A Global Analysis Approach to Passivity Preserving Model Reduction

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***Abstract*— Passivity-preserving model reduction for linear time-invariant systems amounts to approximating a positive-real rational transfer function with one of lower degree. Recently Antoulas and Sorensen have proposed such a model-reduction method based on Krylov projections. The method is based on an observation by Antoulas (in the single-input/single-output case) that if the approximant is preserving a subset of the spectral zeros and takes the same values as the original transfer function in the mirror points of the preserved spectral zeros, then the approximant is also positive real. However, this turns out to be a special solution in the theory of analytic interpolation with degree constraint developed by Byrnes, Georgiou and Lindquist, namely the maximum-entropy (central) solution. By tuning the interpolation points and the spectral zeros, as prescribed by this theory, one is able to obtain considerably better reduced-order models.**

I. Introduction

Consider a time-invariant, continuous-time linear system

$$\xrightarrow{u} \boxed{G(s)} \xrightarrow{y}$$

with a real transfer function

$$G(s) = C(sI-A)^{-1}B + D \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{1}$$

of McMillan degree n such that all eigenvalues of A lie in the open left half of the complex plane, $\mathbb{C}_-$, (A, B) is reachable, (C, A) is observable and $D + D'$ is positive definite. Moreover suppose that the dimension of the input u equals the dimension m of the output y and that the system is *passive* in the sense that

$$\int_0^T u(t)'y(t)dt \geq 0$$

for all $T > 0$ and all square-integrable inputs u. In physical terms, such a system produces no energy internally. Passive systems are important in many applications, such as, for example, in VLSI design and stochastic systems theory.

To say that the system is passive is to say that the $m \times m$ transfer function $G(s)$ is *positive real*; i.e.,

$$G(i\omega) + G(-i\omega)' \geq 0, \quad \omega \in \mathbb{R}. \tag{2}$$

Then $G(s)+G(-s)'$ can be interpreted as a spectral density, and there are rational functions $W(s)$, called *spectral factors*, such that

$$G(s) + G(-s)' = W(s)W(-s)'. \tag{3}$$

Therefore, the zeros of $G(s)+G(-s)'$ are called the *spectral zeros* of $G(s)$. Since

$$G(s) + G(-s)' \sim \left[\begin{array}{cc|c} A & & B \\ & -A' & -C' \\ \hline C & B' & D+D' \end{array}\right],$$

the spectral zeros are precisely the λ for which the matrix $\mathcal{A} - \lambda\mathcal{E}$ is singular, where

$$\mathcal{A} = \begin{bmatrix} A & & B \\ & -A' & -C' \\ C & B' & D+D' \end{bmatrix}, \; \mathcal{E} = \begin{bmatrix} I & & \\ & I & \\ & & 0 \end{bmatrix}. \tag{4}$$

Consequently, the spectral zeros are the generalized eigenvalues of $(\mathcal{A}, \mathcal{E})$.

The problem considered in this paper is to find a passive reduced-order system that approximates the original system and whose transfer function

$$\hat{G}(s) = \hat{C}(sI-\hat{A})^{-1}\hat{B} + D \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & D \end{array}\right] \tag{5}$$

has a lower degree $k < n$ but retains the positive-real property.

Such model reduction is often performed by some projection method that determines matrices $U, V \in \mathbb{R}^{n \times k}$ such that $U'V = I_k$ and

$$\hat{A} = U'AV, \quad \hat{B} = U'B, \quad \hat{C} = CV. \tag{6}$$

The most popular such model reductition methods preserving positive-realness is *stochastically balanced truncation* (or *positive-real balanced truncation*), originally proposed by Desai and Pal [13] in the context of stochastic realization theory. Stochastically balanced model reduction has the advantage that it comes with easily computed bounds; see, e.g., [22].

In this paper, we shall consider another class of model reduction procedures based on interpolation, in which the transfer function $\hat{G}$ of the reduced-order system satisfies the interpolation conditions

$$\hat{G}(s_j) = G(s_j), \quad j = 1, 2, \ldots, k, \tag{7}$$

for some suitable points $s_1, s_2, \ldots, s_k$ in the open right half $\mathbb{C}_+$ of the complex plane. In the scalar case, $m = 1$, Antoulas [1] has recently observed that, if the interpolation points $s_1, s_2, \ldots, s_k$ are mirror images of stable spectral zeros of G, then $\hat{G}$ is positive real. Sorensen [28] has developed an efficient algorithm based on Antoulas' idea [1] that does not

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explicitly use spectral zeros but also works in the case $m > 1$. In [15] we demonstrated that Sorensen's solution amounts to tangential interpolation rather than matricial interpolation involving the condition (7).

However, Antoulas' observation does not come as great surprise to us, since the concept of spectral zeros is a key ingredient in a theory of analytic interpolation developed over the last decades by Byrnes, Georgiou, Lindquist and their coworkers [3]–[12],[14]–[21],[23],[24],[26]. Indeed, given $k+1$ interpolation points and corresponding interpolation values, the class of all analytic interpolants of McMillan degree at most k is completely parameterized by the stable spectral zeros. Moreover, given a specific choice of such spectral zeros, there is a pair of dual convex optimization problems determining the unique corresponding interpolant. We shall demonstrate that Antoulas' solution is essentially the central solution or the maximum entropy solution in this theory. This opens up the questions of whether the full power of the theory of analytic interpolation with degree constraint can be used to obtain better approximations. We shall provide numerical examples showing that this is indeed the case. This is a *global analysis approach*, in which one considers a complete class of solutions as a whole rather than a particular solution, and in which one designs smooth tuning strategies to approximate to specifications.

For simplicity of presentation, from now on, we make the same assumption as in [1], [28], namely that *the spectral zeros are distinct*.

II. The Antoulas-Sorensen approach

The starting point in Sorensen's algorithm is a partial real Schur decomposition

$$\mathcal{A}Q = \mathcal{E}QR \tag{8}$$

for the pair $(\mathcal{A},\mathcal{E})$, where $Q'Q = I_k$ and R is real and quasi-upper triangular. Clearly, the eigenvalues of R are generalized eigenvalues of $(\mathcal{A},\mathcal{E})$; i.e., (selected) spectral zeros. Setting $Q' = (X', Y', Z')$, we have

$$\begin{bmatrix} A & & B \\ & -A' & -C' \\ C & B' & D+D' \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ 0 \end{bmatrix} R. \tag{9}$$

Here we take the decomposition corresponding to k selected spectral zeros in $\mathbb{C}_-$. It can be shown that, in this case, X has full rank if (A,B) is reachable, Y has full rank if (C,A) is observable, and $X'Y = Y'X$; see [28, Lemmas 1 and 2] and also [15, Lemma 1] for a more general statement.

Given this partial real Schur decomposition, in Sorensen's algorithm one performs singular value decomposition on $X'Y$. More precisely, this amounts to determining unitary $k \times k$ matrices Q_x and Q_y such that $Q_x\Sigma^2 Q_y' = X'Y$ is the singular value decomposition of $X'Y$, and setting

$$V := XQ_x\Sigma^{-1}, \quad U := YQ_y\Sigma^{-1}. \tag{10}$$

In [28] Sorensen proves, using the Positive Real Lemma, that the reduced-order transfer function $\hat{G}$ obtained by taking V, U defined by (10) in (6) is positve real.

In [15] we demonstrated that $\hat{G}$ satisfies the right tangential interpolation conditions

$$\hat{G}(s_j)z_j = G(s_j)z_j, \quad j = 1,\dots,k, \tag{11}$$

where $z_j := Zr_j \neq 0$ for $k = 1,2,\dots,k$, and r_j is the right eigenvector of R corresponding to the eigenvalue s_j. In addition, $\hat{G}$ satisfies the left tangential interpolation condition

$$z_j'\hat{G}(-s_j) = z_j'G(-s_j) \tag{12}$$

for each $j = 1,2,\dots,k$ such that $(-s_jI_k - \hat{A})$ is invertible. In particular, if (5) is a minimal realization, (12) holds for all $j = 1,2,\dots,k$ [15].

This is only partly in harmony with Antoulas' result [1]. In fact, in the scalar case, the Antoulas-Sorensen reduced-order transfer function $\hat{G}$ interpolates in the *unstable* spectral zeros. If the reduced-order realization (5) is minimal, it also interpolates in the stable spectral zeros. The minimality of (5) is important, so we pause to consider some consequences of this.

A basic question, raised by Antoulas in [1], is when a rational function G satisfying both the interpolation conditions

$$G(s_j) = w_j, \quad j = 1,2,\dots,k$$

and the corresponding "mirror-image" interpolation conditions

$$G(-\bar{s}_j) = -\bar{w}_j, \quad j = 1,2,\dots,k$$

is positive real. In [1] it is claimed that all minimum-degree interpolants are positive real [1, Lemma 3.1]. This is not correct. A simple first-order counterexample is obtained by taking $(s_1,w_1) = (1,1)$. A function satisfying both $G(1) = 1$ and $G(-1) = -1$ cannot be of degree zero, so the claim in [1, Lemma 3.1] implies that any degree-one function satisfying both $G(1) = 1$ and $G(-1) = -1$ is positive real. A counterexample is $G(s) = (1-2s)/(s-2)$, which is not even analytic in $\mathbb{C}_+$, let alone positive real.

There could be a mistake in transferring the statement of Lemma 3.1 in [1] from that in Theorem 4.2 in the previous paper [2], co-authored by the same author, where there is one less mirror-image interpolation condition. This is more natural, since, in general, $2k-1$ linear equations are required to determine a rational function of degree $k-1$. Transferred into the setting of positive real functions in the right half plane, Theorem 4.2 in [2] implies that, given (s_j, w_j) for $j = 1,\dots,k$ such that the Pick matrix

$$\tilde{P} := \left[\frac{w_i + \bar{w}_j}{s_i + \bar{s}_j}\right]_{i,j=1}^{k} \tag{13}$$

is positive definite, there exists a unique rational function f of degree less or equal to $k-1$ such that $f(s_j) = w_j$, $j = 1,\dots,k$ and $f(-\bar{s}_j) = -\bar{w}_j$, $j = 1,\dots,k-1$, and that this rational function is positive real.

It is true that there exists a positive real function f of degree $\le k-1$ which satisfies $f(s_j) = w_j$, $j = 1,\dots,k$. If $\tilde{P}$ is positive definite, all such solutions are parameterized by Theorem 1 in Section III. However, there does not necessarily exist an interpolant of degree at most $k-1$

which also satisfies the mirror interpolation conditions. In fact, the following is a simple counterexample. If $w_j = 1$ for $j = 1, \dots, k$, there is a unique function of degree at most $k-1$ satisfying $G(s_j) = w_j$ for $j = 1, 2, \dots, k$, namely $G \equiv 1$. However, this function does not satisfy the mirror conditions $G(-\bar{s}_j) = -1$, $j = 1, 2, \dots, k-1$. The following numerical example further elucidates this point.

Example 1: Consider the second-order positive real transfer function

$$G(s) = \frac{6s^2 + 22s + 9}{6s^2 + 15s + 16}, \tag{14}$$

for which $s = \pm 1$ and $s = \pm 2$ are the spectral zeros.

First, we computing the first-order transfer function $\hat{G}$ with the stable spectral zero at $s_1 = -2$, we obtain

$$(\hat{A}, \hat{B}, \hat{C}, D) = (-1.8182, -2.8316, -0.1348, 1),$$

which clearly is minimal. The reduced-degree function $\hat{G}$ is positive real, and both the interpolation conditions $\hat{G}(-2) = G(-2)$ and $\hat{G}(2) = G(2)$ hold.

Next, we computing the first-order transfer function $\hat{G}$ with the stable spectral zero at $s_1 = -1$, we have

$$(\hat{A}, \hat{B}, \hat{C}, D) = (-1, -2, 0, 1),$$

which is not minimal. The reduced-degree transfer function is $\hat{G} \equiv 1$, which is clearly positive real, and satisfies $\hat{G}(1) = G(1) = 1$, but *not* $\hat{G}(-1) = G(-1)$.

III. Analytic Interpolation with Degree Constraint

The reason why the particular choice of interpolation points in the Antoulas-Sorensen solution lead to a positive real reduced-order model is no coincidence. In fact, in the next section we shall demonstrate that this can be interpreted in the context of the theory of analytic interpolation with degree constraint developed by Byrnes, Georgiou and Lindquist.

To this end, we now restate some basic results from this theory in the continuous-time setting. For consistency with the setting in [1] we confine the initial analysis real, scalar interpolants, although, strictly speaking, this is not necessary. Given a set of self-conjugate pairs of complex numbers

$$\{(s_j, w_j) : s_j \in \mathbb{C}_+\}_{j=0}^{k}, \quad \begin{array}{l} s_i \neq s_j \text{ if } i \neq j,\ s_0 \text{ real}, \\ w_i = \bar{w}_j \text{ if } s_i = \bar{s}_j, \end{array} \tag{15}$$

find all functions f with real coefficients that satisfy the following three conditions:

1) *Positive real property*: the function f is analytic in $\mathbb{C}_+$, and
$$\mathrm{Re} f(s) \geq 0, \ \forall s \in \mathbb{C}_+. \tag{16}$$
2) *Interpolation conditions*:
$$f(s_j) = w_j, \quad j = 0, 1, \dots, k. \tag{17}$$
3) *Degree constraint*: f is real rational and
$$\deg f \leq k. \tag{18}$$

A necessary and sufficient condition for the existence of f satisfying these three conditions is the positive semidefiniteness of the Pick matrix

$$P := \left[\frac{w_i + \bar{w}_j}{s_i + \bar{s}_j} \right]_{i,j=0}^{k}. \tag{19}$$

Theorem 1: Suppose that the Pick matrix (19) constructed from the interpolation data (15) is positive definite. Let $\{\lambda_j\}_{j=1}^k \subset \mathbb{C}_-$ be an arbitrary self-conjugate, and define $\sigma(s) := \prod_{j=1}^k (s - \lambda_j)$. Then, there exists a unique (modulo sign) pair of real Hurwitz polynomials (α, β) of degree k such that

(i) $f := \beta/\alpha$ is positive real,
(ii) $f(s_j) = w_j$, $j = 0, 1, \dots, k$, and
(iii) $\alpha(s)\beta(-s) + \alpha(-s)\beta(s) = \sigma(s)\sigma(-s)$.

Conversely, any pair of real polynomials (α, β) of degree k satisfying (i) and *(ii)* determines, via *(iii)*, a unique (modulo sign) Hurwitz polynomial σ of degree k, and the map $\sigma \mapsto (\alpha, \beta)$ is a diffeomorphism. Moreover, setting

$$\Psi(i\omega) := \left| \frac{\sigma(i\omega)}{\tau(i\omega)} \right|^2, \quad \text{where } \tau(s) := \prod_{j=1}^{k} (s + s_j), \tag{20}$$

the problem to maximize

$$\mathbb{I}_\Psi(f) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(i\omega) \log[f(i\omega) + f(-i\omega)] \frac{d\omega}{\omega^2 + s_0^2}, \tag{21}$$

over all positive real functions f satisfying (17), has a unique solution that is precisely the unique f satisfying the conditions (i), (ii) and (iii). Finally, if (α, β) is the corresponding pair of polynomials,

$$Q(i\omega) := |a(i\omega)|^2, \quad \text{where } a(s) = \frac{\alpha(s)}{\tau(s)}, \tag{22}$$

is the unique solution to the convex optimization problem to minimize

$$\begin{aligned} \mathbb{J}_\Psi(Q) \quad := \quad & \frac{1}{2\pi} \int_{-\infty}^{\infty} \{ [w(i\omega) + w(-i\omega)] Q(i\omega) \\ & - \Psi(i\omega) \log Q(i\omega) \} \frac{d\omega}{\omega^2 + s_0^2}, \end{aligned} \tag{23}$$

over all Q in the class $\mathcal{Q}$ of all rational functions of the form (22) with α free to vary over all stable polynomials of degree at most k, and w is any proper, stable (not necessarily positive real) real, rational function satisfying the interpolation condition (ii).

The statements of the theorem have been proven in the discrete time setting in various places: the first part in [4], [19] (also, see [8], [6], and, as for existence only, the early work [16], [17], [18]), the diffeomorphism result in [11], and the optimization results (in various versions) in [9], [7], [4], [10], [6]. Transferring this results to the continuous-time setting via the Möbius transformation

$$s \in \mathbb{C}_+ \mapsto z = \frac{s_0 - s}{s_0 + s} \in \mathbb{D} \tag{24}$$

is quite straight-forward.

This theorem yields a complete smooth parameterization of the whole class of positive real interpolants of degree at most n, where tuning can be done via the k spectral zeros. In particular, if we choose the k spectral zeros at the mirror images of the interpolation points, as suggested by Antoulas and Sorensen,

$$\lambda_j = -\bar{s}_j, \quad j = 1, \ldots, k, \tag{25}$$

then $\Psi \equiv 1$, and the interpolant maximizes the entropy gain

$$\mathbb{I}_1(f) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \log[f(i\omega) + f(-i\omega)] \frac{d\omega}{\omega^2 + s_0^2}. \tag{26}$$

This is the *central* or *maximum entropy solution*, the determination of which can be reduced to a system of linear equations; also see [25]. The proof of the following corollary can be found in [15].

Corollary 1: Let P be the Pick matrix (19), where s_0 is real, let $\tau(s)$ be the Hurwitz polynomial defined in (20), and set

$$\Pi(s) := \left[1, \frac{s+s_0}{s+s_1}, \frac{s+s_0}{s+s_2}, \ldots, \frac{s+s_0}{s+s_k}\right].$$

Moreover, suppose that $P > 0$. Then the maximum entropy solution is

$$f(s) = \frac{\Pi(s)b}{\Pi(s)a},$$

where $a := (a_0, a_1, \ldots, a_k)'$ is given by

$$a = \frac{1}{\sqrt{2s_0 \Pi(s_0) P^{-1} \Pi(s_0)^*}} P^{-1} \Pi(s_0)^*, \tag{27}$$

and $b := (b_0, b_1, \ldots, b_k)'$ is uniquely determined via the linear system of equations

$$a(s)b(-s) + a(-s)b(s) = 1 \tag{28}$$

with $a(s) := \Pi(s)a$ and $b(s) := \Pi(s)b$.

Theorem 1 can be generalized to tangential Nevanlinna-Pick interpolation, as established in [24]. Hence, the theory of analytic interpolation with degree constraint could also be applied in the multivarible case ($m > 1$).

IV. The Antoulas-Sorensen method as the maximum entropy solution

In the Antoulas-Sorensen approach, one interpolates not only at the unstable spectral zeros $s_1, s_2, \ldots, s_k$ but also at $s_0 := \infty$. More specifically,

$$\hat{G}(\infty) = D = G(\infty). \tag{29}$$

However, $s_0 := \infty$ lies on the boundary of the analyticity region $\mathbb{C}_+$ – a situation to which Corollary 1 and Theorem 1 do not immediately apply. However, choosing the interpolation point s_0 to be a positive number, sufficiently large for the Pick matrix (19) to be positive definite, determining the corresponding central solution, and then taking the limit as $s_0 \to \infty$, results precisely in the Antoulas-Sorensen solution. The proof of the following results (Lemma 1 and Theorem 2) can be found in [15].

Lemma 1: Let f_{s_0} be the maximum-entropy solution corresponding to the interpolation conditions (17). Moreover, define $D := w_0$;

$$\hat{C} := (\tilde{P}^{-1} w)', \tag{30}$$

where $w := (w_1 - w_0, w_2 - w_0, \ldots, w_k - w_0)'$ and $\tilde{P}$ is the reduced Pick matrix (13);

$$\hat{A} := -\Lambda + h\hat{C}, \tag{31}$$

where $\Lambda := \mathrm{diag}(s_1, s_2, \ldots, s_k)$ and $h := (1, 1, \ldots, 1)' \in \mathbb{R}^k$; and

$$\hat{B} := 2D(Q\hat{C}^* + h), \tag{32}$$

where Q is the unique solution of the Lyapunov equation

$$\hat{A}Q + Q\hat{A}^* + hh^* = 0. \tag{33}$$

Then, as $s_0 \to \infty$ while all other interpolation data is fixed, $f_{s_0}(s)$ tends to

$$f_{ME}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + D \tag{34}$$

pointwise except in the poles of f_{ME}. Finally, $\hat{A}$ has all its eigenvalues in the open left half plane.

Example 2: Consider a positive real function

$$G(s) = \frac{1/3s + 1}{(s+1)(s+2)} + 1$$

with stable spectral zeros $\lambda_1 = -\sqrt{3},\ \lambda_2 = -\sqrt{2}$. Applying the Antoulas-Sorensen method to approximate this system by a first-order passive system having a transfer function $\hat{G}$ with spectral zeros $\pm\sqrt{3}$ yields

$$\hat{G}(s) = \frac{2s+4}{2s+3}. \tag{35}$$

On the other hand, the maximum entropy solution f_{ME} with interpolation conditions $f_{ME}(\infty) = G(\infty), f_{ME}(\sqrt{3}) = G(\sqrt{3})$ can be determined as in Lemma 1. In fact, $w_0 = 1$, $w = (2\sqrt{3} - 3)/3$, $\tilde{P} = 2/3$, and $\Lambda = \sqrt{3}$, and hence $\hat{C} = \sqrt{3} - 3/2$ and $\hat{A} = -3/2$. Solving the Lyapunov equation (33), we have $Q = 1/3$, and hence $\hat{B} = 2\sqrt{3}/3 + 1$. Inserting this into (34), we have

$$f_{ME}(s) = \frac{2s+4}{2s+3}, \tag{36}$$

which is the same as (35).

Theorem 2: Let G be a scalar, positive real function with spectral zeros in $\{s_j\}_{j=1}^k$, and set $w_j := G(s_j)$ and $w_0 := D = G(\infty)$. Then the reduced order interpolant $\hat{G}$ constructed from these spectral zeros by the method of Antoulas-Sorensen equals the limit function f_{ME} defined in Lemma 1.

V. A large-scale numerical example: a CD player model

We have thus established that the reduced-order model computed by the Antoulas-Sorenson method coincides (in the limit) with the central (maximum-entropy) Nevanlinna-Pick interpolant with interpolation points in the mirror-image of the selected spectral zeros. However, the central solution is quite special, and one would expect to obtain a better approximant by tuning the solution by the global analysis approach described above.

We shall illustrate point by using a high-order model of a portable CD player taken from [29]. The full-order model, the transfer function of which is denoted by F, is a single-input single-output, continuous-time, linear time-invariant, and stable model of order 120 and of relative degree two. We aim at reducing the model order to $k = 12$, as well as maintaining the stability and the uniform upper bound $\rho = 80$ (≈ 38dB) of the gain of F.

The model F can be regarded as a function which is analytic in the right half-plane $\mathbb{C}_+$ and maps $\mathbb{C}_+$ into $\rho\mathbb{D}$. In other words, F is a bounded-real function in continuous-time. To be consistent with the problem setting in this paper, we will introduce a positive real function G in continuous-time by a bilinear transformation:

$$G(s) := \frac{\rho - F(s)}{\rho + F(s)}.$$

After computing the positive real reduced-order model $\hat{G}$ of G, we will obtain the bounded real reduced-order model $\hat{F}$ of F by the inverse transformation

$$\hat{F}(s) := \rho \frac{1 - \hat{G}(s)}{1 + \hat{G}(s)}.$$

We first use the algorithm suggested by Sorensen in [28]. This algorithm is based on the Implicitly Restart Arnoldi (IRA) method [27], and computes automatically the reduced-order system $\hat{G}$ without an explicit computation of all the spectral zeros of G. Following Sorenson, we select $k := 12$ spectral zeros of the system G by determining k eigenvalues of the matrix

$$\mathcal{C}_\mu := (\mu\mathcal{E} - \mathcal{A})^{-1}(\mu\mathcal{E} - \mathcal{A}), \tag{37}$$

where $\mu \geq 0$ has to be chosen properly, and $\mathcal{A}$ and $\mathcal{E}$ are defined by (4) in terms of a minimal realization of the 120th degree positive-real function G.

Using this algorithm, we have freedom in choosing μ and the criteria of selecting the eigenvalues of (37). In this example, we have chosen the 12 eigenvalues of largest magnitude and two different μ, namely $\mu = 260$ (see Fig. 1) and $\mu = 20$ (see Fig. 2). As one can see in these figures, the frequency responce of the reduced-order model matches the original model only in some frequency bands, due to the restriction of placing the interpolation points at the spectral zeros. Moreover, it crucially depends on the choice of μ and of the eigenvalues selection criteria. However, as pointed out in [28], these choices are not trivial.

Instead, applying the theory of Section III, we may choose the twelve spectral zeros and interpolation points arbitrarily. For comparison with the Antoulas-Sorenson method, which requires an interpolation condition $F(\infty) = 0$, we would like to impose the same interpolation condition on the reduced-order system $\hat{F}$. Since F has relative degree two, it can be

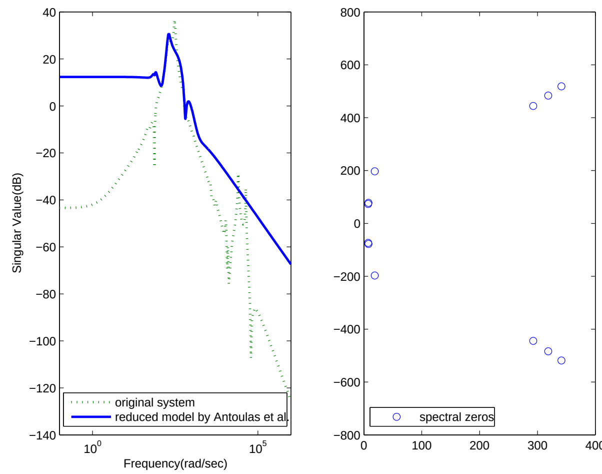

Fig. 1. Model reduction with $\mu = 260$.

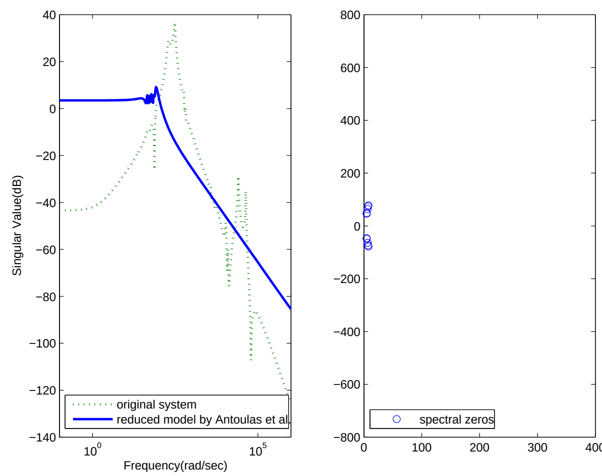

Fig. 2. Model reduction with $\mu = 20$.

factored as $F = F_1 F_2$, where F_2 is of relative degree two and F_1 is of degree 118. In this example

$$F_2 = \frac{1}{(s-p)(s-\bar{p})}, \quad p = -12.2708 + 306.5398i,$$

where p is a pole of F close to the frequency peak $\omega = 300$ rad/s. Hence we can restate the problem of reducing the order of F to $k = 12$ as the problem of reducing the order of F_1 to $t = 10$.

To reduce the order of F_1 with the methods of Section III, we need 11 interpolation conditions

$$\hat{F}_1(s_j) = \frac{F(s_j)}{F_2(s_j)}, \quad j = 0, \ldots 10,$$

which we choose at the points $s_0 = 199, s_{1,2} = 10^{-5} \pm 0.1i, s_{3,4} = 0.2 \pm i, s_{5,6} = 0.01 \pm 74i, s_{7,8} = 0.01 \pm 1.3250 \cdot 10^4 i, s_{9,10} = 0.005 \pm 9.9900 \cdot 10^4 i$; and an uniform upper bound on the gain, which we take to be $\rho_1 = 3.9811 \cdot 10^7 (\approx 140\,\text{dB})$. In the family of all such interpolants $\hat{F}_1$, we select the one that has spectral zeros at $\lambda_{1,2} = -0.0612 \pm 2.3749i, \lambda_3 = -175.0542, \lambda_4 = -351.5899, \lambda_{5,6} = -7.4080 \pm 70.4606i, \lambda_{7,8} = -570.1525 \pm 2.5448 \cdot$

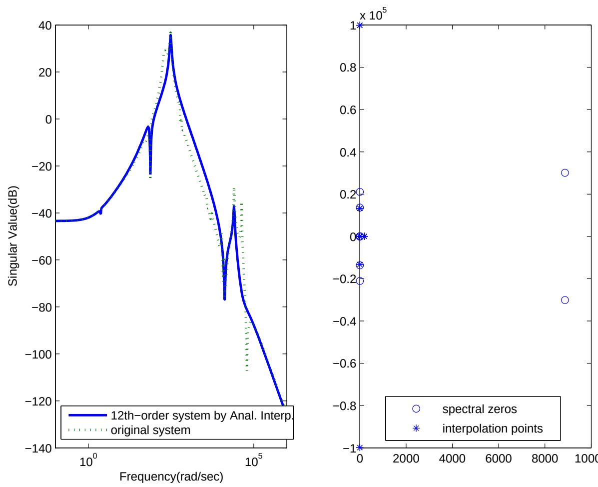

Fig. 3. Solution tuned by the global analysis approach.

$10^4 i, \lambda_{9,10} = -1.6436 \cdot 10^4 \pm 7.5527 \cdot 10^3 i$. In Fig. 3, the interpolation points and the spectral zeros of $\hat{F}$ are plotted together with the reduced-order system $\hat{F}$ and the original system F. One can see that $\hat{F}$ matches the peak around $\omega = 300$rad/s, overlaps the original system at high and low frequencies. Moreover, it matches the ripple around $\omega = 10^4$ rad/s.

VI. Conclusions

Over the last decades a global analysis approach for analytic interpolation has been developed. It is based on a complete smooth parameterization of all positive-real interpolants of degree less than the number interpolation points in terms of spectral zeros. In particular, if the spectral zeros are chosen in the mirror image of the interpolation points, the problem is linear. The corresponding solution is the central (maximum-entropy) solution.

We have demonstrated that the passivity-preserving model reduction method proposed by Antoulas and Sorensen can be identified with the central solution. By applying the global analysis approach, we have demonstrated that it is possible to obtain better approximants by choosing interpolation points that are placed more strategically; i.e., not restricted to the mirror image of the spectral zeros.

It should be noted, however, that the implementation of the central solution provided by Sorensen's algorithm, in which spectral zeros do not have to be determined explicitly, is numerically very efficient, and therefore, for very large problems, preferable even to stochastically balanced truncation, for which there are H^∞ bounds. Therefore, we propose that the high-order model first be reduced by the Sorenson algorithm, and then fine-tuned by moving the interpolation points and spectral zeros to improve the approximation.

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