Linear rate of escape and convergence in direction

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Abstract. This paper describes some situations where random walks (or related processes) of linear rate of escape converge in direction in various senses.

We discuss random walks on isometry groups of fairly general metric spaces, and more specifically, random walks on isometry groups of nonpositive curvature, isometry groups of reflexive Banach spaces, and linear groups preserving a proper cone.

We give an alternative proof of the main tool from subadditive ergodic theory and we make a conjecture in this context involving Busemann functions.

1. Introduction

The well-known classical phenomenon of the nonexistence versus the existence of non-constant bounded harmonic functions in the plane and the unit disk, respectively, may be understood from observing that standard random walks in the euclidean and the hyperbolic geometry behaves quite differently. Brownian motion (or simple symmetric random walk on a lattice) in the euclidean space does not converge in direction as time goes to infinity, while this is the case in the hyperbolic space. See e.g. [12] and [14]. Many contributions have extended this by showing that in many "hyperbolic" geometric situations convergence in direction (almost surely) occurs (e.g. [31, 39, 2, 40, 15, 29, 18, 20, 4, 8, 19, 1, 9, 28]). The present article points out some recent results illustrating that in several situation convergence in direction is a consequence of linear rate of escape of trajectories rather than of hyperbolicity (e.g. the main theorem in [42], as well as Theorems 4.1 and 5.2 below) extending the law of large numbers. We also explain two situations where convergence to points on some hyperbolic-type boundary takes place (sections 6 and 7).

Our contributions are mostly relevant for spaces with large isometry groups, while many important works, some of which are listed above, deal with general, not necessarily homogeneous, situations. We apologize for omitted references.

2. Cocycles of semicontractions

Let S be a semigroup of semicontractions $D \to D$, where D is a nonempty subset of a metric space (Y, d), and fix a point $y \in D$.

Furthermore, let (X, μ) be a measure space with $\mu(X) = 1$ and let $L : X \to X$ be an ergodic and measure preserving transformation. Given a measurable map $w : X \to S$, put

$$u(n, x) = w(x)w(Lx) \cdots w(L^{n-1}x) \tag{2.1}$$

and denote $u(n, x)y$ by $y_n(x)$. Note that by multiplying the transformation in this order makes the orbit $\{y_n(x)\}_{n=0}^{\infty}$ look like a trajectory of some kind of random walk. Assume that

$$\int_X d(y, w(x)y)d\mu(x) < \infty. \tag{2.2}$$

Let $a(n, x) = d(y, y_n(x))$. By the triangle inequality, the equality (2.1) and the semicontraction property,

$$\begin{aligned} a(m+n, x) &\leq a(m, x) + d(u(m, x)y, u(m, x)u(n, L^m x)y) \\ &\leq a(m, x) + a(n, L^m x), \end{aligned}$$

hence a is a subadditive cocycle (see below). Furthermore, by the assumption (2.2),

$$\int_X a^+(1, x)d\mu(x) = \int_X d(y, w(x)y)d\mu(x) < \infty,$$

which means that the cocycle a satisfies the basic integrability condition. The subadditive ergodic theorem (see the next section) then implies that

$$\lim_{n\to\infty} \frac{1}{n} d(y, y_n(x)) = A \geq 0 \tag{2.3}$$

for almost every $x \in X$. This number A is called the *rate of escape* and if $A > 0$ this is refered to as almost every trajectory $y_n(x)$ is of linear rate of escape.

3. Subadditive ergodic theory

Let (X, μ) be a measure space with $\mu(X) = 1$ and L a measure preserving transformation. A *subadditive cocycle* a is a measurable map $a : \mathbb{N}\times X \to \mathbb{R}$ such that

$$a(n+m, x) \leq a(n, x) + a(m, L^n x)$$

for $n, m \geq 1$ and μ-almost every x. Assume that a is integrable, that is

$$\int_X a^+(1, x)d\mu(x) < \infty,$$

where $f^+(x) := \max\{f(x), 0\}$.

Kingman's subadditive ergodic theorem ([32]) asserts that for almost every x, the limit

$$\lim_{n\to\infty} \frac{1}{n} a(n,x)$$

exists. The following lemma will be the basic tool from ergodic theory that we use in most results discussed in this paper. It was proved and used by Margulis and the present author in [24].

Lemma 3.1 ([24]). *For each $\varepsilon > 0$, let E_ε be the set of x in X for which there exist an integer $K = K(x)$ and infinitely many n such that*

$$a(n,x) - a(n-k, L^k x) \geq (A-\varepsilon)k$$

for all k, $K \leq k \leq n$. Then $\mu(\bigcap_{\varepsilon>0} E_\varepsilon) = 1$.

Lemma 3.1 was proved in [24] using the so-called lemma about leaders. Here we describe an alternative proof and raise the question whether a stronger statement is true. Now follows an outline of the alternative proof of Lemma 3.1:

Define $v(n,x)$ through the formula

$$a(n,x) = v(n,x) + \sum_{k=0}^{n-1} a(1, L^k x).$$

It is immediate that $v(n,x)$ is a subadditive cocycle and in addition $v(n,x) \leq 0$. The additive part of a (the above sum) is taken care of with Birkhoff's pointwise ergodic theorem and the subadditive nonpositive part $v(n,x)$ is dealt with using the following lemma. Assume that

$$\gamma(v) := \lim_{n\to\infty} \frac{1}{n} \int_X v(n,x)\, d\mu(x) > -\infty.$$

Lemma 3.2. *Let $\lambda < 0$ and*

$$B = \{x | \exists K : \forall n > K, \min_{1\leq k\leq n} \frac{1}{k}(v(n,x) - v(n-k, L^k x) < \lambda\}.$$

Then

$$\mu(B) \leq \frac{\gamma(v)}{\lambda}.$$

This lemma can be proved in exactly the same way as Lemma 5.10 is proved in [34], but instead setting

$$B_K = \{x | \forall n > K, \min_{1\leq k\leq n} \frac{1}{k}(v(n,x) - v(n-k, L^k x) < \lambda\}$$

and showing that

$$\mu(B_K) \leq \frac{\gamma(v)}{\lambda}$$

for every K.

Combining Lemma 3.2 and Birkhoff's ergodic theorem, we get that $\mu(E_\varepsilon) > 0$ for every ε. It is easy to see that $L^l E_\varepsilon \subset E_{2\varepsilon}$ for all $l \geq 0$, and assuming ergodicity it then follows that $\mu(E_{2\varepsilon}) = 1$. Since this holds for every $\varepsilon > 0$ and $E_\varepsilon \subset E_{\varepsilon'}$, whenever $\varepsilon < \varepsilon'$, Lemma 3.1 is proved.

In view of Sections 2, 5, and also [26], the following question arises. Fix $\varepsilon_i \to 0$ and consider the set F of x for which there are $n_i = n_i(x) \to \infty$ such that

$$a(n_i, x) - a(n_i - k, L^k x) \geq (A - \varepsilon_j)k$$

for all $j \leq i$ and $n_j \leq k \leq n_i$. This set is L-invariant and for any additive cocycle a, $\mu(F) = 1$ by Birkhoff's theorem. Furthermore, for a subadditive sequence $a(n, x) = a_n$, it holds that $\mu(F) = 1$. For a general subadditive cocycle a, can it happen that $\mu(F) = 0$?

4. Nonpositive curvature

A *Hadamard space* is a complete metric space (Y, d) satisfying the following semi-parallelogram law: for any $x, y \in Y$ there exists a point z such that

$$d(x, y)^2 + 4d(z, w)^2 \leq 2d(x, w)^2 + 2d(y, w)^2$$

for any $w \in Y$. For basic facts about these spaces see [7]. A *geodesic ray* is a map $\gamma : [0, \infty) \to Y$ such that

$$d(\gamma(t), \gamma(s)) = |t - s|$$

for every s, t.

The following multiplicative ergodic theorem was proved by Margulis and the author using Lemma 3.1 and some geometric arguments:

Theorem 4.1 ([24]). *Assume that (Y, d) is a Hadamard space. Then for almost every x there exist $A \geq 0$ and a geodesic ray $\gamma(\cdot, x)$ starting at y such that*

$$\lim_{n\to\infty} \frac{1}{n} d(\gamma(An, x), u(n, x)y) = 0.$$

If $A > 0$, then the rays $\gamma(\cdot, x)$s are unique and the orbit $u(n, x)y$ converges to this point on the boundary at infinity. As explained in [24] and [26], this therorem contains as special cases (the convergence statement of) the ergodic theorems of von Neumann, Birkhoff, and Oseledec. Note that the theorem is proved in [24] under the more general condition of a uniformly convex, nonpositively curved in the sense of Busemann, complete metric space Y.

The following remark is taken from [24]: assume that $S = \Gamma$ is a discrete cocompact group of isometries of a Cartan-Hadamard manifold Y. Consider a Markov process on Y/Γ with absolutely continuous transition probabilities, for example Brownian motion. Let X be the space of all bi-infinite trajectories on Y with the measure μ coming from the process and a chosen stationary initial measure on a fundamental domain of Y/Γ. Let $w : X \to \Gamma$ be the map coming from the time 1 map and the chosen fundamental domain. For L we take the time 1 shift operator which is measure preserving. The theorem can then be applied to yield the result that for almost every sample path there is a geodesic ray such that the distance from the sample path to this geodesic grows sublinearly in n. In this context, we refer to Ballmann's paper [4] for comparison. In this paper Ballmann deals with the special case of independent, identically distributed increments of isometries of a space belonging to a certain rank 1 class of locally compact Hadamard spaces). He therefore needs a more sophisticated approximation scheme (following the method of Furstenberg and Lyons-Sullivan [35]) to transfer the Markov process to a random walk on a group of isometries. Then a result of Guivarch ([17]) can be used to guarantee that $A > 0$ whenever the group in question is nonamenable (which most of the time is the case here).

We now establish the link between Theorem 4.1 and the conjecture in section 8. The Busemann function b_γ corresponding to γ is (see also section 8):

$$b_\gamma(z) = \lim_{n\to\infty} d(\gamma(t), z) - d(\gamma(t), y).$$

(The triangle inequality implies that the limit exists.)

Proposition 4.2. *For Y a Hadamard space the conclusion in Theorem 4.1 is equivalent to the conclusion in Conjecture 8.1 below.*

Proof. For Hadamard spaces it is known that every horofunction is a Busemann function corresponding to a geodesic ray as above. Let y_n be an arbitrary sequence of points such that $d(y, y_n)/n \to A > 0$. Assume that $-b_\gamma(y_n) \sim An$ and denote by $\bar{y}_n$ the point on γ closest to y_n. By the cosine law, a property of projections, and the fact that horoballs are geodesically convex:

$$d(y, y_n)^2 \geq d(y, \bar{y}_n)^2 + d(\bar{y}_n, y_n)^2 \geq b_\gamma(y_n)^2 + d(\bar{y}_n, y_n)^2.$$

This implies that $d(\bar{y}_n, y_n) = o(n)$ and by the triangle inequality that $d(\gamma(An), y_n) = o(n)$ as desired. The converse holds for any metric space: assume $d(\gamma(An), y_n) = o(n)$. It is a general fact that

$$b_\gamma(y_n) \leq d(\gamma(An), y_n) - d(\gamma(An), y),$$

which in our case implies that $-b_\gamma(y_n) \sim An$. □

5. Continuous linear functionals

In this section we assume that Y is a normed real vector space and S is a semigroup of semicontractions $D \to D$, where the subset D for convenience is assumed to contain $y = 0$.

Proposition 5.1. *For almost every x and for any $\varepsilon > 0$ there exists an element f_x^ε in the topological dual of Y with norm 1 such that*

$$\liminf_{n\to\infty} \frac{1}{n} f_x^\varepsilon(y(n,x)) \geq A - \varepsilon.$$

Proof. If $A = 0$, then any f would do. If $A > 0$, then consider $x \in E_\varepsilon$ for some $\varepsilon > 0$ (Lemma 3.1). It follows from the Hahn-Banach theorem (see [11, p. 65]) that we can find elements f_n of norm 1 in the dual space such that $f_n(y(n,x)) = a(n,x)$. Take a sequence of n_i and a $k \geq K$ such that the inequality in the lemma holds. By picking subsequences and applying the diagonal process we may assume that $f_{n_i}(y(k,x)$ converges for every $k \geq K$. This defines a linear functional of norm at most 1 on the linear span of the orbit $y(k,x)$, $k \geq K$, which we may extend to a linear functional with the same norm on the whole space again by the Hahn-Banach theorem. We have

$$\begin{aligned} f_{n_i}(y(k,x)) &= a(n_i,x) - f_{n_i}(y(n_i,x) - y(k,x)) \\ &\geq a(n_i,x) - ||y(n_i,x) - y(k,x)|| \\ &\geq a(n_i,x) - a(n_i - k, L^k x) \\ &\geq (A-\varepsilon)k. \end{aligned}$$

Therefore

$$\frac{1}{k} f(y(k,x)) \geq A - \varepsilon$$

for all $k \geq K$. □

Whenever $x \in F$ (see section 3), we can remove ε and replace lim inf by lim in the above proposition. Since it is not clear to the author when this is the case, we can only prove the following by adding assumptions on Y.

Theorem 5.2. *Assume that Y is a reflexive Banach space. For almost every x there exists an element f_x in the dual of Y with norm 1 such that*

$$\lim_{n\to\infty} \frac{1}{n} f_x(y(n,x)) = A.$$

Proof. We may assume that Y is separable as we can, if necessary, replace it with the closed linear span of the orbit. Therefore, and due to reflexivity, the closed unit ball in Y and in Y^* are sequentially compact in the respective weak topology, see [11, p. 68]. Suppress x, pick $\varepsilon_i \to 0$ such that f_{ε_i} converges to some f in the weak*-topologies. Given any infinite subsequence n_j. Pick a weak limit point $\bar{y}$ of $y(n_j,x)/n$ and hence

$$f_{\varepsilon_i}(y(n_j,x)/n_j) \to f_{\varepsilon_i}(\bar{y})$$

along the subsequence of n_j for which the points converge to $\bar{y}$. Therefore $f_{\varepsilon_i}(\bar{y}) \geq A - \varepsilon_i$, but since

$$f_{\varepsilon_i}(y) \to f(y)$$

for any y, we must have that $f(\bar{y}) \geq A$. Finally note that as f has norm 1 it trivially holds that

$$\limsup_{n\to\infty} \frac{1}{n} f(y(n,x)) \leq A,$$

and the theorem is proved. (Instead of arguing with limit points $\bar{y}$ we could have applied S. Mazur's theorem on closures of convex sets.) □

Corollary 5.3 (Cf. [24]). *Assume that Y is a reflexive Banach space whose dual has Fréchet differentiable norm. Then for almost every x*

$$\frac{1}{n} y(n,x)$$

converges in norm.

Proof. It is known (due to Šmulian) and not difficult to show that the dual has Fréchet differentiable norm if and only if every sequence y_n in Y satisfying $||y_n|| = 1$ and $f(y_n) \to 1$ for some $f \in Y^*$ with norm 1, must converge, see [10] for a proof. □

Uniform convexity implies that the dual has Fréchet differentiable norm. The above corollary improves on Theorem 4.1 for Banach spaces. The author believes that the assumption that Y is a reflexive Banach space in the above results may be relaxed, which would have implications for random products of continuous linear operators, see the last section of [26]. One idea of relaxing the conditions on the Banach spaces could be to use the known fact that any separable can be renormed to have a locally uniformly convex norm. Note however that, except possibly for the reflexivity, the above assumption (the differentiability of the norm in the dual) is best possible in Corollary 5.3 in view of a counterexample constructed in [33]. There are several other papers studying the iteration of a single non-expansive map (e.g. [37, 38]).

The random mean ergodic theorem of Beck-Schwartz [6] can be deduced from Corollary 5.3 (although with a less general Y), compare with [26].

6. Conformal or Floyd-type boundaries

The construction here of a hyperbolic type boundary is a restrictive version of the one given by Gromov [16] section 7.2.K "A conformal view on the boundary", which extends Floyd [13], which in turn is "based on an idea of Thurston's and inspired by a construction of Sullivan's".

Assume Y is a complete, geodesic metric space. The *length* of a continuous curve $\alpha : [a,b] \to Y$ is defined to be

$$L(\alpha) = \sup \sum_{i=1}^{k} d(\alpha(t_{i-1}), \alpha(t_i))$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < ... < t_k = b$. When this supremum is finite, α is said to be *rectifiable*. For such α we can define the *arc length* $s : [a,b] \to [0,\infty)$ by

$$s(t) = L(\alpha|_{[a,t]}),$$

which is a function of bounded variation.

Given a continuous, (strictly) positive function f on Y, we define the *f-length* of a rectifiable curve α to be

$$L_f(\alpha) = \int_\alpha f\, ds = \int_a^b f(\alpha(t))ds(t).$$

If $f \equiv 1$, then $L_f = L$.

A new distance d_f is defined by

$$d_f(x,y) = \inf L_f(\alpha)$$

where the infimum is taken over all rectifiable curves α with $\alpha(a) = x$ and $\alpha(b) = y$.

For simplicity we choose $f(z) = d(y,z)^{-2}$, where y is a fixed base point. Let the *f-boundary* of Y be the space $\partial_f Y := \overline{Y_f} - Y$, where $\overline{Y_f}$ denotes the metric space completion of (Y, d_f). In [28] we prove using Lemma 3.1:

Theorem 6.1. *Assume that $A > 0$. Then for almost every x the trajectory $u(n,x)y$ converges to a point $\xi = \xi(x) \in \partial_f Y$.*

Proof. Here is a sketch of a proof somewhat different to the one in [28]. Note that for appropriate k and n in the sense of the Lemma 3.1 we have:

$$\begin{aligned}(y_n(x)|y_k(x))_y &:= \frac{1}{2}(d(y_n(x),y) + d(y_k(x),y) - d(y_n(x),y_k(x)) \\ &\geq \frac{1}{2}(a(n,x) + a(k,x) - a(n-k, L^k x)) \\ &\geq (A-\varepsilon)k.\end{aligned}$$

In view of the lemma in section 5 of [28] it follows from this estimate that for a fixed positive $\varepsilon < A$ and $n_i \to \infty$ for which the inequality in Lemma 3.1 is satisfied, the sequence $\{y_{n_i}(x)\}$ is d_f-Cauchy and hence converges to a point in $\partial_f Y$. Moreover, it then follows that the whole sequence $y_k(x)$ converges to this boundary point as well. □

An interesting special case is a random walk on Y being the Cayley graph of a finitely generated group Γ. In [28] also some visibility properties are shown, in particular we demonstrate that Kaimanovich's conditions (CP), (CS), and (CG) in [21] hold. The arguments in [21] therefore provide an alternative approach (not using Lemma 3.1) to the convergence in direction and which moreover show that if $\partial_f \Gamma$ is non-trivial then it is indeed maximal.

For more on random walks on groups and graphs we refer to the book by Woess [43] and the references therein.

7. Hilbert's projective metric

Assume that (Y, d) is a bounded convex domain in $\mathbb{R}^N$ equipped with Hilbert's metric and let ∂Y be the natural boundary of the domain. Similar to the proof of Theorem 6.1 above, cf. also [25], and in view of the weak hyperbolicity of Hilbert's metric established by Noskov and the author in [27] (extending a result of Beardon) we have:

Theorem 7.1. *Assume that $A > 0$. Then for almost every x, there is a point $\gamma_x \in \partial Y$ such that any other limit point of $y_n(x)$ may be connected by a line segment contained in ∂Y to γ_x. In particular, if Y is strictly convex, then $y_n(x) \to \gamma_x$ for $n \to \infty$.*

In the case of a strictly convex domain and $u(n, x)$ is a random walk (the increments are i.i.d.) taking values in the isometry group, one can probably use Furstenberg's ideas of combining proximality properties with the martingale convergence theorem (without assuming $A > 0$) to show the convergence in direction. In this situation we also have Oseledec's theorem [36] to our disposal since the isometry group is the subgroup of the projective linear group preserving the convex set.

8. Busemann functions

Let (Y, d) be a metric space and let $C(Y)$ denote the space of continuous functions on Y equipped with the topology of uniform convergence on bounded subsets. Fixing a point y, the space Y is continuously injected into $C(Y)$ by

$$\Phi : z \mapsto d(z, \cdot) - d(z, y).$$

A metric space is called *proper* if every closed ball is compact. If Y is a proper metric space, then the Arzela-Ascoli theorem asserts that the closure of the image $\overline{\Phi(Y)}$ is compact. The points on the boundary $\partial Y := \overline{\Phi(Y)} \setminus \Phi(Y)$ are called *Busemann* (or *horo) functions*, see [5] for more on this topic.

In the set-up of Section 2, we formulate the following conjecture:

Conjecture 8.1. *Assume that (Y, d) is a proper metric space. For almost every x there exists a horofunction b_x such that*

$$\lim_{n\to\infty} -\frac{1}{n} b_x(u(n,x)y) = A.$$

Evidence for the truth of this statement: it holds for one transformation $u(n,x) = \phi^n$, see [25]. It holds for complete metric spaces (not necessarily locally compact!) of nonpositive curvature, see section 4. It would hold in general if $\mu(F) = 1$, see section 3. Theorem 5.2 also provides some evidence. Moreover, the above type of limits with respect to Busemann functions should exists fairly generally for the following reason:

Assume that w takes its values in the isometry group of Y. Let $\widehat{X} = X \times \partial Y$ be the product measurable space and define

$$\widehat{L} : (x, \gamma) \mapsto (Lx, w(x)^{-1}\gamma).$$

By a standard argument (using Tychonoff's fixed point theorem) due to the compactness of ∂Y, there exists an ergodic $\widehat{L}$-invariant measure $\widehat{\mu}$ such that $\widehat{\mu}(\widehat{X}) = 1$ and the projection of $\widehat{\mu}$ onto X coincides with μ.

Let $z_i \to \gamma \in \partial Y$ and denote

$$b^y_\gamma(\cdot) = \lim_{i\to\infty} d(\cdot, z_i) - d(z_i, y),$$

the Busemann function centered at γ (and based at y).

Proposition 8.2. *For $\widehat{\mu}$-almost every (x, γ),*

$$\lim_{n\to\infty} \frac{1}{n} b^y_\gamma(u(n,x)y) = \int_{\widehat{X}} b^y_\xi(w(x')y) d\widehat{\mu}(x', \xi).$$

Proof. For any w, the following is a trivial identity:

$$b^y_\gamma(\cdot) = b^y_\gamma(w) + b^w_\gamma(\cdot). \tag{8.4}$$

Let g and h be two isometries. It follows that

$$b^{g(y)}_\gamma(gh(y)) = b^y_{g^{-1}\gamma}(h(y)).$$

In view of this equality and (8.4) we have

$$\begin{aligned} b^y_\gamma(u(n+m,x)y) &= b^y_\gamma(u(m,x)y) + b^{u(m,x)y}_\gamma(u(n+m,x)y) \\ &= b^y_\gamma(u(m,x)y) + b^y_{u(m,x)^{-1}\gamma}(u(n, L^m x)y). \end{aligned}$$

Thus we have an additive cocycle on the skew product system

$$v(n, (x, \gamma)) := b^y_\gamma(u(n,x)y)$$

and it is integrable because $|b_\gamma(w(x)y)| \le d(y, w(x)y)$. The assertion is now just Birkhoff's ergodic theorem. □

9. Random randomness

Recently the subject of random walks in random environment and random walks with random transition probabilities have attracted much attention. (See the books by Kifer [30], L. Arnold [3] and Sznitman [41]). This subject was advertized in some form already by Pitt, von Neumann-Ulam, and Kakutani, see [23]. In particular, they noted that a random individual ergodic theorem follows by a simple trick from the individual ergodic theorem of Birkhoff itself. Another result from the 1950s is the random mean ergodic theorem due to Beck-Schwartz, which in fact can be deduced from Theorem 4.1 or Corollary 5.3 above (note however that their assumption on the Banach space is somewhat weaker), see [26].

The recent paper [22] studies various notions of measure theoretical boundaries and Poisson formulas associated with random walks with random transition probabilities. In the last section of their paper they give some examples of the identification of the Poisson boundary using Theorem 4.1.

We would also like to mention the law of large numbers for certain random walks in random environment obtained by Sznitman-Zerner in [42] as it examplifies the title of the present paper. The proof of their theorem is based on a nice argument establishing, under some transience conditions, a renewal structure: there are times τ_i occuring often enough (integrability), at which the walk reach a new peak in the transience direction and never again returns to the halfplane it just left.

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