorem 5.1 is best possible. Indeed, when $(Y, d) = (\mathbb{R}, |\cdot|)$ and $w(x)$ are translations, the statement coincides with the pointwise ergodic theorem of Birkhoff. Furthermore, for simple random walks on $\mathbb{Z}^k$ there is no convergence in direction (here $A = 0$). See also Remark 2.4. in [KaMa 99] concerning 'bi-infinite' orbits. As far as the question of relaxing the conditions on (Y, d) is concerned, we have the following restriction: in [KoN 81], Kohlberg and Neyman constructed a nonexpanding map $\phi : Y \to Y$ whenever Y is a Banach space whose dual has not Fréchet differentiable norm (so uniform convexity of Y fails), such that

$$\frac{1}{n}\phi^n(0)$$

does not converge in norm.

Let f be a function in $L^2(\Omega)$. Pitt, von Neumann–Ulam, and Kakutani, see [Kk 50], considered

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T_{L^{k-1}x}T_{L^{k-2}x}\cdots T_x\omega)\,,$$

where the T's are some collection of measure preserving transformations of Ω indexed by X. This average equals $\frac{1}{n}u(n, x)f$, cf. [Wi 95], if we let

$$(w(x)g)(\omega) = f(\omega) + g(T_x\omega)\,,$$

which are nonexpanding maps by Koopman's classical observation. We have in this way, by applying Theorem 5.1, the following random mean ergodic theorem:

Corollary 5.4 ([BSc 57]). *Let Y be a Hilbert (or a uniformly convex Banach) space. Let w be a strongly measurable map defined on X and taking values in the Banach space of bounded linear operators of Y. Suppose that*

$||w(x)|| \leq 1$ *for every* $x \in X$. *Then for any* $v \in Y$ *and almost every* x, *there is* $\bar{v}(x) \in Y$ *such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} w(x)w(Lx) \cdots w(L^{k-1}x)v = \bar{v}(x)$$

strongly in Y.

Recall that a sequence of $N \times N$ matrices A_n is *Lyapunov regular* if there exist a filtration of subspaces $0 = V_0 \subset V_1 \subset ... \subset V_s = \mathbb{R}^N$ and numbers $\lambda_1 < ... < \lambda_s$ such that for any $v \in V_i \setminus V_{i-1}$,

$$\lim_{n\to\infty} \frac{1}{n} \log ||A_n v|| = \lambda_i \,,$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log |\det A_n| = \sum_{i=1}^{s} \lambda_i (\dim V_i - \dim V_{i-1}) \,.$$

It is a standard fact that $GL_N(\mathbb{R})$ acts by isometries on the symmetric space $GL_N(\mathbb{R})/O_N(\mathbb{R})$ which is a Hadamard space; the same construction works in infinite dimensions provided that there is a finite trace, therefore we have the following corollary (for details see [KaMa 99] and compare also with [K 89]).

Corollary 5.5 (Cf. [O 68], [R 82]). *Let* $u(n,x)$ *be an integrable cocycle taking values in* $\exp \mathcal{A}$, *where* $\mathcal{A}$ *is the algebra of Hilbert–Schmidt operators. (In dimension* $N < \infty$, $\exp \mathcal{A}$ *consists of all invertible* $N \times N$ *matrices). Then for almost every* x *there is a positive symmetric operator* $\Lambda(x)$ *such that*

$$\frac{1}{n} \left(\sum_i (\log \mu_i(n))^2 \right)^{1/2} \to 0 \,,$$

where $\mu_i(n)$ *denotes the eigenvalues of the positive symmetric part of the operator* $\Lambda(x)^{-n} u(n,x)$. *In particular (or in finite dimensions, equivalently),* $u(n,x)^{-1}$ *is Lyapunov regular (with the* V*'s being the eigenspaces and the* λ*'s being the negative of the logarithm of the eigenvalues of* $\Lambda(x)$*).*

In view of Kaimanovich's ray approximation criterion [K 85], we also have the following application to the subject of random walks, bounded harmonic functions on groups, and Poisson boundaries. We refer to Furstenberg [F 73] for foundations of this subject, to Ballmann–Ledrappier [BL 94] for comparisons with our result, and to Kaimanovich [K 00] for further information.

Corollary 5.6 ([KaMa 99]). *Let* Γ *be a countable group acting by isometries on a Hadamard space* (Y,d) *and let* ν *be a probability measure on* Γ *with finite first moment. Fix a point* y *and assume that for some* $C > 0$,

$$card\{g \in \Gamma : d(y, gy) \leq N\} \leq e^{CN}$$

for all $N \geq 1$. Then the Poisson boundary of (Γ, ν) is either trivial or isomorphic to the boundary of Y with the induced hitting measure.

Finally we mention that Theorem 5.1 holds also if we replace $u(n,x)y$ by trajectories $W(t,x)$ in Y coming from certain Markov processes on Y/Γ, where Y/Γ is assumed to be compact, see Remark 2.3 in [KaMa 99]. Compare this result to [B 89].

Problem 5.7. Let $u(n,x)$ be an integrable cocycle as above, where we assume however that (Y,d) is only a proper metric space. When is it true that for almost every x, there exists $\gamma_x \in \partial Y$ such that

$$\lim_{n\to\infty} -\frac{1}{n} b_{\gamma_x}(u(n,x)y) = A\,,$$

where as usual A is the linear rate of escape? Note that this statement is equivalent to Theorem 5.1 (and hence true) in the case where (Y,d) is a Hadamard space. It would also be true in general if $\mu(F) = 1$ in Problem 3.3.

6 Nonexpansive Iterates in Banach Spaces

Consider nonexpanding (nonlinear) maps of a Banach space into itself. Theorem 4.1 holds (possibly except for the '$k \geq 0$ part' in case $A = 0$) with linear functionals of norm 1 replacing the Busemann function. This can be obtained by a similar method, see [Ka 00], using in addition a diagonal argument and the Hahn–Banach theorem. This has already been proved in [P 71], [KoN 81], and [PlRe 83] with different methods.

Now one can go to the ergodic theoretic context instead using Lemma 3.1. We have a particular application in mind, namely let $\mathcal{A}$ be the C^*-algebra of all bounded linear operators of a Hilbert space and let Sym be the Banach subspace of self-adjoint operators. Sym can be identified with the tangent space at points in the differentiable manifold Pos of all positive invertible elements in $\mathcal{A}$. The group G of all invertible bounded operators acts on Pos by

$$p \longmapsto gpg^*\,,$$

where $p \in Pos$ and $g \in G$. The Banach manifold Pos has a natural G-invariant Finsler metric:

$$||v||_p := ||p^{-1/2} v p^{-1/2}||\,,$$

where $||\cdot||$ denotes the operator norm, $p \in Pos$ and $v \in Sym$. The usual algebraic exponential map

$$\exp : Sym \to Pos$$

is a differential isomorphism and coincides with the differential geometric exponential at the identity, meaning that lines through the origin in Sym maps (distance-preservingly) to geodesics in Pos through Id. Moreover, Segal's inequality

$$||e^{u+v}|| \leq ||e^{u/2}e^{v}e^{u/2}||$$

for $u, v \in Sym$, means in this context that the differential geometric exponential map $T_pPos \to Pos$ semiexpands distances. In other words the distance function along geodesics is convex, or in a yet different terminology: Pos has nonpositive curvature in the sense of Busemann. We refer to [CPR 94] for the proofs of these claims.

A 'subadditive system' of points in Pos, such as the orbit $u(n,x)u(n,x)^*$ of Id under a multiplicative cocycle $u(n,x) \in G$, can be lifted back to Sym via the logarithm and yields, thanks to the convexity, again a 'subadditive system' of points but now in the Banach space Sym. This observation combined with Lemma 3.1, see [Ka 00], yields the following statement:

Theorem 6.1. *Let $u(n,x)$ be an integrable cocycle of elements in G. Then for almost every x and every $\varepsilon > 0$, there is a linear functional f_x^ε of Sym of norm 1 such that*

$$\liminf_{n\to\infty} \frac{1}{n} f_x^\varepsilon(y(n,x)) \geq A - \varepsilon\,,$$

where $A = \lim_{n\to\infty} \frac{2}{n} \log ||u(n,x)||$ and $\exp y(n,x) = u(n,x)u(n,x)^*$.

Remark 6.2. For $x \in F$, ε can be removed from the statement and the liminf can be replaced by lim, see Problem 3.3.

Remark 6.3. Ruelle obtained in [R 82] several general multiplicative ergodic results in infinite dimensions (see also the alternative proofs in [GoMa 89]). The standard trick of using exterior products to get hold of not only the top Lyapunov exponent (which corresponds to A) can be used also in the present arguments.

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