On the Stability and Instability of Padé Approximants

Christopher I. Byrnes and Anders Lindquist

Abstract Over the past three decades there has been interest in using Padé approximants K with $n = \deg(K) < \deg(G) = N$ as "reduced-order models" for the transfer function G of a linear system. The attractive feature of this approach is that by matching the moments of G we can reproduce the steady-state behavior of G by the steady-state behavior of K, for certain classes of inputs. Indeed, we illustrate this by finding a first-order model matching a fixed set of moments for G, the causal inverse of a heat equation. A key feature of this example is that the heat equation is a minimum phase system, so that its inverse system has a stable transfer function G and that K can also be chosen to be stable. On the other hand, elementary examples show that both stability and instability can occur in reduced order models of a stable system obtained by matching moments using Padé approximants and, in the absence of stability, it does not make much sense to talk about steady-state responses nor does it make sense to match moments. In this paper, we review Padé approximants, and their intimate relationship to continued fractions and Riccati equations, in a historical context that underscores why Padé approximation, as useful as it is, is not an approximation in any sense that reflects stability. Our main results on stability and instability states that if $N \geq 2$ and $\ell, r \geq 0$ with $0 < \ell + r = n < N$ there is a non-empty open set $U_{\ell,r}$ of stable transfer functions G, having infinite Lebesque measure, such that each degree n proper rational function K matching the moments of G has ℓ poles lying in $\mathbb{C}^-$ and r poles lying in $\mathbb{C}^+$. The proof is constructive.

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1 Introduction

The power moments

$$E(X^k) = \int_{-\infty}^{+\infty} x^k p(x)dx \tag{1}$$

of a random variable X defined on $\mathbb{R}$ have played a prominent role in probability ever since their use by Chebychev in his proof of the Central Limit Theorem. Their importance is largely due to their interpretation in terms of the Taylor coefficients

$$\phi_X^{(k)}(0) = i^k E(X^k)$$

of the characteristic function

$$\phi_X(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{i\xi x} p(x)dx = \hat{p}(\xi),$$

which is the Fourier transform of the probability density function.

Similarly, if $G(s) = C(sI-A)^{-1}B$ is the transfer function of a strictly proper linear systems (A,B,C), then the moments of G may be defined [14, pp. 112–113] as

$$\eta_k = (-1)^k \frac{d^kG}{ds^k}(0). \tag{2}$$

If $\sigma(A) \subset \mathbb{C}^-$, the moments of the system coincide with the the power moments

$$\eta_k = (-1)^k \frac{d^kG}{ds^k}(0) = \int_0^{\infty} t^k g(t)dt$$

of the impulse response $g(t) = C\mathrm{e}^{At}B$, for $k \geq 0$. For example, η_0 is the DC gain, $-CA^{-1}B$, of the system. In this case, since whenever $\lim_{t\to\infty} f(t)$ exists and $s\hat{f}(s)$ has no poles in the closed right plane we have

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\hat{f}(s),$$

for any other *stable* linear system whose transfer function $K(s)$ satisfies

$$\frac{d^kK}{ds^k}(0) = (-1)^k\eta_k, \;\; 0 \leq k \leq d \tag{3}$$

the difference between the responses to a fixed polynomial input $u(t) = a_0 + \cdots + a_d t^d$ will decay to zero as $t \to \infty$. In particular, any lower order *stable* interpolant K will have the same step response as G. Of course, similar remarks about steady-state behavior apply to the more general moment matching problem for the data

$$\eta_k(s_0) = \int_0^{\infty} t^k g(t)\mathrm{e}^{-s_0 t}dt = (-1)^k \frac{d^kG}{ds^k}(s_0)$$

whenever $s_0 = i\omega_0$ and G and K are stable, as the next example shows.

Example 1. Consider the controlled heat equation system [8]:

$$z_t(x,t) = z_{xx}(x,t) \tag{4}$$
$$z(0,t) = 0, \tag{5}$$
$$z_x(1,t) = u \tag{6}$$
$$z(x,0) = \varphi(x). \tag{7}$$
$$y(t) = z(1,t), \tag{8}$$

with transfer function

$$H(s) = \frac{\sinh(\sqrt{s})}{\sqrt{s}\cosh(\sqrt{s})}.$$

We wish to design a stable controller $K(s)$ so that the cascade interconnection $H(s)K(s)$ provides steady state tracking of the desired output $y_R(t)$ when driven by the input $y_R(t)$. In fact, since the heat equation has a stable, causal inverse system

$$z_t(x,t) = z_{xx}(x,t) \tag{9}$$
$$z(0,t) = 0, \tag{10}$$
$$z_x(1,t) = y_r \tag{11}$$
$$z(x,0) = \psi(x). \tag{12}$$
$$u_r(t) = z(1,t), \tag{13}$$

with transfer function $G(s) = H^{-1}(s)$, one can indeed use G as a feedforward controller. On the other hand, if the reference trajectory is given, for example, by $y_R(t) = A\sin(2t)$ then a *finite dimensional* cascade controller can be obtained by using any rational stable function satisfying the interpolation conditions

$$K(2i) = G(2i) = 1.0856 + 0.6504i, \tag{14}$$
$$K(-2i) = G(-2i) = 1.0856 - 0.6504i, \tag{15}$$

rounding to four decimals. Indeed, driving

$$K(s) = 1.4108\,\frac{s - .1525}{s+1} \tag{16}$$

with $y_R(t)$ produces the steady-state control law, $u_R(t) = 1.2655\sin(2t + 0.5397)$.

In the following simulations, we have taken initial condition $\varphi(x) = -4(1-2x)$. The steady state behavior of the state trajectory is illustrated in Figure 1. The steady state behavior of the output trajectory is illustrated in Figure 2.

In contrast to our first example, however, even interpolation data generated by a stable second order system need not have a stable first order interpolant.

Example 2. Consider the critically damped harmonic oscillator with transfer function

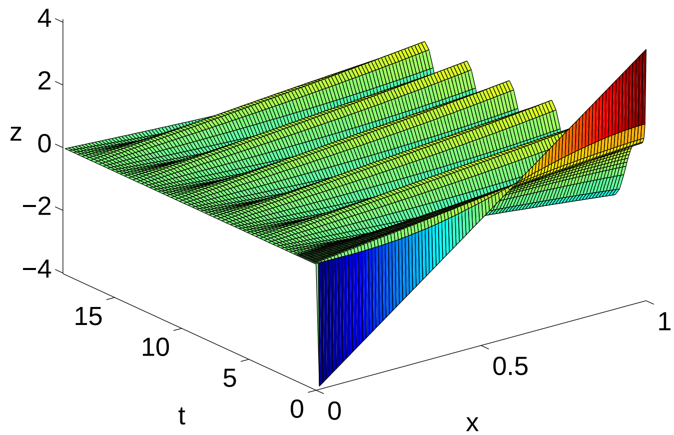

Figure 1. Plot of solution surface for the cascade connection HK driven by y_r.

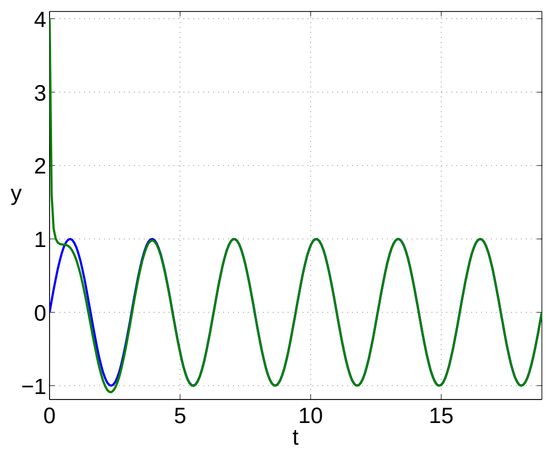

Figure 2. Plot of $y(t)$ compared with $y_r(t)$

$$G(s) = \frac{1}{s^2+2s+1} \tag{17}$$

and the induced one-parameter family of interpolation problems

$$K_\omega(i\omega) = G(i\omega),\ K_\omega(\infty) = G(\infty) = 0, \tag{18}$$

where for any fixed $\omega \in \mathbb{R}$ we seek a first order, stable interpolant K_ω.

First note that $-\pi/2 < \angle G(i\omega) < 0$ for any stable, strictly proper G with a positive high-frequency gain, while $\pi/2 < \angle G(i\omega) < \pi$ for any stable, strictly proper G with a negative high-frequency gain. On the other hand, $-\pi < \angle G(i\omega) < -\pi/2$

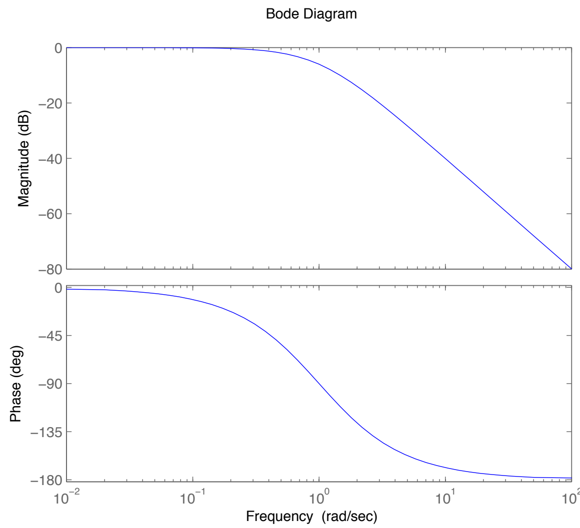

Figure 3. Bode plots for $G(s) = \dfrac{1}{s^2+2s+1}$

for $\omega > 1$, as is illustrated in Figure 3. In particular, the interpolation problem (18) has no stable, first order solution when $\omega > 1$.

Our final example illustrates the existence of stable rational interpolants for an open set of interpolation data.

Example 3. Consider the stable, minimum phase system with transfer function

$$G_\varepsilon(s) = \frac{s+1+\varepsilon}{s^2+2s+1} \tag{19}$$

and the one-parameter family of interpolation problems

$$K_\varepsilon(i) = G_\varepsilon(i+\varepsilon),\ K_\varepsilon(\infty) = G_\varepsilon(\infty) = 0, \tag{20}$$

where for any fixed $\varepsilon \in \mathbb{R}$ we seek a first order, stable interpolant K_ε. Of course, for $\varepsilon = 0$, we can take $K_0(s) = \dfrac{1}{s+1}$. More generally, a stable first-order interpolant exists whenever $-1 < \varepsilon < 1$. Indeed, in this case we have $-\pi/2 < \angle G_\varepsilon(i\omega) < 0$ from which it is easy to construct a stable first order interpolant K_ε.

As Example 1 illustrates, there is potential use for such approximants K with $\deg(K) < \deg(G)$ as "reduced-order models" for G (see, e.g., [1]) when the class of inputs is restricted to sinusoids of a given frequency, *provided the interpolant K is stable*. On the other hand, Examples 2 and 3 show that both stability and instability can occur in reduced order models of a stable system obtained by matching moments. In this paper we shall develop some qualitative results about the stability and instability of strictly proper rational functions which match a sequence of moments of a rational transfer function at $s = 0$. We expect that similar results hold for moments computed along the imaginary axis. Roughly speaking, any transfer function K, stable or not, matching $\eta_k(0)$, for $k = 0, \ldots, \tilde{n} < d$ is a *Padé approximation* to G. In Section 2, we review Padé approximants in more rigorous detail in a historical context that underscores why Padé approximation, as useful as it is, is not an approximation in any sense that reflects stability. In Section 3, we state our main results on stability and instability.

2 Padé approximants, continued fractions and Riccati equations

Over the past three decades there has been interest in using Padé approximants K with $\deg(K) < \deg(G)$ as "reduced-order models" for G (see, e.g., [1]). Rigorously, a Padé form of type (m,n) for G is a pair of polynomials (P,Q) with $\deg(P) \leq m$, $\deg(P) \leq n$ such that

$$Q(s)G(s) - P(s) = O(s^{n+m+1}) \tag{21}$$

as $s \to 0$. If $n = 0$, then (up to constant) P is the Taylor polynomial T_m of degree m. If $n,m \geq 1$ then $K(s) = P(s)/Q(s)$ is the ratio of two polynomials so that one might expect to obtain better approximations to G than T_m and, in many senses, this is true, explaining in part the ubiquity of Padé approximants. We shall be interested in the case $m \leq n$ and note that whenever

$$G(s) - K(s) = O(s^{n+m+1}) \tag{22}$$

as $s \to 0$, then (21) holds. As Example 4 shows, the converse, however, is not true in general.

Padé approximants have found a remarkably wide array of applications in mathematics, engineering and science [18]. In particular, Padé's advisor, Hermite [13], used Padé approximants in 1873 to prove that e is transcendental. Euler [9] had already proved that e is irrational in 1739, by developing a continued fraction expansion for $e^{1/z}$ and evaluating at $z = 1$ to obtain

$$\alpha = \alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \cdots}}} \tag{23}$$

where $(\alpha_0, \alpha_1, \alpha_2, \cdots) = (2, 1, 2, 1, 1, 4, 1, 1, 6, \cdots)$. Since a number is rational if and only if its continued fraction expansion is finite, Euler concludes that e is irrational, but his proof that the continued fraction does not terminate is a remarkable method for summing a continued fraction by solving a Riccati equation. In 1775, Euler [10] returned to this observation in a paper (see also [3]) in which he shows that any continued fraction of the form

$$f(z) = \cfrac{1}{\pi_1(z) + \cfrac{1}{\pi_2(z) + \cfrac{1}{\pi_3(z) + \cdots}}} \tag{24}$$

can be summed by solving a Riccati differential equation and that the solution of any Riccati equation can be expressed as a continued fraction of the form (24). As one of several examples, he gives the continued fraction

$$\frac{e^{2/z}+1}{e^{2/z}-1} = z + \cfrac{1}{3z + \cfrac{1}{5z + \cfrac{1}{7z + \cdots}}} \tag{25}$$

for the hyperbolic function $\coth(1/z)$ which, when evaluated at $z = 2$, gives another proof that e is irrational.

Recall that a best rational approximant to a real number r is a rational number p/q such that $|r - p/q|$ is smaller than any other rational approximation with a smaller denominator. Among the remarkable properties of continued fraction expansions of a real number r is that the rational numbers obtained from the partial sums p_n/q_n obtained from $(\alpha_0, \alpha_1, \ldots, \alpha_n, 0, \ldots)$ turn out to be the sequence of best rational approximants to r and any best rational approximant to r arises in this way. For example, the continued fraction expansion of π yields the sequence $3/1, 22/7, 333/106, \ldots$ of best rational approximants. In general, one can show [12, p. 151] the stronger result that for any $p/q \neq p_n/q_n$

$$0 < q \leq q_n \implies |qr - p| > |q_n r - p_n| \tag{26}$$

Similarly, the partial sums obtained from a continued fraction expansion (24) for a function $f(z)$ form a sequence of Padé approximants (22).

Example 4. Padé approximants can be formed at any point in the extended complex plane, including $s = \infty$ as is treated in [18]. For example, given the Laurent expansion

$$G(s) = \gamma_0 + \gamma_1/s + \gamma_2/s^2 + \ldots\,, \tag{27}$$

consider the problem of finding partial realizations for the sequence of Markov parameters $(\gamma_1, \gamma_2, \gamma_2 \ldots) = (0, 1, 0, 1, 0, 0, \ldots)$ generated by the fourth order linear system with transfer function $G(s) = (s^2+1)/s^4$ having a continued fraction expansion

$$G(s) = \frac{s^2+1}{s^4} = \cfrac{1}{s^2 - 1 + \cfrac{1}{s^2+1}} \tag{28}$$

Indeed while (21) has a solution of type $(1,1)$ the rational form of this expression in (22) does not, reflecting the fact that there is no partial realization fo degree 1. On the other hand, $G_2(s) = 1/(s^2-1)$ is a second order partial realization obtained by truncating the continued fraction expansion. For a generic $G(s)$, the polynomials $\pi_i(s)$ will be linear functions [11, 17].

Remark 1. By analogy with the use of continued fractions in number theory, one might conclude that Padé approximants can be thought of as the "best" rational approximants to $f(z)$. However, while (1) is similar to (21) and $|r - p/q|$ is similar to (22), *best* in the sense of real and rational numbers is measured by absolute values of differences of real numbers while *best* for Padé approximants is measured by degrees of differences of polynomials and rational functions, which in general will not detemine the location of poles or zeros.

3 Main Results

The set of proper rational functions

$$\mathrm{Rat}^*(N) = \{G : G(s) = \frac{p(s)}{q(s)},\ \deg(p) = \deg(q) = N,\ (p,q) = 1\} \tag{29}$$

can be parameterized as an open, dense subset of $\mathbb{R}^{2N+1}$ using the coefficients of the polynomials

$$p(s) = p_N s^N + \cdots + p_1 s + p_0,\ \ q(s) = s^N + q_{N-1}s^{N-1} + \cdots + q_0$$

We call $G \in \mathrm{Rat}^*(N)$ *stable* if all of its poles lie in the open left half plane $\mathbb{C}^-$ and *completely unstable* if all of it poles lie in the open right half plane $\mathbb{C}^+$. We are also interested in the number ℓ of poles of a rational function K lying in $\mathbb{C}^-$ and the number r of poles of K lying in $\mathbb{C}^+$. Thus, $\ell + r = n = \deg(K)$.

Theorem 1. *Suppose $N \geq 2$ and $\ell, r \geq 0$ with $0 < \ell + r = n < N$. For each pair ℓ, r there is a non-empty open cone $U_{\ell,r} \subset \mathrm{Rat}^*(N)$ of stable transfer functions G such that each degree n proper rational function K satisfying* (3) *with $d = 2n$ has ℓ poles lying in $\mathbb{C}^-$ and r poles lying in $\mathbb{C}^+$.*

In particular, for each n there does not exist a stable reduced order model of degree n for an open set of stable G having infinite Lebesgue measure.

Corollary 1. *Suppose $N \geq 2$. For each n satisfying $1 \leq n < N$ there is a non-empty open cone $U_n \subset \mathrm{Rat}^*(N)$ of stable transfer functions G such that each rational K satisfying* (3) *with $d = 2n$ is completely unstable.*

On the other hand, we have the following parallel positive result.

Corollary 2. *Suppose $N \geq 2$. For each n satisfying $1 \leq n < N$ there is a non-empty open cone $V_n \subset \mathrm{Rat}^*(N)$ of stable transfer functions G such that each rational K satisfying* (3) *with $d = 2n$ is stable.*

Proof. Each of the subsets

$$W_1^N = \{G \in \mathrm{Rat}^*(N) : G(0) \neq 0,\ G(\infty) \neq 0\},\ \ W_2^N = \{G \in \mathrm{Rat}^*(N) : q_0 \neq 0\}$$

is open and dense in $\mathrm{Rat}^*(N)$ and so is their intersection $W^N = W_1^N \cap W_2^N$. The function

$$T : W^N \to W^N \ \text{ defined by } \ T(G)(s) = G(1/s)$$

is a homeomorphism since it is continuous and its own inverse. Moreover, the map $s \to 1/s$ leaves both $\mathbb{C}^-$ and $\mathbb{C}^+$ invariant. Therefore, it suffices to prove Theorem 1 on W^N replacing (3) with the partial realization problem

$$\frac{d^kK}{ds^k}(\infty) = (-1)^k \gamma_k,\ \ 0 \leq k \leq 2n, \tag{30}$$

where $\gamma_0, \gamma_1, \gamma_2, \ldots$ are the Markov parameters given by (27). Since solutions to the partial realization theorem are unchanged under multiplication by a non-zero constant, it is clear that the open sets described in Theorem 1 are cones and that it therefore suffices to prove that they are non-empty. Since we are interested only in the number of poles in open half-planes and stability, we can also suppress γ_0 so that we may assume that G is strictly proper. In this case, we are interested in the open dense set $U_N = W^N \cap V^N$ where

$$V^N = \{G : \det(\gamma_{i+j-1})_{i,j=1,\ldots,N} \neq 0\}$$

which is known to be open and dense [5]. For any $G \in V^N$, any degree n rational function K satisfying (3) with $d = 2n$ is unique and can be constructed using the following algorithm.

Following [11], we associate a parameter sequence $\rho = (\rho_1, \ldots, \rho_{2N})$ to each $G \in V^N$, where $\rho \in \mathscr{V}^N = \{\rho : \rho_i \neq 0,\ \ i = 1, \ldots, 2N\}$. In [6, Lemma 1], it is shown that the map $\phi : V^N \to \mathscr{V}^N$ defined by $\phi(G) = \rho$ is a homeomorphism. From ρ one can [11] construct $K(s) = P_n(s)/Q_n(s)$ from the three-term recursions:

$$P_n(s) = (s - \rho_{2n})P_{n-1}(s) - \rho_{2n-1}P_{n-2}(s);\ \ P_0 = 0,\ \ P_{-1} = -1 \tag{31}$$

$$Q_n(s) = (s - \rho_{2n})Q_{n-1}(s) - \rho_{2n-1}Q_{n-2}(s);\ \ Q_0 = 1,\ \ Q_{-1} = 0. \tag{32}$$

Finally, in [6, Lemma 3], an open dense subset $\mathscr{U}_N \subset U_N$ is constructed so that the map $\rho \to (Q_N, Q_{N-1}, \rho_1)$ is a global, continuous change of coordinates.

Matters being so, we are now prepared to conclude the proof of Theorem 1 by induction on N. For $N = 2$, we have $n = 1$ and so either $\ell = 1,\ r = 0$ or $\ell = 0,\ r = 1$. In the first case, we construct the open subset

$$U_{1,0} = \{(Q_2, Q_1, \rho_1) \in \mathscr{U}_2 : Q_2 \text{ is stable}, \ \ Q_1 \text{ is stable}, \ \rho_1 \neq 0\}.$$

In the second case, we construct the open subset

$$U_{0,1} = \{(Q_2, Q_1, \rho_1) \in \mathscr{U}_2 : Q_2 \text{ is stable}, \ \ Q_1 \text{ is unstable}, \ \rho_1 \neq 0\}.$$

The latter construction is a special case of [6, Theorem 1], which was done in the case $n = r = N-1$.

We now assume Theorem 1 is true for $N-1$. In particular, for every $1 \leq \ell + r = n \leq N-2$ there exists an open subset $U_{\ell,r} \subset \mathscr{U}_{N-1}$ of stable rational functions P_{N-1}/Q_{N-1} so that the degree n partial realization has ℓ poles in $\mathbb{C}^-$ and r poles in $\mathbb{C}^+$. In the parameter sequence coordinates, we need to supplement the open set of corresponding $(\rho_1, \ldots, \rho_{2N-2})$ by adding two more coordinates $\widetilde{\rho}_{2N-1}, \widetilde{\rho}_{2N}$ is such a way that Q_N is stable and the corresponding subset $U_{\ell,r} \subset \mathscr{U}_N$ of points $(\rho_1, \ldots, \rho_{2N-2}, \widetilde{\rho}_{2N-1}, \widetilde{\rho}_{2N})$ is open. We first choose $\widetilde{\rho}_{2N} < 0$, so that the first term $d(s) = (s - \widetilde{\rho}_{2N})Q_{N-1}(s)$ appearing in the expression (32) for Q_N is a Hurwitz polynomial. We next write $n(s) = Q_{n-2}(s)$ and $k = -\widetilde{\rho}_{2N-1}$ so that (32) is the closed loop denominator $d(s) + kn(s)$ for the feedback system consisting of the stable open-loop systeme $g(s) = n(s)/d(s)$ with the feedback law $u = -ky$. In particular, for $\widetilde{\rho}_{2N-1}$ sufficiently small, the closed-loop system is stable and Q_N is a Hurwitz polynomial. Therefore, we have proved Theorem 1 for $n \leq N-2$.

Finally, suppose $n = N-1$. For any decomposition $\ell + r = n$, in the (Q_N, Q_{N-1}, ρ_1) coordinates on $\mathscr{U}_N$ we shall choose Q_N to be a Hurwitz polynomial and Q_{N-1} to have ℓ poles in $\mathbb{C}^-$ and r poles in $\mathbb{C}^+$. The corresponding subset $U_{\ell,r} \subset \mathscr{U}_N$ is again clearly open.

Remark 2. The importance of continued fractions in the deterministic partial realization problem was recognized in [15] and developed more comprehensively in [11], using the results in [16, 17]. These results were used in [6] to study the stability and instability properties of partial realizations, early results which are now generalized by Theorem 1. The inductive proof of Theorem 1 is constructive in each step and is phrased in terms of basic facts about root-loci. This is intimately related to the stability and instability proofs given in [6] using the Nyquist stability criterion. The geometry of the deterministic partial realization problem and its smooth parameterizations were studied in [5] using differential topology. The stochastic realization problem, which has proven much harder to analyze, was most recently studied using methods from algebraic geometry and differential topology in [7] in which it is shown, among other things, that there is no generic value for the degree of a minimal partial stochastic realization of a given covariance sequence $(\gamma_0, \ldots, \gamma_n)$, in contrast to the deterministic partial realization problem.

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