Important Moments in Systems, Control and Optimization

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***Abstract*— The moment problem matured from its various special forms in the late 19th and early 20th Centuries to a general class of problems that continues to exert profound influence on the development of analysis and its applications to a wide variety of fields. In particular, the theory of systems and control is no exception, where the applications have historically been to circuit theory, optimal control, robust control, signal processing, spectral estimation, stochastic realization theory and the use of the moments of a probability density. Many of these applications are also still works in progress. In this paper, we consider the generalized moment problem, expressed in terms of a basis of a finite-dimensional subspace $\mathfrak{P}$ of the Banach space $C[a, b]$ and a "positive" sequences c, but with a new wrinkle inspired by the applications to systems and control. We seek to parameterize solutions which are positive "rational" measures, in a suitably generalized sense. Our parameterization is given in terms of smooth objects. In particular, the desired solution space arises naturally as a manifold which can be shown to be diffeomorphic to a Euclidean space and which is the domain of some canonically defined functions. Moreover, on these spaces one can derive natural convex optimization criteria which characterize solutions to this new class of moment problems.**

I. Introduction

Given a sequence of complex numbers, $(c_0, c_1, \cdots, c_n)$, and a basis, $(u_0, u_1, \cdots, u_n)$, of a (finite-dimensional) subspace $\mathfrak{P}$ of the Banach space $C[a, b]$ of complex-valued continuous functions defined on the real interval $[a, b]$, the generalized moment problem [24] is to find a positive measure $d\mu$ such that

$$\int_a^b u_k(t) d\mu(t) = c_k, \quad k = 0, 1, \cdots, n. \tag{1}$$

This problem is a beautiful generalization of several important classical moment problems, including the power moment problem, the trigonometric moment problem and the moment problem arising in Nevanlinna-Pick interpolation. There are, of course, necessary conditions stemming from the positivity of $d\mu$ and whether a particular u_k is real-valued or not; these will be summarized in Section II.

Among the pioneers in the use of power moments, where $u_k(t) = t^k$, we should mention Chebyshev and his students, particularly Markov and Lyapunov, who used them in connection with the classical Central Limit Theorem in the 19th Century. We refer to [24], especially pages 166–171 and the references therein, and to [9] for a more detailed historical and technical treatment.

Remark 1.1: In classical treatments of the power moment problems [24] it is typical to take $\mathfrak{P}$ to be the real subspace $\text{span}_{\mathbb{R}}\{u_0, \cdots, u_n\}$. While in this case the role of the functions u_k are clear and familiar to any student of probability, it is reasonable to ask why we need $\mathfrak{P}$. One of many good reasons for this is that $\mathfrak{P}$ is a natural space of "test functions" with which to develop necessary and sufficient conditions on the candidate moments for the solvability of the moment equations. For example, if $p(t) = p_0 + p_1 t + \cdots + p_n t^n > 0$ for all $t \in [a, b]$, then solvability of the moment equations for a positive measure $d\mu$ implies that

$$\sum_{i=0}^n p_i c_i = \int_a^b p(t) d\mu > 0. \tag{2}$$

This has been refined, in a neat way, to give necessary and sufficient conditions for the solvability of the generalized moment problem (see [24] and the discussion in Section II).

In the trigonometric moment problem, where $u_k(t) = e^{ikt}$ defined on $[-\pi, \pi]$, the constants c_k are, of course, the first $n + 1$ Fourier coefficients of $d\mu$. The corresponding moment problem was classically considered by Carathéodory in potential theory, where the moment conditions place a constraint on the boundary value data for Laplace's equation on the unit disc. Through subsequent classical work by Schur, Toeplitz, Nevanlinna, Pick and many others, this has been influential in the development of modern analysis (see e.g. [18]). Applications of the trigonometric moment problem to systems and control also have a long and fruitful history, including the rational covariance extension problem originally posed by Kalman [20] and later observed to be related to the trigonometric moment problem in [14]. However, to be applicable to problems in spectral estimation and stochastic realization theory there are systems theoretic constraints that must be added to the trigonometric moment problem, relating to rationality of, and the degree of, a solution. These challenges were noticed early on [20], [21], [15] and the ultimate breakthroughs relied (and still do rely) on the nontrivial use of topology, nonlinear convex optimization or a combination thereof (see [15], [2], and the SIGEST paper [3] and references therein).

Remark 1.2: In this setting, the classical theory was developed for a complex subspace $\mathfrak{P}$ of "test functions" and follows, *mutatis mutandis*, the real case. Explicitly, in order to develop the corresponding necessary conditions it is

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necessary to take those $p \in \mathfrak{P}$ for which the trigonometric polynomial $P := \mathrm{Re}(p)$ is positive on $[-\pi, \pi]$. The complex-valued enhacement of condition (2) is then

$$\mathrm{Re}\left(\sum_{i=0}^{n} p_i c_i\right) = \frac{1}{2}\sum_{i=0}^{n}(\bar{p}_i c_i + p_i \bar{c}_i) = \int_{-\pi}^{\pi} P d\mu > 0. \quad (3)$$

One of the many gems in this classical literature is the use [24, p. 65] of the Riesz-Fejér Theorem to evaluate the quadratic form on the right hand side of (3), where $P > 0$, as

$$\sum_{i,j=0}^{n} c_{i-j} z_i \bar{z}_j = \bar{z}^{\mathsf{T}} T_n z > 0,$$

where $z = (z_0, \ldots, z_n) \in \mathbb{C}^n \setminus 0$ and T_n the standard Toeplitz form fashioned out of the moment sequence $c = (c_i)$. For a general moment problem, the form on the left hand side of (3) is classically denoted by $\langle c, p \rangle$.

Remark 1.3: In both the power and the trigonometric moment problems we were led to consider the "polynomials," $P = \mathrm{Re}(p)$, for $p \in \mathfrak{P}$. For this reason, the functions $P := \mathrm{Re}(p)$, for $p \in \mathfrak{P}$ in an arbitrary generalized moment problem are referred to as "polynomials" for $\mathfrak{P}$. Following this precedent, we shall refer to the ratio P/Q with $p, q \in \mathfrak{P}$ as a "rational functions" for $\mathfrak{P}$.

In the Nevanlinna-Pick interpolation problem for distinct interpolation points $z_0, z_1, \ldots, z_n$, the basis functions are given by

$$u_k(t) = \frac{1}{2\pi}\frac{e^{ikt} + z_k}{e^{ikt} - z_k}, \quad k = 0, 1, \cdots, n,$$

which coincide on $[-\pi, \pi]$, modulo an additive constant, with Cauchy kernels. Higher order kernels can of course be used for multiple points. As for the case of trigonometric polynomials, it turns out that it is more helpful to identify the interval with the unit circle and, in this case, to think of $\mathfrak{P}$ in terms of Hardy spaces. This has also led to profound developments in several complex variables and in operator theory as well as in the applications of mathematics to circuit theory [13], [19] and to robust control [27], [22], [16], [12], [23]. For this problem as well, the applications to systems and control impose additional constraints to the classical moment problem whose treatment still requires nonlinear methods drawn from geometry, topology and/or optimization [19], [16], [6], [3].

Remark 1.4: For the classical Nevanlinna-Pick interpolation problem, using the Riesz-Fejér Theorem, the quadratic form (3) can also be evaluated, with some work [24, pp. 67-69] as the value of the celebrated Pick form. Moreover, it turns out that $\mathfrak{P}$ is a finite-dimensional coinvariant subspace of H^2 so that the elements of $\mathfrak{P}$ are rational functions σ/τ, where τ is fixed, and the "polynomials" are the real parts of elements in $\mathfrak{P}$. This of course implies that the "rational functions" are rational in the usual sense.

The generalized moment problem is about measures and combining these two concepts leads us to the following definition.

Definition 1.5: Any measure of the form

$$d\mu = \frac{P(t)}{Q(t)} dt, \quad (4)$$

where P and Q are positive polynomials for $\mathfrak{P}$, is a *rational positive measure*.

Problem 1.6: Given a sequence of complex numbers $c_0, c_1, \cdots, c_n$ and a subspace $\mathfrak{P}$, the generalized moment problem for rational measures is to parameterize all positive rational measures $\frac{P(t)}{Q(t)}dt$ such that

$$\int_a^b u_k(t)\frac{P(t)}{Q(t)}dt = c_k, \quad k = 0, 1, \cdots, n. \quad (5)$$

The problem itself is motivated by classical applications and examples, in both finite and infinite dimensions, and also reflects the importance of rational functions in systems and control. In this paper we give a concise description of all solutions of this generalized moment problem for a broad class of subspaces $\mathfrak{P}$.

II. The main result

In order to state our result, we first need to compute the dimension of $\mathfrak{P}$ as a real vector space, taking into account the cases where a basis element is real, purely imaginary or neither. In order for the moment equations to hold it is necessary that c_k be real whenever u_k is real. Moreover, a purely imaginary moment condition can always be reduced to a real one, and henceforth we shall assume that $u_0, \ldots, u_{r-1}$ are real functions and $u_r, \ldots, u_n$ are complex-valued functions whose real and imaginary parts, taking together with $u_0, \ldots, u_{r-1}$, are linearly independent over $\mathbb{R}$. In particular, we may regard $\mathfrak{P}$ as a real vector space of dimension $2n - r + 2$. Since we have chosen a fixed basis, we may regard each

$$p := \sum_{k=0}^{n} p_k u_k \in \mathfrak{P} \quad (6)$$

also as $(n+1)$-tuple of points $(p_0, p_1, \ldots, p_n)$, where $p_0, p_1, \ldots, p_{r-1}$ are real and $p_r, p_{r+1}, \ldots, p_n$ are complex. Moreover, p is determined by its real part $P := \mathrm{Re}(p)$, a notation we shall keep throughout. Next we define the subset $\mathfrak{P}_+$ of those elements $p \in \mathfrak{P}$ such that $P > 0$.

Throughout this paper we will assume the following.

Hypothesis 2.1: $\mathfrak{P}_+$ is nonempty.

In particular, $\mathfrak{P}_+$ is an open, convex set having dimension $2n - r + 2$. The rational measures we seek as solutions have the property that $d\mu(E) > 0$ for every Borel measurable subset $E \subset [a, b]$ having nonzero measure. For this reason, we will seek a necessary condition expressible in terms of the slightly larger space $\overline{\mathfrak{P}}_+ \setminus \{0\}$ of "test" functions. That is, we define $\mathfrak{C}_+$ as the set of sequences $c = (c_0, c_1, \cdots, c_n)$ such that

$$\langle c, p \rangle := \mathrm{Re}\left\{\sum_{k=0}^{n} p_k c_k\right\} = \int_a^b P\, d\mu > 0 \quad (7)$$

for all $p \in \overline{\mathfrak{P}}_+ \setminus \{0\}$. We will call such a sequence *positive*. In particular, $\mathfrak{C}_+$ is also a nonempty, open convex subset of $\mathbb{R}^{2n-r+2}$ of dimension $2n-r+2$.

Remark 2.2: A sequence c is positive *in the classical sense* if it satisfies $\langle c, p\rangle \geq 0$ for all $p \in \overline{\mathfrak{P}}_+$ and the main result [24] for the generalized moment problem is that there exists a positive measure satisfying (1) for any sequence c that is positive in the classical sense. Since we are seeking a solution to the moment problem for a smaller class of positive measure than in the classical treatment, our necessary conditions are, not surprisingly, stronger than the classical conditions. In particular, a sequence c which we call positive is referred to as *strictly positive* in [24].

We shall now fix $c \in \mathfrak{C}_+$ and consider the set $\mathfrak{M}_c$ of all pairs of polynomials (p, q) for which the rational measure $P(t)/Q(t)dt$ solves Problem 1.6 for the positive sequence c. There is a natural parameterization of $\mathfrak{M}_c$ as a submanifold of the product space $\mathfrak{P}_+ \times \mathfrak{P}_+$ and, as a subset of a product space, $\mathfrak{M}_c$ comes with two mappings:

$$\pi_1 : \mathfrak{M}_c \to \mathfrak{P}_+ \quad \text{and} \quad \pi_2 : \mathfrak{M}_c \to \mathfrak{P}_+,$$

where π_1 and π_2 are the restrictions to $\mathfrak{M}_c$ of the two mappings

$$\text{proj}_1 : \mathfrak{P}_+ \times \mathfrak{P}_+ \to \mathfrak{P}_+ \quad \text{and} \quad \text{proj}_2 : \mathfrak{P}_+ \times \mathfrak{P}_+ \to \mathfrak{P}_+,$$

defined by $\text{proj}_1(p, q) = p$ and $\text{proj}_2(p, q) = q$.

Theorem 2.3: Suppose that $\mathfrak{P}$ consists of Lipschitz continuous functions. Then, for each $c \in \mathfrak{C}_+$, $\mathfrak{M}_c$ is a smooth submanifold of $\mathfrak{P}_+ \times \mathfrak{P}_+$, diffeomorphic to $\mathbb{R}^{2n-r+2}$. Moreover, each of the maps π_1, π_2 is a diffeomorphism of $\mathfrak{M}_c$ onto its image, which is an open submanifold of $\mathfrak{P}_+$. Finally, $\pi_1 : \mathfrak{M}_c \to \mathfrak{P}_+$ is surjective.

The formulations of Definition 1.5 and Problem 1.6 for generalized rational measures and of Theorem 2.3 were developed in [9], which is summarized and enhanced in this paper. We believe this has some appeal both for the intrinsic simplicity of the formulation and as a unification of a variety of specific applications and more general results on the moment problem. In particular, to say that π_1 is a bijection is to say for every $c \in \mathfrak{C}_+$ and each $p \in \mathfrak{P}_+$, there exists a unique $q \in \mathfrak{P}_+$ such that (5) is satisfied. On the other hand, to say that that π_2 is an injection is to say for every $c \in \mathfrak{C}_+$ and each $q \in \mathfrak{P}_+$, there is at most one $p \in \mathfrak{P}_+$ such that (5) is satisfied. However, as we shall discuss below, in general there is a non-empty open set of q for which no p exists.

Remark 2.4: To the best of our knowledge, all instances of the generalized moment problem that arise in systems and control involve subspaces of $C[a, b]$ consisting of Lipschitz continuous functions. Moreover, this class of subspaces have been of considerable classical interest. For example, an important class of spaces considered in the classical literature on the generalized moment problem [24] consists of those spaces $\mathfrak{P}$ spanned by a Chebyshev system (or T-system), which are characterized by a bound on the number of zeros for any nonzero polynomial in $\mathfrak{P}$. These spaces arise in important applications of the generalized moment problem; e.g., the power moment problem and the trigonometric moment problem of odd order and have remarkable approximation properties in the Banach space $C[a, b]$. We remark that [24] contains a neat application, generalizing Feldbaum's Theorem on the number of switchings, of Chebyshev systems to the time-optimal control of scalar-input linear control systems. For our present purposes, we recall the classical result that, if $\mathfrak{P}$ is spanned by a Chebyshev system and contains a constant function, then, after a reparameterization, $\mathfrak{P}$ consists of Lipschitz continuous functions [24, p. 37].

Remark 2.5: We have remarked that the finite-dimensional Nevanlinna-Pick problem can be recast in a Hardy space setting, where the space $\mathfrak{P}$ is a coinvariant subspace (defined, in fact, by a finite Blaschke product) in $H^2(\mathbb{D})$. In a seminal paper [25], Sarason developed a vast generalization of this problem to one involving liftings of a partial isometry T, defined on an arbitrary coinvariant subspace, which commute there with the restriction of the shift operator. Among many other results, Sarason showed that, under general conditions, the lift of T has an H^∞ symbol which is *rational* with respect to the coinvariant subspace. The corresponding problem for T being a strict contraction was studied in [11], where optimization methods were used to show that the lifting of such a T always had such a generalized rational symbol. Moreover, it was shown that this symbol is completely parameterized by its numerator in parallel with the conclusion in Theorem 2.3 that π_1 is a bijection. In this light, it is interesting to enquire whether a general version of Problem 1.6 can be formulated, and solved, in a meaningful infinite dimensional setting.

III. An Outline of the Proof

The proof of our main result can be reduced to several steps. The first part involves establishing some smoothness results for $\mathfrak{M}_c$ and the maps π_1 and π_2. The map

$$M : \mathfrak{P}_+ \times \mathfrak{P}_+ \to \mathfrak{C}_+,$$

defined via

$$M(p, q) = \int_a^b \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix} \frac{P(t)}{Q(t)} dt$$

has $\mathfrak{M}_c$ as its level set $M^{-1}(c)$.

For simplicity, we view $\mathfrak{P}$ and $\mathfrak{C}$ as real vector spaces, so that $\mathfrak{P}$ is spanned by the real basis (u_i), where we have replaced a complex-valued (u_k) by its real and imaginary parts. The Jacobian, $\text{Jac}(M)_{(p_0, q_0)}$, of M at a point (p_0, q_0) takes the form

$$\text{Jac}(M) = (\partial M/\partial p, \partial M/\partial q) = (M_p, M_q) \tag{8}$$

where M_p is the square matrix whose (i, j)-th entry is

$$(M_p)_{(i,j)} = \int_a^b u_i(t) u_j(t) \frac{1}{Q(t)} dt \tag{9}$$

and where M_q is defined by

$$(M_q)_{(i,j)} = -\int_a^b u_i(t)u_j(t)\frac{P(t)}{Q^2(t)}dt, \qquad (10)$$

each being evaluated at the point (p_0, q_0). Thus, M_p (or $-M_q$) is the gramian matrix of the real basis (u_i) with respect to the positive definite inner product defined by $Q(t)^{-1}dt$ (or $P(t)/Q^2(t)dt$) on $C[a,b]$. Therefore, Jac(M) has rank $2n-r+2$ at each point (p,q) so that, by the Implicit Function Theorem, we obtain the following result.

Proposition 3.1: For each $c \in \mathfrak{C}_+$, $\mathfrak{M}_c$ is either empty or a submanifold of $\mathfrak{P}_+ \times \mathfrak{P}_+$ of real dimension $2n - r + 2$.

As restrictions of a smooth map to a smooth submanifold of the product, both π_1 and π_2 are smooth maps from $\mathfrak{M}_c$ to $\mathfrak{P}_+$. Suppose that $M(p_0, q_0) = c$ so that, in particular, $\mathfrak{M}_c$ is nonempty. An argument similar to the one given above shows:

Proposition 3.2: [9] Whenever $\mathfrak{M}_c$ is nonempty, each of the maps π_1 and π_2 is a local diffeomorphism.

The final steps in the proof are to demonstrate that $\mathfrak{M}_c$ is nonempty for each positive sequence c, that π_1 is a bijection and that π_2 is an injection.

Since the map

$$L_+ : \mathfrak{P}_+ \to \mathfrak{C}_+, \quad p \mapsto \int_b^a \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{pmatrix} \frac{P}{Q}dt, \quad k = 0, 1, \ldots, n,$$

is linear in p for a fixed $q \in \mathfrak{P}_+$, the inverse image $\pi_2^{-1}(q)$ in $\mathfrak{M}_c$ is convex for each fixed $q \in \mathfrak{P}_+$. If $\pi_2^{-1}(q)$ is nonempty, then Proposition 3.2 implies that it consists of a single point.

Lemma 3.3: The map π_2 is an injection.

The question of whether $\pi_2^{-1}(q)$ is nonempty is more interesting. To say that $L_+(p) = c$ is to say that $(p,q) \in \mathfrak{M}_c$, so we are interested in the image of L_+. This is also of independent interest. For example, if $\mathfrak{P}$ contains the constant functions, choosing $q = 1$ leads to a special case of Problem 1.6.

Problem 3.4: Given a sequence of complex numbers $c_0, c_1, \cdots, c_n$ and a subspace $\mathfrak{P}$, the generalized moment problem for polynomial measures is to parameterize all positive polynomial measures $P(t)dt$ such that

$$\int_a^b u_k(t)P(t)dt = c_k, \quad k = 0, 1, \cdots, n. \qquad (11)$$

More generally, to say that L_+ is surjective for a fixed $q \in \mathfrak{P}_+$ is equivalent to asserting that $L_+ : \partial\mathfrak{P}_+ \to \partial\mathfrak{C}_+$. This is trivially true for $\dim(\mathfrak{P}) = 1$. In order to analyze the image of L_+, we shall assume an auxiliary hypothesis, introduced in [8] in a similar context.

Hypothesis 3.5: The zero set of any $p \in \mathfrak{P}$ has Lebesgue measure zero.

Proposition 3.6: [9] If Hypothesis 3.5 holds, then the generalized moment problem for polynomial measures is unsolvable for a set of positive sequences having positive measure, whenever $\mathfrak{P}$ has dimension at least two.

Remark 3.7: Every $\mathfrak{P}$ spanned by a Chebyshev system (or T-system) satisfies Hypothesis 3.5. In particular, this applies to the power moment problem. The spaces $\mathfrak{P}$ corresponding to the trigonometric moment problem and the Nevanlinna-Pick interpolation problem satisfy Hypothesis 3.5. More generally, finite-dimensional spaces of analytic functions always satisfy Hypothesis 3.5.

In contrast, the analysis of π_1 is nonlinear and the result is nicer. To say that for every c, and for each p, there exists a unique q is to say that for each fixed p in $\mathfrak{P}_+$ and, as before, for an arbitrary $c \in \mathfrak{C}_+$, there exists a unique q in $\mathfrak{P}_+$ so that the corresponding rational measure solves the moment problem for c. This leads to a related constrained moment problem, defined as follows. Consider the function

$$F^p : \mathfrak{P}_+ \to \mathfrak{C}_+ \qquad (12)$$

defined componentwise via

$$F_k^p(q) = \int_a^b u_k(t)\frac{P(t)}{Q(t)}dt,$$

and a given positive sequence $c = (c_0, ..., c_n)$.

Proposition 3.8: If $\mathfrak{P}$ consists of Lipschitz continuous functions, the map (12) is a surjective local diffeomorphism.

Proof: We want to show that the image of the map (12) is both open and closed in $\mathfrak{C}_+$. We begin with a lemma ensuring that $F^p(\mathfrak{P}_+)$ is open.

Lemma 3.9: For each $q \in \mathfrak{P}_+$, $\det \mathrm{Jac}(F^p)|_q \neq 0$.

Proof: This follows immediately from the fact that Jac$(F^p)|_q) = M_q$ where M_q is the invertible matrix defined in (10). ■

In particular, by the Inverse Function Theorem, F^p is a local diffeomorphism and by the Implicit Function Theorem $F^p(\mathfrak{P}_+) \subset \mathfrak{C}_+$ is open. Since $\mathfrak{C}_+$ is connected, the proposition follows from the next result which implies that $F^p(\mathfrak{P}_+)$ is also closed in $\mathfrak{C}_+$.

Lemma 3.10: [9] If $\mathfrak{P}$ consists of Lipschitz continuous functions, the map (12) is proper; i.e., for any compact set K in $\mathfrak{C}_+$, $(F^p)^{-1}(K)$ is bounded.

Since the image of a proper map is closed, this concludes the proof of the proposition. ■

This result has several interesting consequences.

Corollary 3.11: If $\mathfrak{P}$ consists of Lipschitz continuous functions, then $\mathfrak{M}_c \neq \emptyset$ for each $c \in \mathfrak{C}_+$.

We have shown that (12) is a proper, local diffeomorphism onto the convex set $\mathfrak{C}_+$. Since any open convex subset of $\mathbb{R}^n$ is itself diffeomorphic to $\mathbb{R}^n$ (see, e.g., [5, p. 771]), (12) is a diffeomorphism by Hadamard's Global Inverse Function Theorem. We record this important fact as follows.

Theorem 3.12: If $\mathfrak{P}$ consists of Lipschitz continuous functions, the mapping

$$F^p : \mathfrak{P}_+ \to \mathfrak{C}_+$$

is a diffeomorphism.

Remark 3.13: An alternative proof of this result is given in the next section using convex optimization methods.

Corollary 3.14: If $\mathfrak{P}$ consists of Lipschitz continuous functions, for each $c \in \mathfrak{C}_+$ the restriction

$$\pi_1 : \mathfrak{M}_c \to \mathfrak{P}_+$$

of the first projection is bijective. That is, for every positive sequence c and every choice of p in $\mathfrak{P}_+$, there is a unique q such that (p,q) lies in $\mathfrak{M}_c$.

The conclusion of Corollary 3.14 defines, for each fixed $c \in \mathfrak{C}_+$, a map

$$g^c : \mathfrak{P}_+ \to \mathfrak{P}_+$$

where $g^c(p)$ is the unique q such that $(p,q) \in \mathfrak{M}_c$. This map was also studied in [8]. In more explicit terms, q is the unique function in $\mathfrak{P}_+$ such that Q is the denominator in the rational measure with numerator P solving the moment equations

$$\int_a^b u_k(t)\frac{P(t)}{Q(t)}dt = c_k, \quad k = 0, 1, \cdots, n$$

for c. Moreover, in the language of Theorem 2.3, we see that

$$g^c = \pi_2 \circ \pi_1^{-1}.$$

We summarize these observations in the following result.

Corollary 3.15 ([8]): The mapping

$$g^c : \mathfrak{P}_+ \to \mathfrak{P}_+$$

is a diffeomorphism onto its image.

IV. A Dirichlet Principle for the moment problem with rational positive measures

In this section, using a convex optimization argument derived in [7], [10], we outline an independent proof that the surjection F^p is injective, and we characterize the unique rational measure as the solution of a variational problem. In fact, we derive both a primal optimization problem and its dual. Remarkably, the moment problem for rational positive measures *is* the set of critical point equations for the dual variational problem. In this classical sense, a nonlinear convex optimization provides an illustration of the Dirchlet Principle for this class of moment problems.

Let $\mathbb{I}_p : C_+[a,b] \to \mathbb{R} \cup \{-\infty\}$ be the relative entropy functional

$$\mathbb{I}_p(\Phi) = \int_a^b P(t)\log\Phi(t)dt, \tag{13}$$

which is a generalization of the entropy functional obtained by setting $P = 1$. From Jensen's inequality we see that $\mathbb{I}_p(\Phi) \le \log\left(\int_a^b P\Phi dt\right) \le \int_a^b P\Phi dt < \infty$.

Theorem 4.1: Suppose $\mathfrak{P}$ consists of Lipschitz continuous functions and $c \in \mathfrak{C}_+$. Then, for any $p \in \mathfrak{P}_+$, the constrained optimization problem to maximize (13) over $C_+[a,b]$ subject to the moment constraints

$$\int_a^b u_k(t)\Phi(t)dt = c_k, \quad k = 0, 1, \ldots, n, \tag{14}$$

has a unique solution, and it has the form

$$\Phi = \frac{P}{Q}, \quad Q := \text{Re}\{q\}, \tag{15}$$

where $q \in \mathfrak{P}_+$.

The optimization problem of Theorem 4.1, to which we shall refer as the *primal problem*, can be solved by Lagrange relaxation. In fact, we have the Lagragian

$$L(\Phi, q) = \mathbb{I}(\Phi) + \text{Re}\sum_{k=0}^n q_k\left[c_k - \int_a^b u_k\Phi dt\right],$$

where $(q_0, q_1, \ldots, q_n) \in \mathbb{R}^r \times \mathbb{C}^{n-r+1}$ are Lagrange multipliers. Then,

$$L(\Phi, q) = \int_a^b P\log\Phi\; dt + \langle c, q\rangle - \int_a^b Q\Phi dt,$$

where $Q = \text{Re}\{q\}$ with $q := \sum_{k=0}^n q_k u_k \in \mathfrak{P}$. Clearly, comparing linear and logarithmic growth, we see that the dual functional

$$\psi(q) = \sup_{\Phi \in C_+[a,b]} L(\Phi, q)$$

takes finite values only if $q \in \mathfrak{P}_+$, so we may restrict our attention to such Lagrange multipliers. For any $q \in \mathfrak{P}_+$ and any $\Phi \in C_+[a,b]$ such that P/Φ is integrable, the directional derivative

$$d_{(\Phi,q)}L(h) = \int_a^b\left[\frac{P}{\Phi} - Q\right]h\,dt = 0$$

for all $h \in C[a,b]$ if and only if $\Phi = \frac{P}{Q} \in C_+[a,b]$, which inserted into the dual functional yields

$$\psi(q) = \mathbb{J}_p(q) + \int_a^b P(\log P - 1)dt, \tag{16}$$

where $\mathbb{J}_p : \overline{\mathfrak{P}}_+ \to \mathbb{R} \cup \{\infty\}$ is the strictly convex functional

$$\mathbb{J}_p(q) = \langle c, q\rangle - \int_a^b P\log Q dt. \tag{17}$$

As the last term in (16) is constant, the dual problem to minimize ψ over $\overline{\mathfrak{P}}_+$ is equivalent to the convex optimization problem

$$\min_{q \in \overline{\mathfrak{P}}_+} \mathbb{J}(q). \tag{18}$$

Since

$$\frac{\partial\mathbb{J}_p}{\partial q_k} = c_k - \int_a^b u_k\frac{P}{Q}\,dt, \quad k = 0, 1, \ldots, n,$$

it follows from Proposition 3.8 that the optimization problem (18) has an optimal solution $\hat{q} \in \mathfrak{P}_+$, satisfying the moment equations (5). Moreover, since the functional (17) is strictly convex, this optimum is unique.

Consequently,

$$\hat{\Phi} := \frac{P}{\hat{Q}} \in C_+[a,b] \tag{19}$$

is the unique optimal solution of the primal problem. To see this, observe that $\Phi \mapsto L(\Phi, \hat{q})$ is strictly concave and that $dL_{(\hat{\Phi},\hat{q})}(h) = 0$ for all $h \in C_+[a,b]$. Therefore,

$$L(\Phi, \hat{q}) \le L(\hat{\Phi}, \hat{q}), \quad \text{for all } \Phi \in C_+[a,b] \tag{20}$$

with equality if and only if $\Phi = \hat{\Phi}$. However, $L(\Phi, \hat{q}) = \mathbb{I}_p(\Phi)$ for all Φ satisfying the moment conditions (14). In particular, since (14) holds with $\Phi = \hat{\Phi}$, $L(\hat{\Phi}, \hat{q}) = \mathbb{I}_p(\hat{\Phi})$. Consequently, (20) implies that $\mathbb{I}_p(\Phi) \leq \mathbb{I}_p(\hat{\Phi})$ for all $\Phi \in C_+[a,b]$ satisfying the moment conditions, with equality if and only if $\Phi = \hat{\Phi}$. Hence, $\mathbb{I}_p$ has a unique maximum in the space of all $\Phi \in C_+[a,b]$ satisfying the constraints (14), and it is given by (19).

This concludes the proof of Theorem 4.1, but we have also proven the following theorem.

Theorem 4.2: Suppose $\mathfrak{P}$ consists of Lipschitz continuous functions. Let $(c,p) \in \mathfrak{C}_+ \times \mathfrak{P}_+$, and set $P := \text{Re}\{p\}$. Then the functional (17) has a unique minimizer $\hat{q} \in \mathfrak{P}_+$, and $\hat{Q} := \text{Re}\{\hat{q}\}$ is the unique solution to the moment equations

$$\int_a^b u_k \frac{P}{Q} dt = c_k, \quad k = 0, 1, \ldots, n. \tag{21}$$

Corollary 4.3: If $\mathfrak{P}$ consists of Lipschitz continuous functions, the moment mapping $F^p : \mathcal{P}_+ \to \mathfrak{C}_+$ is a bijection.

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