A PROOF OF THE SUBADDITIVE ERGODIC THEOREM

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Abstract. This is a presentation of the subaditive ergodic theorem. A proof is given that is an extension of F. Riesz 1945 proof of the Birkhoœ ergodic theorem.

1. Introduction

Let throughout this paper $(X, \mathcal{A}, \mu)$ be a probability space and $L : X \to X$ a measure preserving transformation.

Recall the following result.

Theorem 1.1. (Birkhoœ 1931) Let $f \in L^1$, then there is an integrable, a.e. L-invariant function $\bar{f}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(L^k x) = \bar{f}(x)$$

for a.e. x (the convergence also takes place in L^1). In fact $||\bar{f}||_1 \leq ||f||_1$ and for any L-invariant set A, $\int_A f = \int_A \bar{f}$, in particular $A = X$.

Note that if

$$c(n,x) = \sum_{k=0}^{n-1} f(L^k x) \tag{1.1}$$

then

$$c(n+m,x) = c(n,x) + c(m, L^n x),$$

such a condition is usually expressed as c is an additive cocycle. They are all of the form (1.1), for $f(x) = c(1,x)$.

If for a sequence of functions $a(n,.) \in L^1$ we instead require

$$a(n+m,x) \leq a(n,x) + a(m, L^n x),$$

then a is called a subadditive cocycle. Assume that

$$\inf \frac{1}{n} \int_X a(n,x) d\mu > -\infty.$$

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Then the following generalisation of the Birkhoœ ergodic theorem holds.

Theorem 1.2. (Kingman 1968) Under the above conditions, there is an integrable, a.e. L-invariant function $\bar{a}$ such that

$$\lim_{n\to\infty} \frac{1}{n} a(n,x) = \bar{a}(x)$$

for a.e. x (the convergence also takes place in L^1). Moreover

$$\lim \frac{1}{n} \int_A a(n,x) d\mu = \int_A \bar{a}(x) d\mu$$

for all L invariant measurable sets A.

2. Examples of subadditive cocycles

The aim of this section is to illustrate that Kingman's theorem is a signiøcant extension of Birkhoœ's theorem and that it has many applications.

2.1. Random products in a Banach algebra. Let $A : X \to B$ be a measurable map into a Banach algebra. Let $u(n,x) = A(L^{n-1}x)A(L^{n-2}x)...A(x)$, then

$$a(n,x) = \log ||u(n,x)||$$

is a subadditive cocycle, because $||AB|| \leq ||A||||B||$. The corresponding convergence was ørst proved by Furstenberg and Kesten in 1960 for random products of matrices, of course without the use of the subadditive ergodic theorem. This application is used in some proofs of Osceledts' multiplicative ergodic theorem.

2.2. Random walks. Let G be a topological group and $h : X \to G$ a Borel measurable map. Let $v(n,x) = h(L^{n-1}x)...h(Lx)h(x)$, if $\{h \circ L^k\}$ are independent, then this is usually called a random walk. The range, that is how many points visited in G,

$$a(n,x) := Card\{v(i,x) : 1 \leq i \leq n\}$$

is a L-subadditive cocycle.

Assume that d is a left invariant metric on G, (e.g. a word metric in the case G is ønitely generated) then

$$b(n,x) := d(e, v(n,x))$$

is a L-subadditive cocycle, by the triangle inequality and the invariance of d.

2.3. Metric theory of continued fractions. See [Bi] and [Barbolosi, J. Num. Th. 66 172-182 (1997)].

Let $X = [0, 1)$ and $\mathcal{A}$ be the Borel σ-algebra. For any x write it as

$$x = \frac{1}{\frac{1}{x}} = \frac{1}{[\frac{1}{x}] + \{\frac{1}{x}\}}.$$

Contiuing this scheme gives the continued fraction expansion of x, $a(x) := [\frac{1}{x}]$, etc. The relevant transformation of X is

$$Tx = \{\frac{1}{x}\}$$

unless $x = 0$ in which case $Tx = 0$. So $a_n(x) = a(T^{n-1}x)$.

The corresponding approximants $p_n(x)/q_n(x)$ are deøned for all n if x is irrational, they are given by recusion formulas. Also

$$\frac{p_n(x)}{q_n(x)} = 1/(a_1(x) + 1/(a_2(x) + ... + 1/a_n(x)))...).$$

There is a unique T-invariant probability measure absolutely continuous with respect to Lebesgue measure, namely

$$\nu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx$$

called the Gauss measure.

By use of the subadditive ergodic theorem, Barbolosi proves that

$$\lim_{n\to\infty} \frac{1}{n} Card\{1 \leq i \leq n : |xq_i(x) - p_i(x)| \leq \frac{C}{q_i(x)}\}$$

exists for a.e. x. The functions are not simply a subadditive cocycle, it is necessary to divide into odd and even index, due to the fact that approximants lie on alternating sides of x.

2.4. Percolation.

2.5. Entropy.

3. A proof

The Banach uniform boundedness principle.

3.1. Three elementary observations.

Proposition 3.1. Let v_n be a subadditive sequence of real numbers, that is $v_{n+m} \leq v_n + v_m$. Then the following limit exists

$$\lim_{n\to\infty} \frac{1}{n} v_n = \inf_{m>0} \frac{1}{m} v_m \in \mathbb{R} \cup \{-\infty\}.$$

Proof. Given $\varepsilon > 0$, pick M such that $v_M/M \le \inf v_n/n + \varepsilon$. Decompose $n = k_n M + r_n$, where $0 \le r_n < M$. Hence $k_n/n \to 1/M$. Using the subadditivity and considering n big enough $(n > N(\varepsilon))$

$$\begin{aligned}\inf \frac{1}{m} v_m &\le \frac{1}{n} v_n \le \frac{1}{n} v_{k_n M + r_n} \le \frac{1}{n}(k_n v_M + v_{r_n}) \\ &\le \frac{1}{M} v_M + \varepsilon \le \inf \frac{1}{m} v_m + 2\varepsilon.\end{aligned}$$

Since ε is at our disposal, the lemma is proved. □

Note that this proposition implies that for any L-invariant set A that

$$\lim_{n\to\infty} \frac{1}{n} \int_A a(n,x) d\mu(x) = \inf_n \frac{1}{n} \int_A a(n,x) d\mu(x),$$

when $X = A$ denote this value by $\gamma(a)$. [We might not need this, it comes out of the theorem.]

F. Riesz realized in 1945 that the following simple lemma can be used to prove Birkhoœ's theorem, the lemma is sometimes called F. Riesz's combinatorial lemma.

Lemma 3.2. Call the term c_u a leader in the ønite sequence $c_0, c_1, ..., c_{n-1}$ if one of the sums

$$c_u, c_u + c_{u+1}, ..., c_u + ... + c_{n-1}$$

is negative. Then the sum of the leaders is seminegative. (An empty sum is by convention 0).

Proof. Proof by induction. If $n = 1$, then either $c_0 \ge 0$, in which case the sum is empty, or $c_0 < 0$, in which case the sum equals $c_0 < 0$. Assume that the statement is true for integers smaller than n. Consider the two cases, c_0 is or is not a leader. If c_0 is not a leader then all leaders are among $c_1, ..., c_{n-1}$ in which case the induction hypothesis apply. If c_0 is a leader, then pick k smallest integer such that $c_0 + ... + c_k < 0$, then each c_i, $i \le k$ is a leader. If not then $c_i + .. + c_k \ge 0$, but by minimality $c_0 + ... + c_{i-1} \ge 0$, which is a contradiction. Hence $c_0, ..., c_k$ are all leaders and $c_0 + ... + c_k < 0$, the remaining leaders (if any) are leaders of $c_k, ...c_n$, for which the induction hypothesis applies. □

Proposition 3.3. The functions

$$f(x) = \limsup \frac{1}{n} a(n,x)$$

and

$$g(x) = \liminf \frac{1}{n} a(n,x)$$

are a.e L-invariant.

Proof. Note that $f(Lx) \geq f(x)$ and $g(Lx) \geq g(x)$ because of the subadditivity

$$a(n, Lx) \geq a(n+1, x) - a(1, x)$$

and in the case of limsup (same for liminf)

$$\limsup(\frac{1}{n}a(n+1, x) - \frac{1}{n}a(1, x)) = f(x).$$

Now integrate

$$\int_X f(Lx) - f(x) d\mu(x) = 0$$

by the L-invariance, but the integrand is semipositive, hence $f(Lx) - f(x) = 0$ a.e. □

3.2. The maximal ergodic inequality. The following key lemma will be proved by en extension of the argument of F. Riesz. It thus avoid use of the usual maximal ergodic inequality and it is simple.

Lemma 3.4 (Derriennic 1975). Let

$$B = \{x : \lim_{n\to\infty} \inf \frac{1}{n} a(n, x) < 0\}$$

then

$$\lim_{n\to\infty} \frac{1}{n} \int_B a(n, x) d\mu \leq 0.$$

Proof. For each n, let

$$\Psi_n = \{x : \inf_{1\leq k\leq n} a(n, x) - a(n-k, L^k x) < 0\}.$$

Note that

$$A_n := \{x : \inf_{1\leq k\leq n} a(k, x) < 0\} \subset \Psi_n$$

by subadditivity. Note also that $A_n \subset A_{n+1}$ and $B \subset \bigcup A_n$. For each n, let

$$b_n(x) = a(n, x) - a(n-1, Lx).$$

Because of telescoping we have that

$$a(n, x) - a(n-k, L^k x) = b_n(x) + b_{n-1}(Lx) + ... + b_{n-k+1}(L^{k-1}x)$$

and in particular

$$a(n, x) = \sum_{0\leq k\leq n-1} b_{n-k}(L^k x).$$

By the deønition, $L^k x \in \Psi_{n-k}$ means that there is a j, $k \le j \le n-1$ such that

$$b_{n-k}(L^k x) + ... + b_{n-j}(L^j x) < 0.$$

Hence by the lemma about leaders

$$\sum_{0 \le k \le n-1, L^k x \in \Psi_{n-k}} b_{n-k}(L^k x) \le 0.$$

Hence, using the L-invariance of μ and B,

$$\begin{aligned} 0 \ &\ge \int_B \sum_{0 \le k \le n-1, L^k x \in \Psi_{n-k}} b_{n-k}(L^k x) d\mu(x) = \\ &= \sum_{0 \le k \le n-1} \int_{B \cap L^{-k} \Psi_{n-k}} b_{n-k}(L^k x) d\mu(x) = \\ &= \sum_{0 \le k \le n-1} \int_{L^k B \cap \Psi_{n-k}} b_{n-k}(x) d\mu(x) = \\ &= \sum_{i=1}^{n} \int_{B \cap \Psi i} b_i(x) d\mu(x). \end{aligned}$$

On the other hand, again by the L-invariance

$$\frac{1}{n} \int_B a(n,x) d\mu(x) = \frac{1}{n} \sum_{i=1}^{n} \int_B b_i(x) d\mu(x) =$$

$$= \frac{1}{n} \sum_{i=1}^{n} \int_{B \cap \Psi_i} b_i(x) d\mu(x) + \frac{1}{n} \sum_{i=1}^{n} \int_{B - (B \cap \psi_i)} b_i(x) d\mu(x) \le$$

$$\le 0 + \frac{1}{n} \sum_{i=1}^{n} \int_{B - (B \cap A_i)} a^+(1,x) d\mu(x)$$

since $b_j(x) \le a(1,x) \le a^+(1,x)$, which is positive and $A_i \subset \Psi_i$. Now since $a^+ \in L^1$, $A_i \subset A_{i+1}$ and $B \subset \bigcup_{i \ge 1} A_i$ it follows when taking $\limsup$, that

$$\limsup \frac{1}{n} \int_B a(n,x) d\mu \le 0.$$

Since B is invariant, the $\limsup$ actually is the limit, by proposition (3.1). □

Corollary 3.5. Let

$$B = \{x : \liminf \frac{1}{n} a_n(x) < \lambda\}$$

then

$$\lim_{n\to\infty}\frac{1}{n}\int_B a_n d\mu \leq \lambda\mu(B).$$

Proof. Apply the lemma to $a_n - n\lambda$, which is a subadditive cocycle. □

3.3. The proof of a.e. convergence. First we establish the result for an additive cocycle c_n. The point is that $-c_n$ is again additive, hence in particular the corollary applies to $-c_n$ as well.

Let

$$E_{\alpha,\beta} = \{x : \liminf \frac{1}{n}c_n < \alpha < \beta < \limsup \frac{1}{n}c_n\}$$

and by the proposition (3.3) this set is L-invariant. Hence we can apply the corollary (3.5) for $X = E_{\alpha,\beta}$. Since if $E := \{x : \liminf \frac{1}{n}c_n < \alpha\}$, then $E \cap E_{\alpha,\beta} = E_{\alpha,\beta}$, this gives

$$\lim_{n\to\infty}\frac{1}{n}\int_{E_{\alpha,\beta}} c_n d\mu = \int_{E_{\alpha,\beta}} c_1 d\mu \leq \alpha\mu(E_{\alpha,\beta}).$$

And similarily for $-c_n$,

$$\lim_{n\to\infty}\frac{1}{n}\int_{E_{\alpha,\beta}} -c_n d\mu = -\int_{E_{\alpha,\beta}} c_1 d\mu \leq -\beta\mu(E_{\alpha,\beta}).$$

This yields a contradiction unless $\mu(E_{\alpha,\beta}) = 0$, because

$$\beta\mu(E_{\alpha,\beta}) \leq \int_{E_{\alpha,\beta}} c_1 d\mu \leq \alpha\mu(E_{\alpha,\beta})$$

but $\beta > \alpha$.

Now for any subadditive cocycle a_n, let

$$v_n(x) = a_n(x) - \sum_{i=0}^{n-1} a_1(L^i x)$$

and note that v_n is a subadditive cocycle and $v_n \leq 0$.

Fix $\varepsilon > 0$. Pick M big enough so that

$$\frac{1}{m}\int_X v_m \leq \gamma(v) + \varepsilon$$

for all $m \geq M$. Let $g(x) = \liminf \frac{1}{n}v_n(x)$ and $f(x) = \limsup \frac{1}{n}v_n(x)$. Note that by subadditivity

$$v_{nM}(x) \leq v_n(x) + v_n(L^n x) + ... + v_n(L^{nM}x)$$

and by L-invariance of g that

$$g^M(x) := \lim_{n\to\infty}\inf \frac{1}{nM}v_{nM}(x) = g(x)$$

a.e. By the seminegativity of v_n,

$$v_{nM+k}(x) \leq v_{nM}(x) + v_k(L^{nM}x) \leq v_{nM}(x),$$

it also hold that

$$f^M(x) := \lim_{n\to\infty} \sup \frac{1}{nM} v_{nM}(x) = f(x).$$

This has the following consequence,

$$f - g = f^M - g^M \leq - \lim_{n\to\infty} \inf \frac{1}{nM} v_n^M,$$

because $v_n \leq 0$ and the established convergence for additive cocycles and where

$$v_n^M(x) := v_{nM}(x) - \sum_{i=0}^{n-1} v_M(L^{iM}x)$$

which again is subadditive and seminegative. Note that

$$0 \geq \gamma(v^M) = M\gamma(v) - \int_X v_M \geq -M\varepsilon.$$

This means that

$$B := \{x : f - g > \alpha\} \subset \{x : \liminf \frac{1}{n} v_n^M < -M\alpha\} =: E$$

and applying the corollary of the lemma we have

$$-M\alpha\mu(E) \geq \lim \frac{1}{n} \int_E v_n^M \geq \lim \frac{1}{n} \int_X v_n^M \geq -M\varepsilon.$$

Hence

$$\mu(E) \leq \frac{\varepsilon}{\alpha}$$

and letting $\varepsilon \to 0$ we conclude that $\mu(B) = 0$ for any $\alpha > 0$ as required.

The limit is of course a.e. L-invariant, by the proposition (3.3).

The limit is integrable by Fatou's lemma

$$\int \liminf -\frac{1}{n} v_n \leq \liminf \int -\frac{1}{n} v_n < \infty.$$

In general, $\int |a(n,x)| \leq n \int |a(1,x)|$.

4. Appendix: Garsia's proof of the maximal ergodic lemma

This is not used above.

Lemma 4.1. Let

$$E_n = \{x : \sup_{1 \le k \le n} a(k,x) \ge 0\}$$

then

$$\int_{E_n} a(1,x) d\mu \ge 0 \text{ and } \int_{E_\infty} a(1,x) d\mu \ge 0.$$

Proof. Let

$$h_n(x) = \sup_{1 \le k \le n} a(k,x),$$

so

$$h_n^+(x) - a(k,x) \ge 0$$

for $k \le n$. By positivity and linearity of T

$$Th_n^+ \ge Ta_k$$

for $k \le n$. $(a_k(x) = a(k,x))$.

Hence

$$a_1 + Th_n^+ \ge a_1 + Ta_k \ge a_{k+1}$$

by subadditivity. So

$$a_1 \ge a_{k+1} - Th_n^+$$

for all $k, 1 \le k \le n$ and trivially for $k = 0$. Therefore

$$a_1 \ge \sup_{0 \le k \le n-1} a_{k+1} - Th_n^+ = h_n - Th_n^+.$$

Now integrate

$$\int_{E_n} a_1 \ge \int_{E_n} h_n - \int_{E_n} Th_n^+ = \int_X h_n^+ - \int_{E_n} Th_n^+$$
$$\ge \int_X h_n^+ - \int_X Th_n^+ \ge 0,$$

because T is contractive and $h_n^+ \ge 0$.

The statement about $E_\infty = \bigcup E_n$ follows from passing to the limit.

[There is nothing special about a_1?, look at $(\frac{1}{k} a_{kn}(x))_n$.]