

Multivariable analytic interpolation with complexity constraints: A modified Riccati approach

Yufang Cui¹, and Anders Lindquist²

Abstract—Analytic interpolation problems with rationality and derivative constraints occur in many applications in systems and control. In this paper we present a new method for the multivariable case, which generalizes our previous results on the scalar case. This turns out to be quite nontrivial, as it poses many new problems. A basic step in the procedure is to solve a Riccati type matrix equation. To this end, an algorithm based on homotopy continuation is provided.

I. INTRODUCTION

A common problem in robust control and spectral estimation is to find an $\ell \times \ell$ matrix-valued real rational function F , analytic in the unit disc $\mathbb{D} = \{z \mid |z| < 1\}$, such that

$$F(e^{i\theta}) + F(e^{-i\theta})' > 0, \quad -\pi \leq \theta \leq \pi, \quad (1)$$

which also satisfies the interpolation condition

$$\frac{1}{k!} F^{(k)}(z_j) = W_{jk}, \quad j = 0, 1, \dots, m, \quad (2)$$

$$k = 0, \dots, n_j - 1,$$

where $'$ denotes transposition, $F^{(k)}(z)$ is the k th derivative of $F(z)$, and z_0, z_1, \dots, z_m are distinct points in \mathbb{D} , occurring in conjugate pairs if not real, and $W_{jk} \in \mathbb{C}^{\ell \times \ell}$ for each (j, k) . We restrict the complexity of the rational function $F(z)$ by requiring that its McMillan degree be at most ℓn , where

$$n = \sum_{j=0}^m n_j - 1. \quad (3)$$

Without loss of generality we may assume that $z_0 = 0$ and $W_0 = \frac{1}{2}I$. Then $F(z)$ has a realization

$$F(z) = \frac{1}{2}I + zH(I - zF)^{-1}G, \quad (4)$$

where $H \in \mathbb{R}^{\ell \times \ell n}$, $F \in \mathbb{R}^{\ell n \times \ell n}$, $G \in \mathbb{R}^{\ell n \times \ell}$, the matrix F has all its eigenvalues in \mathbb{D} and (H, F) is an observable pair.

Let W be the $\ell(n+1) \times \ell(n+1)$ matrix

$$W := \begin{bmatrix} W_0 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & W_m \end{bmatrix} \quad (5)$$

with

$$W_j = \begin{bmatrix} W_{j0} & & & \\ W_{j1} & W_{j0} & & \\ \vdots & \ddots & \ddots & \\ W_{jn_{j-1}} & \cdots & W_{j1} & W_{j0} \end{bmatrix} \in \mathbb{C}^{\ell n_j \times \ell n_j} \quad (6)$$

¹Department of Automation, Shanghai Jiao Tong University, Shanghai, China. cui-yufang@sjtu.edu.cn

²Department of Automation and School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai, China. alq@math.kth.se

for each $j = 0, 1, \dots, m$. Moreover, let Z be the $(n+1) \times (n+1)$ matrix

$$Z := \begin{bmatrix} Z_0 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & Z_m \end{bmatrix} \quad (7)$$

where, for each $j = 0, 1, \dots, m$, Z_j is the $n_j \times n_j$ matrix

$$Z_j = \begin{bmatrix} z_j & & & \\ 1 & z_j & & \\ & \ddots & \ddots & \\ & & 1 & z_j \end{bmatrix}. \quad (8)$$

Finally define the $n+1$ -dimensional column vector

$$e := [e_{n_0}^1, e_{n_1}^1, \dots, e_{n_m}^1]', \quad (9)$$

where $e_{n_j}^1 = [1, 0, \dots, 0] \in \mathbb{R}^{n_j}$ for each $j = 0, 1, \dots, m$, and let S be the unique solution of the Lyapunov equation

$$S = ZSZ^* + ee'. \quad (10)$$

Note that the eigenvalues of Z are all located in the open unit disc \mathbb{D} .

The problem of determining the interpolant $F(z)$ is an inverse problems which has a solution if and only if

$$W(S \otimes I_\ell) + (S \otimes I_\ell)W^* > 0, \quad (11)$$

where \otimes denotes Kronecker product (see, e.g., [1]), and then there are an infinite number of solutions. We would like to find a parametrization of these solutions.

The special case when $\ell = 1$, $m = 0$ and $n_0 = n + 1$ is called the *rational covariance extension problem* and was first formulated by Kalman [2] and then solved in steps in [3], [4], [5], [6], where a complete parameterization in terms of spectral zeros was obtained, and in [7], [8], where a convex optimization approach was introduced. This problem have occurred in many applications in systems and control such as in signal and speech processing [9] and in identification [10]. The case $n_0 = n_1 = \dots = n_m = 1$ and $m = n$ is called the *Nevanlinna-Pick interpolation problem with degree constraint* and was early considered in robust control [11] and many other applications in systems and control [12], [13]. It was completely parameterized, again in steps, in [14], [15], [16], [17], and a convex optimization approach was introduced in [16], [17]. Since then a large number of papers on the more general scalar problem has appeared [18], [19], [20], [21], [22]. We refer to [10] for further references.

The multivariable case ($\ell > 1$) is much harder, and the nice spectral-zero assignability present in the scalar case

appears to be lost or at least elusive. Restrictive classes of such problems have been considered in large number of papers [3], [1], [23], [24], [25], [26], [28], [27], [29], [30], but the theory remains incomplete, and many problems have been left open.

In [31] we presented a complete parameterization of the problem presented above for the scalar case ($\ell = 1$) in terms of a modified Riccati equation, which was first introduced for more restricted classes of interpolation problems in [6] and [32]. As [6] studied the rational covariance extension problem, the modified Riccati equation was named the *Covariance Extension Equation* (CCE), and we retain this name although the problems now considered are much more general.

In the present paper we take a first step in generalizing the results in [31] to the multivariable case ($\ell > 1$). In Section II we provide the basic tools for the multivariable problem. To describe our ultimate goal we provide in Section III a brief review of the scalar results in [31], and then in Section IV we develop the multivariable case in the same spirit. In Section V we present our main results and an algorithm based on homotopy continuation in the style of [33], [31]. The results fall somewhat short of what the scalar case promises, and, given some results in [27], we suspect that this is due to problems introduced by the nontrivial Jordan structure of the multivariable case. In Section VI-C we provide some simulations to illustrate this and also an example of model reduction. Finally, in Section VII we give some concluding remarks and suggestions for future research.

II. PRELIMINARIES

Defining $\Phi_+(z) := F(z^{-1})$ we have

$$\Phi_+(z) = \frac{1}{2}I + H(zI - F)^{-1}G, \quad (12)$$

which has all its poles in the unit disc \mathbb{D} . In view of (1)

$$\Phi_+(e^{i\theta}) + \Phi_+(e^{-i\theta})' > 0, \quad -\pi \leq \theta \leq \pi,$$

and hence $\Phi_+(z)$ is positive real [10, Chapter 6]. By a coordinate transformation $(H, F, G) \rightarrow (HT^{-1}, TFT^{-1}, TG)$ we can choose (H, F) in the observer canonical form

$$H = \text{diag}(h_{t_1}, h_{t_2}, \dots, h_{t_\ell}) \in \mathbb{R}^{\ell \times n\ell}$$

with $h_\nu := (1, 0, \dots, 0) \in \mathbb{R}^\nu$, and

$$F = J - AH \in \mathbb{R}^{n\ell \times n\ell} \quad (13)$$

where $J := \text{diag}(J_{t_1}, J_{t_2}, \dots, J_{t_\ell})$ with J_ν the $\nu \times \nu$ shift matrix

$$J_\nu = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

and $A \in \mathbb{R}^{n\ell \times \ell}$. The numbers t_1, t_2, \dots, t_ℓ are the *observability indices* of $\Phi_+(z)$, and

$$t_1 + t_2 + \dots + t_\ell = n\ell. \quad (14)$$

Next define $\Pi(z) := \text{diag}(\pi_{t_1}(z), \pi_{t_2}(z), \dots, \pi_{t_\ell}(z))$, where $\pi_\nu(z) = (z^{\nu-1}, \dots, z, 1)$, and the $\ell \times \ell$ matrix polynomial

$$A(z) = D(z) + \Pi(z)A, \quad (15)$$

where

$$D(z) := \text{diag}(z^{t_1}, z^{t_2}, \dots, z^{t_\ell}). \quad (16)$$

Lemma 1: $H(zI - F)^{-1} = A(z)^{-1}\Pi(z)$

Proof: Since

$$\Pi(z)(zI - J) = \text{diag}(z^{t_1}, z^{t_2}, \dots, z^{t_\ell})H,$$

$$\Pi(z)(zI - F) = \Pi(z)(zI - J) + \Pi(z)AH = A(z)H$$

as claimed. \blacksquare

Then it directly follows from Lemma 1 and (15), that (12) takes the form

$$\Phi_+(z) = \frac{1}{2}A(z)^{-1}B(z), \quad (17)$$

where

$$B(z) = D(z) + \Pi(z)B$$

with

$$B = A + 2G. \quad (18)$$

Moreover let $V(z)$ be the minimum-phase spectral factor of

$$V(z)V(z^{-1})' = \Phi(z) := \Phi_+(z) + \Phi_+(z^{-1})'.$$

We know [10, Chapter 6] that $V(z)$ has a realization of the form

$$V(z) = H(zI - F)^{-1}K + R,$$

which, by Lemma 1, can be written

$$V(z) = A(z)^{-1}\Sigma(z)R, \quad (19)$$

where

$$\Sigma(z) = D(z) + \Pi(z)\Sigma \quad (20)$$

with

$$\Sigma = A + KR^{-1}. \quad (21)$$

From stochastic realization theory [10, Chapter 6] we have

$$K = (G - FPH')(R')^{-1} \quad (22)$$

$$RR' = I - HPH' \quad (23)$$

where P is the unique minimum solution of the algebraic Riccati equation

$$P = FPF' + (G - FPH')(I - HPH')^{-1}(G - FPH)'. \quad (24)$$

Now, from (13), (22) and (23) we have

$$\begin{aligned} G &= JPH' - AHPH' + KR^{-1}(I - HPH') \\ &= \Gamma PH' + KR^{-1}, \end{aligned}$$

where, in view of (21),

$$\Gamma = J - \Sigma H. \quad (25)$$

Hence

$$G = \Gamma PH' + \Sigma - A. \quad (26)$$

Since $F = \Gamma + KR^{-1}H$ and $G - \Gamma PH' = KR^{-1}$, (24) can be written

$$\begin{aligned} P &= (\Gamma + KR^{-1}H)P(\Gamma + KR^{-1}H)' + KK' \\ &= \Gamma P\Gamma' + \Gamma PH'(KR^{-1})' + KR^{-1}HP\Gamma' \\ &\quad + KR^{-1}(KR^{-1})', \end{aligned}$$

where we have also used (23). Inserting $KR^{-1} = G - \Gamma PH'$ we have

$$P = \Gamma(P - PH'HP)\Gamma' + GG'. \quad (27)$$

III. A REVIEW OF THE SCALAR CASE

To motivate our approach to the multivariable problem presented in Section I we shall briefly review some results on the scalar case ($\ell = 1$) presented in [31]. To stress the fact that the matrices H, G, A, B and Σ are now n -vectors we shall here denote them h, g, a, b and σ instead, and the scalar R will be denoted ρ .

Introducing the interpolation conditions (2) into the calculation we obtain

$$g = u + U\sigma + U\Gamma Ph,$$

where P is the unique solution of the algebraic Riccati equation (27) and $u \in \mathbb{R}^n$ and $U \in \mathbb{R}^{n \times n}$ are given by

$$\begin{bmatrix} u & U \end{bmatrix} := \begin{bmatrix} 0 & I_n \end{bmatrix} M \quad (28)$$

with

$$M = \begin{bmatrix} e & V \end{bmatrix}^{-1} \left(W + \frac{1}{2}I \right)^{-1} \left(W - \frac{1}{2}I \right) \begin{bmatrix} e & V \end{bmatrix} \quad (29a)$$

and

$$V := \begin{bmatrix} Ze & Z^2e & \cdots & Z^ne \end{bmatrix}. \quad (29b)$$

Eliminating g we then have the modified Riccati equation

$$\begin{aligned} P &= \Gamma(P - Phh'P)\Gamma' \\ &\quad + (u + U\sigma + U\Gamma Ph)(u + U\sigma + U\Gamma Ph)', \end{aligned} \quad (30)$$

We showed in [31] that there is a map sending W to u which is a diffeomorphism and that there is a linear map L such that $U = Lu$.

Let \mathcal{S}_n be the space of Schur polynomials (i.e., polynomials with all zeros in the open unit disc \mathbb{D}) of the form

$$a(z) = z^n + a_1z^{n-1} + \cdots + a_n, \quad (31)$$

and let \mathcal{P}_n be the $2n$ -dimensional space of pairs $(a, b) \in \mathcal{S}_n \times \mathcal{S}_n$ such that $b(z)/a(z)$ is positive real. Moreover, for each $\sigma \in \mathcal{S}_n$, let $\mathcal{P}_n(\sigma)$ be the submanifold of \mathcal{P}_n for which

$$a(z)b(z^{-1}) + b(z)a(z^{-1}) = 2\rho^2\sigma(z)\sigma(z^{-1}) \quad (32)$$

holds, where ρ^2 is the appropriate normalizing factor. It was shown in [34] that $\{\mathcal{P}_n(\sigma) \mid \sigma \in \mathcal{S}_n\}$ is a *foliation* of \mathcal{P}_n , i.e., a family of smooth nonintersecting submanifolds, called *leaves*, which together cover \mathcal{P}_n . Moreover, for any polynomial (31), let $a_*(z) = z^n a(z^{-1})$ be the reversed polynomial of $a(z)$. Finally, let \mathcal{W}_+ be the space of all W such that the generalized Pick matrix $WS + SW^*$ is positive definite, where S the unique solution of the Lyapunov equation (10).

The following result was proved in [31].

Theorem 2: Let $\ell = 1$. For each $(\sigma, W) \in \mathcal{S}_n \times \mathcal{W}_+$, the modified Riccati equation (30) has a unique positive definite solution P such that $hPh' < 1$, and the problem to find a rational function $b_*(z)/a_*(z)$ satisfying the interpolation conditions (2) and the positivity condition (32) has a unique solution given by

$$\begin{aligned} a &= (I - U)(\Gamma Ph + \sigma) - u \\ b &= (I + U)(\Gamma Ph + \sigma) + u \end{aligned} \quad (33)$$

In fact, the map sending $(a, b) \in \mathcal{P}_n(\sigma)$ to $W \in \mathcal{W}_+$ is a diffeomorphism. Finally, the degree of the interpolant equals the rank of P .

Consequently, by (32), for each $\sigma \in \mathcal{S}_n$ there is a unique interpolant $b_*(z)/a_*(z)$ with the prescribed properties such that

$$\rho^2 \frac{\sigma(z)\sigma(z^{-1})}{a(z)a(z^{-1})} = \frac{1}{2} \left[\frac{b(z)}{a(z)} + \frac{b(z^{-1})}{a(z^{-1})} \right].$$

Hence

$$V(z) = \rho \frac{\sigma(z)}{a(z)}$$

is the corresponding spectral factor.

In [31] we solved (30) by homotopy continuation by taking $u(\lambda) = \lambda u$ with λ varying from 0 to 1. We showed that this provides an efficient and robust algorithm for analytic interpolation with degree constraint that can handle situations which are difficult with the optimization approach, especially when system poles are close to the unit circle.

IV. THE MATRIX CASE

Next we turn to the general multivariable case and introduce the interpolation condition (2) in the matrix setting of Section II.

Lemma 3: Let the matrices W and Z be given by (5) and (7), respectively. Then the interpolation condition (2) can be written

$$F(Z \otimes I_\ell) = W. \quad (34)$$

Proof: Since $F(z)$ is analytic in \mathbb{D} , it has the representation

$$F(z) = \sum_{k=0}^{\infty} C_k z^k$$

for all $z \in \mathbb{D}$, where $C_0 = \frac{1}{2}I_\ell$. A straight-forward but tedious calculation, omitted here for lack of space, yields

$$F(Z_j \otimes I_\ell) = \sum_{k=0}^{\infty} (Z_j)^k \otimes C_k = W_j,$$

where W_j , defined by (6), is given by (2). Then (34) follows from (7) and (5). \blacksquare

Let $A_*(z)$ be the reversed matrix polynomial

$$A_*(z) = D(z)A(z^{-1}) = I_\ell + D(z)\Pi(z^{-1})A, \quad (35)$$

where $D(z)$ is given by (16), and let $B_*(z)$ be defined in the same way in terms of $B(z)$. Then

$$F(z) = \frac{1}{2}A_*(z)^{-1}B_*(z) \quad (36)$$

and the interpolation condition (34) can be written

$$2A_*(Z \otimes I_\ell)W = B_*(Z \otimes I_\ell). \quad (37)$$

Moreover, let the $\ell \times \ell n$ matrices N_1, N_2, \dots, N_t be defined by

$$D(z)\Pi(z^{-1}) = N_1z + N_2z^2 + \dots + N_tz^t, \quad (38)$$

where t is the largest observability index. Then

$$A_*(z) = I_\ell + A_1z + A_2z^2 + \dots + A_tz^t,$$

where $A_k = N_k A$. For later use we observe that

$$N = \begin{bmatrix} N_1 \\ \vdots \\ N_t \end{bmatrix} \in \mathbb{R}^{\ell t \times \ell n}, \quad N_k = \begin{bmatrix} e_{t_1}^k & & & \\ & e_{t_2}^k & & \\ & & \ddots & \\ & & & e_{t_\ell}^k \end{bmatrix} \quad (39)$$

where e_j^k is a $1 \times j$ row vector with the k :th element 1 and the others 0 whenever $k \leq j$, and a zero row vector of dimension $1 \times j$ when $k > j$. We also define the $\ell(n+1) \times \ell t$ matrix

$$V := [Ze \otimes I_\ell \quad \dots \quad (Z^t e) \otimes I_\ell]. \quad (40)$$

Next we want to derive the matrix version of the modified Riccati equation (30) by using the interpolation condition (37) to eliminate G in (24). To this end, we define

$$Q := \Gamma P H' + \Sigma \quad (41)$$

in terms of which we have $Q = G + A$ from (26). We refer to the longer version [35] of the present paper for the proof of the following lemma.

Lemma 4: Let $T := (W - \frac{1}{2}I)(W + \frac{1}{2}I)^{-1}$. Then the interpolation condition (37) is equivalent to

$$\begin{aligned} & Z \otimes G_1 + Z^2 \otimes G_2 + \dots + Z^t \otimes G_t \\ &= (I_{\ell(n+1)} + Z \otimes Q_1 + Z^2 \otimes Q_2 + \dots + Z^t \otimes Q_t)T. \end{aligned}$$

In particular, setting $\hat{T} := T(e \otimes I_\ell)$, we have

$$VNG = \hat{T} + (Z \otimes N_1 Q + \dots + Z^t \otimes N_t Q)\hat{T}. \quad (42)$$

Now, VN is an $\ell(n+1) \times \ell n$ matrix in which the top ℓ rows are zero, since $z_0 = 0$. Therefore it takes the form

$$VN = \begin{bmatrix} 0_{\ell \times \ell n} \\ L \end{bmatrix}. \quad (43)$$

For the moment assuming that the square matrix L is nonsingular – we shall later see that this is not always true – VN has a pseudo-inverse $(VN)^\dagger$, and hence (42) yields

$$G = (VN)^\dagger \hat{T} + (VN)^\dagger (Z \otimes N_1 Q + \dots + Z^t \otimes N_t Q)\hat{T}, \quad (44)$$

and therefore, by (41),

$$G = u + U(\Gamma P H' + \Sigma), \quad (45)$$

where $u := (VN)^\dagger \hat{T}$ and $U : \mathbb{R}^{\ell n \times \ell} \rightarrow \mathbb{R}^{\ell n \times \ell}$ is the linear operator

$$Q \mapsto (VN)^\dagger (Z \otimes N_1 Q + \dots + Z^t \otimes N_t Q)\hat{T}.$$

Inserting (45) into (27) we obtain the modified Riccati equation

$$\begin{aligned} P &= \Gamma(P - P H' H P)\Gamma' \\ &+ (u + U\Sigma + U\Gamma P H')(u + U\Sigma + U\Gamma P H')'. \end{aligned} \quad (46)$$

It was first introduced in [6] for the scalar case $\ell = 1$ and for the special case of covariance extension. Therefore it has been called the Covariance Extension Equation (CEE).

V. MAIN RESULTS

Next we generalize the results of Section III to the general multivariable problem, which is considerably more difficult. Therefore several key questions will be left unanswered at this time. Nevertheless the theory in its present (preliminary) form does give an workable algorithm for large classes of problems. To save space we refer the reader to the longer version [35] of this paper for all the proofs.

A. Basic results

Now redefine \mathcal{S}_n to be the class of $\ell \times \ell$ matrix polynomials (15) such that $\det A(z)$ has all its zeros in the open unit disc \mathbb{D} . Clearly \mathcal{S}_n consists of subclasses with different Jordan structure J defined via (13). In each such subclass $D(z)$ and $\Pi(z)$ in (15), as well as N_1, N_2, \dots, N_t in (38), are the same. Let \mathcal{W}_+ be the values in (2) that satisfy the generalized Pick condition (11).

Lemma 5: Let the $\ell n \times \ell n$ matrix L be defined by (43). Then L is nonsingular if and only if all observability indices are the same, i.e., $t_1 = t_2 = \dots = t_\ell = n$.

In the present matrix case, the relation (32) reads

$$A(z)B(z^{-1})' + B(z)A(z^{-1})' = 2\Sigma(z)R R' \Sigma(z^{-1})'. \quad (47)$$

Let \mathcal{P}_n be the space of pairs $(A, B) \in \mathcal{S}_n \times \mathcal{S}_n$ such that $A(z)^{-1}B(z)$ is positive real. Then the problem at hand is to find, for each $\Sigma \in \mathcal{S}_n$, a pair $(A, B) \in \mathcal{P}_n$ such that (47) and (2) hold.

Theorem 6: Given $(\Sigma, W) \in \mathcal{S}_n \times \mathcal{W}_+$, where $\Sigma(z)$ has all its observability indices equal. Then to any positive definite solution P of the Covariance Extension Equation (46) such that $H P H' < I$, there corresponds a unique analytic interpolant (36), where $A(z)$ and $B(z)$ have the same Jordan structure as $\Sigma(z)$, the matrices A and B are given by

$$\begin{aligned} A &= (I - U)(\Gamma P H' + \Sigma) - u \\ B &= (I + U)(\Gamma P H' + \Sigma) + u \end{aligned} \quad (48)$$

and $A(z)$ and $B(z)$ satisfy (47) with

$$R = (I - H P H')^{\frac{1}{2}}. \quad (49)$$

Finally,

$$\deg F(z) = \text{rank } P. \quad (50)$$

Let us stress that these results are considerably weaker than the corresponding theorem for the scalar case reviewed in Section III. In fact, Theorem 6 does not guarantee that there exists a unique solution to (46). In fact, if there were

two solutions to (46), there would be two interpolants, a unique one for each solution P . Moreover, the condition on the observability indices restricts the classes of Jordan structures that are feasible.

However, Theorem 6 can be combined with other partial results on existence and uniqueness. There are multivariable problems for which we already know that there is a unique solution to the interpolation problem, and then existence and uniqueness of a solution to (46) will follow. A case in point is when $\Sigma(z) = \sigma(z)I$, where $\sigma(z)$ is a stable scalar polynomial [1], [25], in which case the observability indices are all equal, as required in Theorem 6. In this case the analytic interpolation problem will have a unique solution, and thus, tracing the calculations in Section II backwards, so will (46). The same is true when $\Sigma(z) = \sigma(z)C$, where C is full rank [25].

On the other hand, in recent years there have been a number of results [25], [26], [27], [29], [30] on the question of existence and uniqueness of the multivariate analytic interpolation problem, mostly for the covariance extension problem ($m = 0, n_0 = n + 1$), but there are so far only partial results and for special structures of the prior (in our case $\Sigma(z)$). Especially the question of uniqueness has proven elusive. Perhaps, as suggested in [27], this is due to the Jordan structure, and this could be the reason for the condition on the observability indices required in Theorem 6. In any case, as long as our algorithm delivers a solution to the Covariance Extension Equation, we will have a solution to the analytic interpolation problem, unique or not. An advantage of our method is that (50) can be used for model reduction, as will be illustrated in Section VI-C.

B. Solving CEE by homotopy continuation

We shall provide an algorithm for solving (46) based on homotopy continuation. We assume from now on that $t := t_1 = t_2 = \dots, t_\ell = n$. Whenever this algorithm delivers a solution P , the interpolant is obtained via (48).

When $u = 0, \hat{T} = 0$, and hence $U = 0$. Then the modified Riccati equation (46) becomes $P = \Gamma(P - PH'HP)\Gamma'$, which has the solution $P = 0$. We would like to make a continuous deformation of u to go from this trivial solution to the solution of (46), so we choose $u(\lambda) = \lambda u$ with $\lambda \in [0, 1]$. The corresponding deformation of U is λU , and T is deformed to λT . Since $T = (W - \frac{1}{2}I)(W + \frac{1}{2}I)^{-1}$, we have $W = (I - T)^{-1} - \frac{1}{2}I$, and therefore the value matrix (5) will then vary as

$$W(\lambda) = (I - \lambda T)^{-1} - \frac{1}{2}I.$$

It can be shown [35] that $W(\lambda) \in \mathcal{W}_+$ for all $\lambda \in [0, 1]$ whenever $W \in \mathcal{W}_+$. Consequently, $W(\lambda)$ satisfies generalized Pick condition (11) along the whole trajectory.

Now, note that equation (47) can be written as

$$\begin{aligned} S(A)M(B) + S(B)M(A) \\ = 2S(\Sigma)(I_{n+1} \otimes RR')M(\Sigma) \end{aligned} \quad (51)$$

where

$$S(A) = \begin{bmatrix} I & A_1 & \cdots & A_n \\ & I & \cdots & A_{n-1} \\ & & \ddots & \vdots \\ & & & I \end{bmatrix} \quad M(A) = \begin{bmatrix} I \\ A'_1 \\ \vdots \\ A'_n \end{bmatrix}.$$

Moreover it can be shown [35] that

$$A_n + B_n = 2N_n \Sigma R R'. \quad (52)$$

Applying (52) and deleting the zero row in (51), it can be reduced to $n\ell \times \ell$ equations

$$\begin{aligned} [I_{n\ell} \quad 0_{n\ell \times \ell}] (S(A)M(B) + S(B)M(A)) \\ = 2 [I_{n\ell} \quad 0_{n\ell \times \ell}] S(\Sigma)(I_{n+1} \otimes R R')M(\Sigma) \end{aligned}$$

Therefore, introducing the $n\ell \times \ell$ matrix

$$p = PH', \quad (53)$$

we use the homotopy

$$\begin{aligned} \mathcal{H}(p, \lambda) := [I_{n\ell} \quad 0_{n\ell \times \ell}] (S(A)M(B) + S(B)M(A) \\ - 2S(\Sigma)(I_{n+1} \otimes (I - Hp))M(\Sigma)) = 0, \end{aligned} \quad (54)$$

where

$$\begin{aligned} A = A(p, \lambda) &:= \Gamma p + \Sigma - \lambda u - \lambda U(\Gamma p + \Sigma) \\ B = B(p, \lambda) &:= \Gamma p + \Sigma + \lambda u + \lambda U(\Gamma p + \Sigma) \end{aligned} \quad (55)$$

depend on (p, λ) . Then the problem reduces to solving the differential equation

$$\begin{aligned} \frac{d}{d\lambda} \text{vec}(p(\lambda)) &= \left[\frac{\partial \text{vec}(\mathcal{H}(p, \lambda))}{\partial \text{vec}(p)} \right]^{-1} \frac{\partial \text{vec}(\mathcal{H}(p, \lambda))}{\partial \lambda} \\ \text{vec}(p(0)) &= 0 \end{aligned} \quad (56)$$

[36], which has the solution $\hat{p}(\lambda)$ for $0 \leq \lambda \leq 1$. The solution of (46) is then obtained by finding the unique solution of the Lyapunov equation

$$\begin{aligned} P - \Gamma P \Gamma' &= -\Gamma \hat{p}(1) \hat{p}(1)' \Gamma' \\ &+ (u + U \Gamma \hat{p}(1) + U \Sigma)(u + U \Gamma \hat{p}(1) + U \Sigma)'. \end{aligned} \quad (57)$$

VI. SOME SIMPLE ILLUSTRATIVE EXAMPLES

To illustrate our theory we provide a few simulations.

A. Example 1

We consider a problem with the interpolation constraints

$$\begin{aligned} F(0) &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ F(0.5) &= \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix} \quad F^{(1)}(0.5) = \begin{bmatrix} 2 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \end{aligned}$$

where $n = 2, \ell = 2$. This yields

$$Z = \begin{bmatrix} 0 & & \\ & 0.5 & \\ & & 1 \end{bmatrix} \quad e = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad (58)$$

Taking $\Sigma(z)$ of the form

$$\Sigma(z) = \begin{bmatrix} z^2 & 0 \\ 0 & z^2 \end{bmatrix} + \Pi(z)\Sigma \quad (59)$$

we have $t_1 = t_2 = 2$, and thus

$$VN = \begin{bmatrix} 0_{2 \times 4} \\ L \end{bmatrix}, \quad L = \begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

where clearly L is nonsingular as required. Then choosing

$$\Sigma = \begin{bmatrix} 1 & 0.3 \\ 0.2 & 0.3 \\ 0.1 & 0.4 \\ 0.7 & 0.2 \end{bmatrix}, \quad (60)$$

the matrix polynomial (59) is stable, and we can use our algorithm to obtain

$$\begin{aligned} A(z) &= \begin{bmatrix} z^2 & 0 \\ 0 & z^2 \end{bmatrix} + \Pi(z)A \\ B(z) &= \begin{bmatrix} z^2 & 0 \\ 0 & z^2 \end{bmatrix} + \Pi(z)B, \end{aligned} \quad (61)$$

where

$$A = \begin{bmatrix} 0.9467 & -0.1737 \\ 0.3603 & 0.3583 \\ -0.0445 & 1.0925 \\ 0.2147 & 0.7364 \end{bmatrix} \quad B = \begin{bmatrix} -0.0533 & 0.2263 \\ -0.3517 & -0.2893 \\ -0.2445 & 0.0925 \\ 0.2406 & -0.9739 \end{bmatrix}$$

If instead we choose $\Sigma(z)$ of the form

$$\Sigma(z) = \begin{bmatrix} z^3 & 0 \\ 0 & z \end{bmatrix} + \Pi(z)\Sigma,$$

then $t = t_1 = 3, t_2 = 1$, and thus

$$VN = \begin{bmatrix} 0_{2 \times 4} \\ L \end{bmatrix}, \quad L = \begin{bmatrix} 0.5 & 0.25 & 0.125 & 0 \\ 0 & 0 & 0 & 0.5 \\ 1 & 1 & 0.75 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where L is singular as anticipated by Lemma 5. Hence the algorithm cannot be used.

B. Example 2

Next consider the multivariable covariance extension problem

$$F(0) = \frac{1}{2}I_2, \quad F^{(1)}(0) = C_1, \quad F^{(2)}(0) = C_2,$$

where

$$C_1 = \begin{bmatrix} -0.5 & 0.2 \\ -0.1 & -0.5 \end{bmatrix}, \quad C_2 = 2 \begin{bmatrix} 0.1 & -0.6 \\ 0.1 & -0.3 \end{bmatrix}.$$

In this case

$$Z = \begin{bmatrix} 0 \\ 1 & 0 \\ & 1 & 0 \end{bmatrix} \quad e = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

By Lemma 5, we need to choose $t_1 = t_2 = 2$ and this yields

$$VN = \begin{bmatrix} 0_{2 \times 4} \\ L \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where L is nonsingular. Choosing the $\Sigma(z)$ in (60), our algorithm delivers the solution (61) with

$$A = \begin{bmatrix} 0.9467 & -0.1737 \\ 0.3603 & 0.3583 \\ -0.0445 & 1.0925 \\ 0.2147 & 0.7364 \end{bmatrix} \quad B = \begin{bmatrix} -0.0533 & 0.2263 \\ -0.3517 & -0.2893 \\ -0.2445 & 0.0925 \\ 0.2406 & -0.9739 \end{bmatrix}.$$

Again the choice $t = t_1 = 3, t_2 = 1$ of observability indices does not work (Lemma 5).

C. Model reduction

Consider a system with a 2×2 transfer function

$$V(z) = A(z)^{-1}\Sigma(z) \quad (62)$$

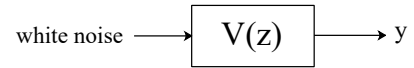
of dimension 10 and with observability indices $t_1 = t_2 = 5$, where $A(z)$ and $\Sigma(z)$ are given by (15) respectively (20) with

$$A = \begin{bmatrix} -0.11 & -0.02 \\ -0.08 & -0.15 \\ 0.05 & 0.10 \\ -0.05 & -0.09 \\ -0.13 & -0.09 \\ 0.11 & 0.07 \\ 0.09 & 0.19 \\ -0.03 & -0.03 \\ -0.10 & -0.13 \\ 0.12 & 0.05 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.1500 & 0 \\ -0.6900 & 0 \\ 0.1025 & 0 \\ 0.0306 & 0 \\ -0.0034 & 0 \\ 0 & 0.1500 \\ 0 & -0.6900 \\ 0 & 0.1025 \\ 0 & 0.0306 \\ 0 & -0.0034 \end{bmatrix}.$$

Here $\Sigma(z) = \sigma(z)I_2$, where

$$\sigma(z) = (z - 0.1)(z - 0.3)(z - 0.6)(z + 0.2)(z + 0.95).$$

We pass (normalized) white noise through the system



to obtain the output y_0, y_2, \dots, y_N , and from this output data we estimate the 2×2 matrix valued covariance sequence

$$\hat{C}_k = \frac{1}{N - k + 1} \sum_{t=k}^N y_t y_{t-k}' \quad (63)$$

Then we solve the problem (2) with $\ell = 2, m = 0, n_0 = 6$, and $W_{0k} = \hat{C}_k$ for $k = 0, 2, \dots, 5$. This is a matrix-valued covariance extension problem. The modified Riccati equation (46) has a solution P with eigenvalues

$$1.5 \times 10^{-6}, 2.7 \times 10^{-5}, 0.0007, 0.0041,$$

$$0.0104, 0.0338, 0.1993, 0.3457, 0.6535, 0.7138.$$

The first four eigenvalues are very small, so we can reduce the degree of this system from ten to six by choosing the first four covariance lags $\hat{C}_0, \hat{C}_1, \hat{C}_2, \hat{C}_3$ and removing two zeros of $\Sigma(z)$. Since the reduced matrix A will be 6×2 , we can only match 12 covariance elements. Consequently we can only match \hat{C}_j up to $j = 3$. We choose to remove the zeros at -0.2 and 0.6 . The reduced-order system will have observability indices $t_1 = t_2 = 3$. The singular values of the true system (62) are shown in the multivariable Bode-type plot of Fig. 1 together with those of the estimated systems of degree ten and six, respectively. As can be seen, the reduced-order system is a good approximation.

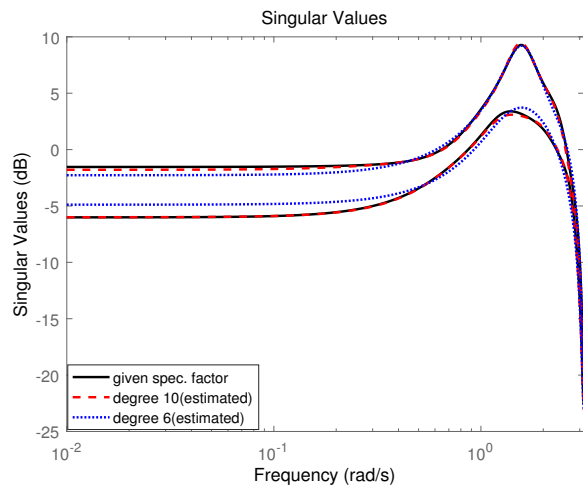


Fig. 1: Estimated singular values and the true ones

VII. CONCLUDING REMARKS

We have extended our previous results [31] for the scalar case to the matrix case. However, multivariable versions of analytic interpolation with rationality constraints have been marred by difficulties to establish existence and, in particular, uniqueness in the various parameterizations [3], [1], [23], [24], [25], [26], [27], [29], [30], and we have encountered similar difficulties here. Our approach attacks these problems from a different angle and might put new light on these challenges. Therefore future research efforts will be directed towards settling these intriguing open questions in the context of the modified Riccati equation (46).

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