Scaling limits of Anisotropic Hastings-Levitov clusters

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Abstract

We summarize some of our recent work [3] on an anisotropic version of the Hastings-Levitov model.

Conformal mapping and random growth

We describe an approach to modelling random growth by composing conformal maps. Let D_0 denote the exterior unit disk in the complex plane $\mathbb{C}$. Let $K_0 = \mathbb{C} \setminus D_0$ be the closed unit disk. Consider a simply connected set $D_1 \subset D_0$, such that $P = D_1^c \setminus K_0$ has diameter $d \in (0, 1]$ and $1 \in \overline{P}$. The set P models an incoming particle, which is attached to the unit disk at 1. We use the unique conformal mapping $f_P : D_0 \to D_1$ as a mathematical description of the particle.

Figure 1: *Basic conformal mapping from the exterior disk.*

Let $P_1, P_2, \dots$ be a sequence of particles (or, equivalently, let $f_{P_1}, f_{P_2}, \dots$ be the sequence of associated conformal mappings) with $\mathrm{diam}(P_j) = d_j$. Let $\theta_1, \theta_2, \dots$ be a sequence of angles. Define rotated copies $f_{P_j}^{\theta_j}(z)$ of the maps $\{f_{P_j}\}$ so that $f_{P_j}^{\theta_j}(D_0) = e^{i\theta_j} f_{P_j}(D_0)$. Take $\Phi_0(z) = z$, and recursively define

$$\Phi_n(z) = \Phi_{n-1} \circ f_{P_n}^{\theta_n}(z), \quad n = 1, 2, \dots.$$

This generates a sequence of conformal maps $\Phi_n : D_0 \to D_n = \mathbb{C} \setminus K_n$, where $K_{n-1} \subset K_n$ are growing compact sets, or clusters.

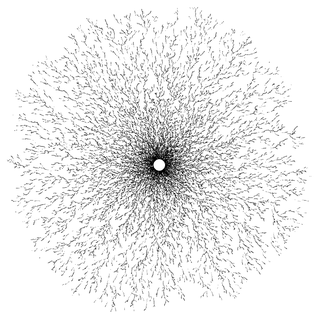

Figure 2: *HL(0) cluster with 25000 particles.*

By choosing the sequences $\{\theta_j\}$ and $\{d_j\}$ in different ways, it is possible to describe a wide class of growth models. For example, in the Hastings-Levitov family of models $\mathrm{HL}(\alpha)$, $\alpha \in [0, 2]$, the θ_j are chosen to be independent uniform random variables on the unit circle which corresponds to the attachment point at the nth step being distributed according to harmonic measure at infinity for K_{n-1}, see [2] and [6]. The particles are usually taken to be "slits" with diameters taken as $d_j = d/|\Phi'_{j-1}(e^{i\theta_j})|^{\alpha/2}$.

The motivation for studying these clusters comes from growth processes that arise in physics, such as diffusion-limited aggregation (DLA), anisotropic diffusion-limited aggregation and the Eden model. Heuristically, the case $\alpha = 1$ corresponds to the Eden model and the case $\alpha = 2$ is a candidate for off-lattice DLA.

Anisotropic Hastings-Levitov model

We have studied a variant of the $\mathrm{HL}(0)$ model in which $\theta_1, \theta_2, \dots$ are i. i. d. random variables on the unit circle $\mathbb{T}$ with common law ν and $d_j = d$. We shall refer to this growth model as anisotropic Hastings-Levitov, $\mathrm{AHL}(\nu)$.

The use of more general distributions for the angles is a way of introducing anisotropy or localization in the growth. This is similar in spirit to the work in [5], where an anisotropic variant of $\mathrm{HL}(2)$ is studied numerically.

Two natural scaling limits

From the physical point of view, it is natural to consider particle sizes that are very small compared to the overall size of the cluster. We consider two different scaling limits where we scale the particle sizes and let the number of particles grow at a rate depending on the particle diameters (so that the total diameter is roughly constant). We will see that the anisotropy in the angular distributions is reflected in these scaling limits.

A shape theorem

If μ_t is a family of probability measures on the unit circle $\mathbb{T}$, the Loewner equation

$$\partial_t f_t(z) = z f'_t(z) \int_{\mathbb{T}} \frac{z + \zeta}{z - \zeta} \mathrm{d}\mu_t(\zeta)$$

produces a family of conformal maps $f_t \colon D_0 \to \mathbb{C} \setminus K_t$, where K_t is a growing sequence of compact sets (see [1] for a discussion).

For simplicity, we assume the particles $P_1, P_2, \dots$ in $\mathrm{AHL}(\nu)$ are chosen to be identical and symmetric, with $\mathrm{diam}(P) = d$. Assume additionally that the particle shape is chosen with capacity of order d^2. If $O(d^{-2})$ particles are added, the diameter of the cluster is roughly constant.

We prove that the macroscopic shape of the cluster converges almost surely as $d \to 0$ and the limit can be realized as the image of the solution to the Loewner equation driven by the angle measure ν evaluated at a suitable time.

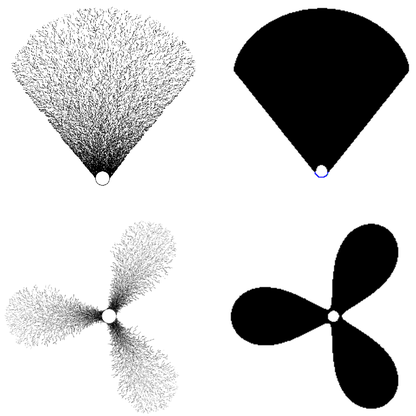

Figure 3: *Simulations of* $\mathrm{AHL}(\nu)$ *and associated limiting shape, for* $d = 0.02$ *after* 25000 *repetitions, corresponding to* $\mathrm{d}\nu(x) = 2\chi_{[0,1/2]}\mathrm{d}x$ *and* $\mathrm{d}\nu(x) = 2\sin^2(3\pi x)\mathrm{d}x$.

The boundary evolution of harmonic measure

For the mapping associated to a particle P, write g_P for the inverse mapping from $D_1 \to D_0$. There exists a unique γ_P that restricts to a continuous map from the interval $(0, 1)$ to itself, such that

$$g_P(e^{2\pi i x}) = e^{2\pi i \gamma_P(x)}, \quad x \in (0, 1).$$

Set $\Gamma_n = g_{P_n} \circ \cdots \circ g_{P_1}$, where $g_{P_n} = (f_{P_n}^{\theta_n})^{-1}$, so that $\Gamma_n : D_n \to D_0$. The extension of Γ_n to the boundary $\partial K_n = \partial D_n$, gives a natural parametrization of the boundary of the nth cluster by the unit circle. The mappings Γ_n induce a flow on the unit circle and this flow describes the evolution of the harmonic measure on the cluster boundary, as particles are added to the cluster.

We establish a scaling limit for this flow where particles are added at rate $O(d^{-2})$, and we show that it can be approximated by the flow arising from a deterministic ordinary differential equation closely related to the Loewner equation and determined by the ordinary differential equation

$$\dot{\phi}_{(s,t]}(x) = H[h_\nu](\phi_{(s,t]}(x)), \quad \phi_{(s,s]}(x) = x;$$

where $H[h_\nu](\xi) = \mathrm{p.v.}\frac{1}{2\pi}\int_0^1 \cot(\pi(\xi - z))h_\nu(z)dz$ is the Hilbert transform of the measure $\mathrm{d}\nu = h_\nu \mathrm{d}x$. We also characterise the stochastic fluctuations around this deterministic limit flow.

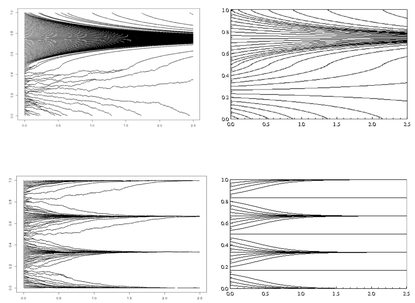

Figure 4: *Simulations of flows associated to* $\mathrm{AHL}(\nu)$ *and their limit, for* $d = 0.02$ *after* 25000 *repetitions, corresponding to* $\mathrm{d}\nu(x) = 2\chi_{[0,1/2]}\mathrm{d}x$ *and* $\mathrm{d}\nu(x) = 2\sin^2(3\pi x)\mathrm{d}x$.

We remark that the present situation is different from the isotropic case where the limiting object is "purely stochastic". In the later case, particles must be added at rate $O(d^{-3})$ and the flow converges to the so-called Brownian web of coalescing Brownian motions as was proved in [4].

References

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