The Inverse Problem of Analytic Interpolation With Degree Constraint and Weight Selection for Control Synthesis

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***Abstract*—The minimizers of certain weighted entropy functionals are the solutions to an analytic interpolation problem with a degree constraint, and all solutions to this interpolation problem arise in this way by a suitable choice of weights. Selecting appropriate weights is pertinent to feedback control synthesis, where interpolants represent closed-loop transfer functions. In this paper we consider the correspondence between weights and interpolants in order to systematize feedback control synthesis with a constraint on the degree. There are two basic issues that we address: we first characterize admissible shapes of minimizers by studying the corresponding inverse problem, and then we develop effective ways of shaping minimizers via suitable choices of weights. This leads to a new procedure for feedback control synthesis.**

***Index Terms*—Analytic interpolation, controller synthesis, degree constraint, loop shaping, model reduction, robust control, weight selection.**

I. Introduction

THE topic of this paper relates to the framework and the mathematics of modern robust control. The foundational work [45] of George Zames in the early 1980's casts the basic robust control problem as an analytic interpolation problem, where interpolation constraints ensure stability of the feedback scheme, and a norm bound guarantees performance and robustness. In this context, the analytic interpolant represents a particular transfer function of the feedback system. The work of Zames and the fact that the degree of the interpolant relates to the dimension of the closed-loop system motivated a program to investigate analytic interpolation with degree constraint (see [12], [13]). This led to an approach based on convex optimization, in which interpolants of a certain degree are obtained as minimizers of *weighted* entropy functionals. In this paper we study the correspondence between weights and such interpolants, and we develop a theory which allows for systematic shaping of interpolants to specification.

The basic issue of how the choice of weights and indices in optimization problems affects the final design is by no means new. It was R.E. Kalman [25] who, in the context of quadratic optimal control, first raised the question of what it is that characterizes optimal designs and, further, how to describe all performance criteria for which a certain design is optimal. Following Kalman's example we study here the analogous inverse problem for analytic interpolation with complexity constraint, namely the problem to decide when a particular interpolant is a minimizer of some weighted entropy functional, and if so, to determine the set of all admissible weights.

The analysis of the inverse problem leads to a new procedure for feedback control synthesis. More specifically, the quality of control depends on the frequency characteristics of the interpolant, which in turn dictates the loop shape of the feedback control system. The theory of [12], [13] provides a parametrization of all interpolants, having degree less than the number of interpolation constraints, in terms of weights in a suitable class. These admissible weights are specified by their roots. These roots coincide with the *spectral zeros* of the corresponding minimizers of the weighted entropy functionals [12]. The choice of weights for feedback control design via this procedure has been the subject of several papers (see, e.g., [34], [35]). The challenge stems from the fact that the correspondence between weights and the shape of interpolants is nonlinear. One of the contributions of this paper is to develop a systematic procedure for the selection of weights based on the inverse problem.

The synthesis proceeds in two steps. We first obtain an interpolant with the required shape, but without any restriction on the degree. Then, via the inverse problem, we identify all weights for which the given interpolant is a minimizer of the corresponding entropy functional. The problem of approximating the interpolant by one of lower degree is then replaced by approximating weights in a suitable class. This approximation problem is quasi-convex and can be solved by standard methods. Hence we have replaced a non-convex problem by one that is tractable.

In Section II we establish notation and review basic facts on bounded analytic interpolation and complexity-constrained interpolation. We only discuss interpolation in the unit disc $\mathbb{D} = \{z : |z| < 1\}$, but the theory applies equally well to interpolation in the half plane. In Section III we consider two motivating examples in the context of robust control. In Section IV

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we first provide the characterization of minimizers of weighted entropy functionals and describe the set of weights which give interpolants of a prespecified bounded degree. We then formulate and solve the inverse problem which is one of the key tools needed in the paper, and study the possible shapes of minimizers without any bound on their degree. In Section V we develop a method for degree reduction of interpolants via the corresponding weights and describe the steps of the synthesis procedure. Finally, in Section VI, we revisit the motivating examples of Section III and apply the procedure of Section V.

II. Background

Given complex numbers $z_0, z_1, \ldots, z_n$ in $\mathbb{D} := \{z : |z| < 1\}$, which we assume to be distinct for simplicity, and given complex numbers $w_0, w_1, \ldots, w_n$, the classical Pick interpolation problem asks for a function f in the *Schur class*

$$\mathcal{S} = \{f \in H_\infty(\mathbb{D}) : \|f\|_\infty \leq 1\}$$

which satisfies the interpolation condition

$$f(z_k) = w_k, \quad k = 0, 1, \ldots, n \tag{1}$$

where $H_\infty(\mathbb{D})$, or simply H_∞, is the Hardy space of bounded analytic functions on $\mathbb{D}$. It is well-known (see, e.g., [17]) that such a function exists if and only if the Pick matrix

$$P = \left[\frac{1 - w_k \bar{w}_\ell}{1 - z_k \bar{z}_\ell}\right]_{k,\ell=0}^{n} \tag{2}$$

is positive semi-definite. The solution is unique if and only if P is singular, in which case f is a Blaschke product of degree equal to the rank of P. In this paper, throughout, we assume that P is positive definite and hence that there are infinitely many solutions to the Pick problem. A complete parameterization of all solutions was given by Nevanlinna (see e.g., [2]), and for this reason the subject is often referred to as Nevanlinna-Pick interpolation.

In engineering applications f usually represents the transfer function of a feedback control system or of a filter, and therefore the McMillan degree of f is of significant interest. Thus, it is natural to require that f be rational and of bounded degree. Such a constraint completely changes the nature of the underlying mathematical problem.

The classical Nevanlinna-Pick theory provides one particular solution that is rational and of a generic degree equal to n – the so-called *central solution* – as well as a parameterization of all solutions. However, it provides no insight and no help in determining other possible solutions *of degree n or less*. The central solution is also referred to as the *maximum-entropy solution* (see, e.g., [4], [14], [33]), because it maximizes the functional

$$\int_{\mathbb{T}} \log(1 - |f|^2) dm$$

subject to (1), where $\mathbb{T} = \{z = e^{i\theta} : \theta \in (-\pi, \pi]\}$ is the unit circle and $dm = d\theta/2\pi$ is the (normalized) Lebesgue measure on $\mathbb{T}$. In recent work (see [13], [22] and references therein) it has been shown that all solutions of degree at most n are in fact minimizers of certain weighted entropy functionals.

Following this recent development, we consider the weighted entropy functional $\mathbb{K}_\Psi : \mathcal{S} \to \mathbb{R} \cup \infty$, given by

$$\mathbb{K}_\Psi(f) = -\int_{\mathbb{T}} \Psi \log(1 - |f|^2) dm \tag{3}$$

where Ψ is a non-negative log-integrable function on $\mathbb{T}$. We study how the minimizer of

$$\min\{\mathbb{K}_\Psi(f) : f \in \mathcal{S},\ f(z_k) = w_k,\ k = 0, \ldots, n\} \tag{4}$$

depends on the weighting function Ψ, and then we determine when a given interpolant f can be obtained as a minimizer of (4) for a suitable choice of Ψ.

The interpolants of degree at most n correspond to a very specific choices of Ψ. In fact, let ϕ be the Blaschke product

$$\phi(z) = \prod_{k=0}^{n} \frac{z_k - z}{1 - \bar{z}_k z} \tag{5}$$

and let $U : f(z) \mapsto z f(z)$ denote the standard shift operator on H_2. Then ϕH_2 is a shift invariant subspace; i.e., $f \in \phi H_2$ implies that $U(f) = zf \in \phi H_2$. Denote by $\mathcal{K}$ the co-invariant subspace $H_2 \ominus \phi H_2$. Then $\mathcal{K}$ is invariant under U^*, where U^* denotes the adjoint of U. Let $\mathcal{K}_0$ denote the set of outer functions [37, p. 370] in $\mathcal{K}$ that are positive at the origin. The following result is taken from [13].

Theorem 1: Suppose that the Pick matrix (2) is positive definite, and let σ be an arbitrary function in $\mathcal{K}_0$. Then there exists a unique pair of elements $(a, b) \in \mathcal{K}_0 \times \mathcal{K}$ such that

(i) $f = b/a \in H_\infty$ with $\|f\|_\infty \leq 1$;
(ii) $f(z_k) = w_k, \quad k = 0, 1, \ldots, n$;
(iii) $|a|^2 - |b|^2 = |\sigma|^2$ a.e. on $\mathbb{T}$.

Conversely, any pair $(a, b) \in \mathcal{K}_0 \times \mathcal{K}$ satisfying (i) and (ii) determines, via (iii), a unique $\sigma \in \mathcal{K}_0$. Moreover, setting $\Psi = |\sigma|^2$, the optimization problem

$$\min \mathbb{K}_\Psi(f) \text{ s.t. } f(z_k) = w_k, \quad k = 0, \ldots, n$$

has a unique solution f that is precisely the unique $f \in \mathcal{S}$ satisfying conditions (i), (ii) and (iii).

We define $\tau(z) := \prod_{k=0}^{n}(1 - \bar{z}_k z)$ and note that

$$\mathcal{K} = \left\{\frac{p(z)}{\tau(z)} : p \in \mathrm{Pol}(n)\right\}$$

where $\mathrm{Pol}(n)$ denotes the set of polynomials of degree at most n, while

$$\mathcal{K}_0 = \left\{\frac{p(z)}{\tau(z)} : p \in \mathrm{Pol}_+(n)\right\}$$

where $\mathrm{Pol}_+(n)$ denotes the subset of polynomials $p \in \mathrm{Pol}(n)$ such that $p(z) \neq 0$ for all $z \in \mathbb{D}$ and $p(0) > 0$. With $\sigma = p/\tau \in \mathcal{K}_0$ as in the theorem, we refer to the roots of $z^n \bar{p}(z^{-1})$ as the *spectral zeros* corresponding to the pair (a, b). Since $1 - |f|^2 = |\sigma/a|^2$ on $\mathbb{T}$, then, except for possible cancellations between σ and a, the spectral zeros are the roots of a minimum phase spectral factor of $1 - |f|^2$. When such a cancellation occurs, the degree of f is less than n, and a, b, and σ, have a common root.

Theorem 1 has two parts: the first part states that interpolants of degree at most n are completely parameterized in terms of spectral zeros in the sense that there is bijection between the pairs (a, b) such that $f = b/a$ is an interpolant of degree at most n and sets of n spectral zeros. Given a choice of n spectral zeros, the second part of the theorem provides a convex optimization problem, the unique solution of which provides the corresponding interpolant.

Remark 1: The theorem, as stated in [13], allows for considerably more general interpolation conditions than (ii). In the case where the points $\{z_0, \ldots, z_n\}$ are not necessarily distinct, condition (ii) needs to be replaced by $f = f_0 + \phi q$ with $q \in H_\infty(\mathbb{D})$, which encapsulates interpolation of derivatives as well. The special case where $\phi(z) = z^n$ is analogous to the so-called covariance extension problem with degree constraints, which is usually stated for Carathéodory functions rather than Schur functions. The theorem is also valid when ϕ is an arbitrary inner function. The background to the derivation of Theorem 1 has a long history. The existence part of the parameterization was first proved in the covariance extension case in [18], [19] and in the Nevanlinna-Pick case in [20]. The uniqueness part (as well as well-posedness) in [11]. The optimization approach was initiated in [9] (also, see the extended version [10]) and further developed in, e.g., [8], [12], [21].

III. Motivating Examples

We present two basic examples of robust control design to highlight the relevance of the theory. The first one deals with sensitivity minimization and revisits an (academic) example which was discussed in [12]. The second addresses a more typical (and well-conditioned) design which is formulated in the context of H_∞-loopshaping and optimal robustness in the gap metric.

A. Sensitivity Minimization

Consider a standard feedback system where $P(z)$ and $C(z)$ are the transfer functions of plant and controller, respectively. The stability and tracking/disturbance-rejection qualities of the feedback system are reflected in the sensitivity function $S(z) = (1 - P(z)C(z))^{-1}$. It is well-known (see, for example, [46]) that the internal stability of the feedback system is equivalent to $S(z)$ being analytic outside the unit disc and satisfying the interpolation conditions

$$\begin{aligned} S(z_k) &= 1, \quad k = 1, 2, \ldots, n_z, \\ S(p_\ell) &= 0, \quad \ell = 1, 2, \ldots, n_p \end{aligned}$$

where $z_1, \ldots, z_{n_z}$ and $p_1, \ldots, p_{n_p}$ are the zeros and poles outside the unit disk, respectively, of the plant $P(z)$. Conversely, if $S(z)$ is any stable, proper rational function which satisfies these conditions, then it can be realized as the sensitivity function of such a feedback system with the given $P(z)$ and a choice of a suitable control transfer function $C(z)$.

The maximal disturbance-amplification depends on the choice of the controller and is equal to the H_∞-norm of the sensitivity function $\|S\|_\infty$. To illustrate this, consider the simple

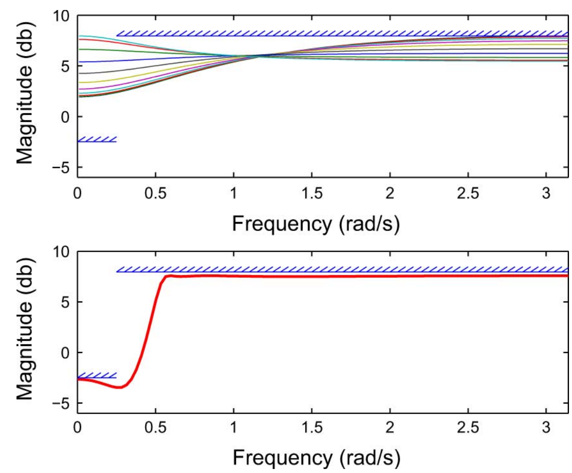

Fig. 1. Subplot 1: Possible degree-one shapes. Subplot 2: A degree 5 function satisfying the constraints.

example $P(z) := 1/(z-2)$. The optimal value $\|S_{\rm opt}\|_\infty$ in this example turns out to be equal to two (see [12]), and the magnitude of the optimal sensitivity function $S_{\rm opt}$ is constant across frequencies. However, in general, the magnitude of disturbances and the modeling uncertainty are *not* uniform across frequencies. Thus, the design ought to differentiate between frequency bands so as to achieve desired levels of performance and robustness. Therefore, we need to relax the requirement of uniformly minimal sensitivity gain to a pre-specified upper bound

$$\|S\|_\infty \le \gamma.$$

In this case, provided $\gamma > \|S_{\rm opt}\|_\infty$, there is large family of controllers which achieve such a specification. Naturally, the size of the family and the ability to "shape" $|S|$ increases with γ. On the other hand, the degree of the controller and the complexity of the feedback system depend on the degree of S. Thus, for a given value of γ, our goal is to not only control the shape of S, but its degree as well.

Adhering to a typical design specification for disturbance rejection, we require

$$\begin{aligned} |S(e^{i\theta})| &\le \epsilon, \quad \text{on } (0, \theta_c), \\ |S(e^{i\theta})| &\le \gamma, \quad \text{on } (\theta_c, \pi) \end{aligned} \tag{6}$$

where we take $\gamma = 2.5, \epsilon = 0.75$ and $\theta_c = 0.25$. In Fig. 1, the degree 1 sensitivity functions which satisfy $\|S\|_\infty \le \gamma$ are depicted in Subplot 1. It is observed that the design specifications are not met by any such function. Subplot 2, in the same figure, shows a sensitivity function of degree 5 which satisfies the constraints. As expected, this shows that by relaxing the degree constraint to degree 5, we are able to find a function that satisfies the constraint. The design is accomplished with the method in Section V-B, utilizing the theory in the first part of the paper. Details will be provided in Section VI where the example will be revisited.

B. Frequency-Dependent Robustness Margin

Let P_0 denote the transfer function of a single-input, single-output finite-dimensional linear system with stable coprime factorizations

$$P_0 = \frac{N_0}{M_0}$$

i.e., $M_0, N_0 \in H_\infty$, normalized to satisfy

$$M_0^* M_0 + N_0^* N_0 = 1, \quad \text{on } \mathbb{T} \tag{7}$$

where $f(z)^* := \overline{f(\bar{z}^{-1})}$. Then, as is well-known, all stabilizing controllers for P_0 are parameterized by $q \in H_\infty$ via

$$C = \frac{U_0 + M_0 q}{V_0 + N_0 q} \tag{8}$$

where $U_0,\ V_0 \in H_\infty$ satisfy $V_0 M_0 - U_0 N_0 = 1$, see, e.g., [15], [42]. To model perturbations of the coprime factors for frequency-dependent uncertainty, we consider plants $P = N/M$ such that

$$\left\| \begin{pmatrix} M(z) - M_0(z) \\ N(z) - N_0(z) \end{pmatrix} \right\| < \alpha |w(z)| \quad \text{for } z \in \mathbb{T} \tag{9}$$

where $\|\cdot\|$ denotes Euclidean vector norm and w is an outer function shaping the radius. Moreover, the size of the radius is controlled by a separate scaling parameter $\alpha \in \mathbb{R}_+$. Thus, we consider the problem of robust stabilization of the ball of plants

$$\mathcal{B}(P_0, \alpha w) := \left\{ P = \frac{N}{M} : (9) \text{ holds} \right\}$$

around the center P_0.

As shown in [43], a controller specified by q stabilizes $\mathcal{B}(P_0, \alpha w)$ provided

$$\left\| \begin{pmatrix} U_0 + M_0 q \\ V_0 + N_0 q \end{pmatrix} \alpha w \right\|_\infty \le 1. \tag{10}$$

This condition can be expressed as a Nevanlinna-Pick problem. Indeed, taking advantage of the normalization of the coprime factors as in [32] (see also [23]), we define the transformation

$$Z := \begin{pmatrix} M_0^* & N_0^* \\ -N_0 & M_0 \end{pmatrix}$$

which is unitary, i.e., $ZZ^* = Z^* Z = I$. We also denote by ϕ the Blaschke product that vanishes at the conjugate inverse of the poles of M_0, N_0. Hence, $\phi M_0^*, \phi N_0^* \in H_\infty$. Then, the left hand side of (10) is

$$\left\| \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix} Z \begin{pmatrix} U_0 + M_0 q \\ V_0 + N_0 q \end{pmatrix} \alpha w \right\|_\infty = \left\| \begin{pmatrix} F \\ 1 \end{pmatrix} \alpha w \right\|_\infty \tag{11}$$

where $F_0 = \phi M_0^* U_0 + \phi N_0^* V_0 \in H_\infty$ and

$$F = F_0 + \phi q. \tag{12}$$

It can be seen that the values of F at the roots of ϕ are independent of q and are specified by the plant. Moreover, as seen from (11), condition (10) holds provided $F \in H_\infty$ satisfies (12) and

$$\sqrt{|F|^2 + 1} \le \frac{1}{\alpha |w|}, \text{ on } \mathbb{T}. \tag{13}$$

Conversely, for any $F \in H_\infty$ satisfying (12), there corresponds a unique parameter q and a controller C, where C stabilizes the ball of plants $\mathcal{B}(P_0, \alpha w)$ with radius $\alpha |w| = (|F|^2 + 1)^{-1/2}$. Furthermore, if the degree of F is small, so is the degree of the controller C. This is stated in the following proposition proved in the Appendix.

Proposition 2: Let $F \in H_\infty$ satisfy (12) and C be the controller specified via (12) and (8). Then $\deg C \le \deg F$.

Now we consider

$$\begin{aligned} \Pi_{P//C} &:= \begin{pmatrix} \frac{P_0}{1 - P_0 C} & \frac{-P_0 C}{1 - P_0 C} \\ \frac{1}{1 - P_0 C} & \frac{-C}{1 - P_0 C} \end{pmatrix} \\ &= Z^* \begin{pmatrix} 1 & -\phi^* F \\ 0 & 0 \end{pmatrix} Z \end{aligned}$$

which is the matrix of transfer functions from disturbances at the input and output ports of the plant to the plant input and output. This is a rank-one matrix function (see [23]) with singular value $\sqrt{|F|^2 + 1}$. Thus, the shape of $|F|$ relates directly to amplification of external disturbances in the loop, and it also dictates how robust the control system is to plant uncertainty in the coprime factor (or, in the gap metric; cf. [23], [32]). In fact

$$b_{\text{opt}}(P) := \max_{C \text{ stabilizing}} \|\Pi_{P//C}\|_\infty^{-1}$$

is precisely the optimal robustness radius for gap-ball uncertainty (see [23]) and coincides with $1/\sqrt{|F|^2 + 1}$ for the smallest $\|F\|_\infty$ consistent with (12).

The use of a frequency-dependent weight w allows shaping the loop-gain [32] as well as the performance and the robustness of the closed-loop system over different frequency bands [7], [43], [44]. By scaling α in (13) one can maximize the radius of $\mathcal{B}(P_0, \alpha w)$ for which a stabilizing controller exists (as in [23], [32], [43]). The maximal value $\alpha_{\max}$ and the optimal interpolant F, consistent with (12) and (13), satisfy

$$|F|^2 = \frac{1 - \alpha_{\max}^2 |w|^2}{\alpha_{\max}^2 |w|^2}. \tag{14}$$

Thus, the use of a nontrivial weight w forces the interpolant to have a nontrivial outer factor as well. This causes a corresponding increase in the degree of the closed-loop system and of the controller. Therefore, the purpose of this work is to develop techniques for reducing the degree of the control system while relaxing design requirements in a controlled fashion. In Section VI we revisit the issue of frequency-dependent robustness margin in order to demonstrate the application of the framework.

IV. Inverse Problem of Analytic Interpolation With Degree Constraint

We formulate the inverse problem of analytic interpolation with a degree constraint and explain how this can be used to shape interpolants to specifications. However, we begin by characterizing the minimizers of $\mathbb{K}_\Psi$ for general weights Ψ.

A. Characterization of $\mathbb{K}_\Psi$-Minimizers

Theorem 1 provides a (complete) parametrization of Nevanlinna-Pick interpolants of degree $\le n$. It states that such interpolants are in correspondence with $\Psi = |\sigma|^2$ for $\sigma \in \mathcal{K}_0$ and that they are minimizers of the generalized entropy integral $\mathbb{K}_\Psi$ specified by the weight Ψ. In this paper we are interested in interpolants of higher degree. Thus, we are led to consider $\mathbb{K}_\Psi$-entropy minimizers for more general choices of Ψ. The relevant generalization is stated next.

Theorem 3: Suppose that the Pick matrix (2) is positive definite and Ψ is a log-integrable nonnegative function on the unit circle. A function f is a minimizer of (4) if and only if the following three conditions hold:
(i) $f(z_k) = w_k$ for $k = 0, \dots, n$;
(ii) $f = b/a \in \mathcal{S}$ where $b \in \mathcal{K}$ and a is outer;
(iii) $\Psi = |a|^2 - |b|^2$.
Any such minimizer is necessarily unique.

Proof: See the Appendix. ■

As seen from Theorem 1, any choice of $\Psi = |\sigma|^2$ with $\sigma \in \mathcal{K}_0$ gives rise to $a, b \in \mathcal{K}$, and hence to an interpolant f with a degree bounded by n. Theorem 3 allows for a Ψ which is rational of an arbitrarily high degree or even irrational. In this case, b still belongs to $\mathcal{K}$, while the additional "complexity" is absorbed in a. Naturally, in such a case, the interpolant $f = b/a$ will also be rational of a high degree or irrational, respectively. This observation allows us to characterize all minimizers of $\mathbb{K}_\Psi$ of degree at most $n + m$ for any given $m \in \mathbb{N}_+$. More specifically, let

$$\mathcal{K}_m := \left\{\sigma = \sigma_0\rho : \sigma_0 \in \mathcal{K}_0, \deg\rho \le m, \right.$$
$$\left. \rho \text{ outer}, \rho(0) > 0\right\}$$
$$= \left\{\frac{q}{\tau\pi} : \pi \in \mathrm{Pol}_+(m), q \in \mathrm{Pol}_+(n+m)\right\}. \quad (15)$$

Next we state the sought characterization.

Corollary 4: Let $\Psi = |\sigma|^2$ with $\sigma \in \mathcal{K}_m$. Then the minimizing function f in (4) satisfies
(i) $f(z_k) = w_k$ for $k = 0, \dots, n$;
(ii) f has at most n zeros in $\mathbb{D}$;
(iii) the degree of f is at most $n + m$.
Conversely, for any $f \in \mathcal{S}$ which satisfies (i), (ii), and (iii), there exists a corresponding choice of $\sigma \in \mathcal{K}_m$ so that f is the minimizer of (4).

Proof: By Theorem 3, the minimizer f satisfies (ii), where

$$|a|^2 = |\sigma|^2 + |b|^2, \quad \sigma \in \mathcal{K}_m, b \in \mathcal{K}.$$

Then, in view of (15), $a \in \mathcal{K}_m$, and hence $\deg f \le n + m$. Conversely, suppose $f = (\beta\pi)/\alpha$, where $\beta \in \mathrm{Pol}(n)$, $\pi \in \mathrm{Pol}_+(m)$ and $\alpha \in \mathrm{Pol}_+(n+m)$, then $f = b/a$, where $b \in \mathcal{K}$ and $a \in \mathcal{K}_m$. Then

$$\Psi = |a|^2 - |b|^2 = |\sigma|^2$$

where $\sigma \in \mathcal{K}_m$. If, in addition, f satisfies the interpolation conditions in (i), then, by Theorem 3, f is the minimizer of (4), as claimed. ■

Corollary 5: If $f \in \mathcal{S}$ is a minimizer of $\mathbb{K}_\Psi$ for some choice of a log-integrable non-negative function Ψ, then f has at most n zeros in $\mathbb{D}$.

Proof: This follows directly from condition (ii) in Theorem 3, since $f = b/a$ with a outer and $b \in \mathcal{K}$. ■

These two corollaries underscore the significance of Theorem 3 for understanding the structure of minimizers.

B. Inverse Problem

It turns out that the number of roots in $\mathbb{D}$ determine whether an interpolant f is a minimizers of $\mathbb{K}_\Psi$ for some choice of Ψ. This is stated next together with the characterization of all such choices of Ψ.

Proposition 6: Any function $f \in \mathcal{S}$ that satisfies
(i) $f(z_k) = w_k$ for $k = 0, \dots, n$;
(ii) f has at most n zeros in $\mathbb{D}$;
(iii) $\log(1 - |f|^2) \in L_1(\mathbb{T})$;
is the unique minimizer of (4) with

$$\Psi = (|f|^{-2} - 1)|b|^2$$

for any $b \in \mathcal{K}$ chosen so that bf^{-1} is outer. Conversely, a (nonzero) function f having more than n zeros in $\mathbb{D}$ cannot arise as the minimizer of (4) for any choice of Ψ.

Proof: Suppose that $f \in \mathcal{S}$ satisfies the interpolation constraint (i). Then, by Theorem 3, f is the minimizer of (4) if and only if $f = b/a$, where $b \in \mathcal{K}$, a is outer, and $\Psi = |a|^2 - |b|^2$, which in turn holds if and only if bf^{-1} is outer, $b \in \mathcal{K}$, and $\Psi = (|f|^{-2} - 1)|b|^2$. This condition fails when f has more than n zeros in $\mathbb{D}$. In fact, for bf^{-1} to be outer, the zeros of f in $\mathbb{D}$ must be canceled by zeros of b. However, $b \in \mathcal{K}$ can have at most n zeros. ■

The choice of $b \in \mathcal{K}$ in Theorem 6 is not unique, in general. The selection of b must prevent bf^{-1} from having poles in $\mathbb{D}$, and hence any zero of f must also be a zero of b. If f has more than n zeros in $\mathbb{D}$, there is no such b, whereas if f has exactly n zeros in $\mathbb{D}$, then b is uniquely defined. In all other cases, when f has $n_f < n$ zeros in $\mathbb{D}$, the family of possible choices of b, and hence the family of possible weights

$$\{\Psi : \Psi = (|f|^{-2} - 1)|b|^2, b \in \mathcal{K},\ bf^{-1}\text{outer}\} \quad (16)$$

has dimension $n - n_f$. The design freedom offered by this nonuniqueness will be exploited in Section V for model reduction.

C. Shaping the Interpolants

Suppose that we are given an outer function g in $\mathcal{S}$. We address the following two questions that pertain to admissible shapes of analytic interpolants:

a) Does there exist an $f \in \mathcal{S}$ which satisfies the interpolation condition (1) and

$$|f(e^{i\theta})| \le |g(e^{i\theta})|, \quad \theta \in (-\pi, \pi]? \quad (17)$$

b) Does there exist a Ψ such that the corresponding (unique) minimizer f of (4) satisfies

$$|f(e^{i\theta})| = |g(e^{i\theta})|, \quad \theta \in (-\pi, \pi]? \quad (18)$$

The first question is equivalent to

$$\{f \in \mathcal{S} : \|fg^{-1}\|_\infty \le 1, f(z_k) = w_k, \quad k = 0, \dots, n\} \quad (19)$$

being nonempty, which holds if and only if the associated Pick matrix

$$\mathbf{Pick}(g) := \left[\frac{1 - w_k g(z_k)^{-1}\overline{w_l g(z_l)^{-1}}}{1 - z_k \overline{z_l}}\right]_{k,l=0}^{n} \tag{20}$$

is positive semi-definite. This answers the first question.

If $\mathbf{Pick}(g)$ is positive definite, there exist interpolants $\hat{f}$ such that $|\hat{f}(z)| < |g(z)|$ for all $z \in \mathbb{T}$ and the design specifications (17) may be satisfied with strict inequality. Therefore, any minimizing interpolant f cannot satisfy (18), since $|\hat{f}(z)| < |f(z)|$ for all $z \in \mathbb{T}$ implies that $\mathbb{K}_\Psi(\hat{f}) < \mathbb{K}_\Psi(f)$, which contradicts the claim that f is the minimizer. Therefore in order for (4) to have a solution satisfying(18), $\mathbf{Pick}(g)$ must be positive semidefinite and singular. In this case, the set (19) is a singleton. This provides an answer to the second question, which we summarize next.

Proposition 7: Let $g \in \mathcal{S}$ be an outer function such that $\log(1 - |g|^2) \in L_1$. Then there exists a pair (Ψ, f) of functions on $\mathbb{T}$ such that

(i) $\log \Psi \in L_1$;

(ii) f is the solution of (4);

(iii) $|f| = |g|$ on $\mathbb{T}$;

if and only if $\mathbf{Pick}(g)$ is positive semidefinite and singular. Furthermore, f is uniquely determined.

Proof: Sufficiency is shown in the text leading to the proposition, so it remains to prove necessity. Since the matrix in (20) is nonnegative definite and singular, there is a unique f satisfying $\|fg^{-1}\| \leq 1$ and (1). Then $f = g\varphi$ where φ is inner and of degree $\leq n$. Since f is rational with at most n zeros in $\mathbb{D}$, by Proposition 6 there exists a function Ψ such that f minimizes (4). ■

V. Approximation With Interpolants of Low Degree

Assume that f is the minimizer of the entropy functional, as in (4), for a suitable weight selected without regard to the degree. In this section we develop an approach for approximating f, indirectly, via approximation of the corresponding weight Ψ in order to obtain an interpolants f_r of a similar shape and of a reduced degree. To this end we begin by showing that f is continuous in Ψ in a suitable sense, i.e., using appropriate metrics to measure perturbations in f and Ψ. Then we formulate a quasi-convex optimization problem to approximate Ψ by a *reduced-order* weight Ψ_r which we use to obtain f_r as the minimizer of the corresponding functional.

A. Continuity of the Map From Weights to Minimizers

The use of the H_∞-norm to quantify perturbations of interpolants is rather strong and renders

$$\varphi : \Psi \mapsto f \tag{21}$$

with f being the minimizer in (4), discontinuous (see Example 1 below). Moreover, since f may represent a transfer function of a feedback system, from a practical standpoint perturbations of f are naturally quantified via either weighted norms (i.e., in a weak sense) or in the H_2 norm. Below we use the later. Similarly, when considering perturbations of a given weight Ψ the L_1-norm as well as the L_1 norm on logarithms of weights, seem inappropriate. Instead, for practical as well as technical reasons we choose to work with the metric

$$d(\Psi, \Psi_r) := \|\log(\Psi) - \log(\Psi_r)\|_\infty = \left\|\log\left(\frac{\Psi}{\Psi_r}\right)\right\|_\infty$$

which quantifies perturbations in a multiplicative fashion. The proof of the following proposition is given in the Appendix.

Proposition 8: Let Ψ, Ψ_r be nonnegative log-integrable functions on $\mathbb{T}$ such that $d(\Psi, \Psi_r) = \epsilon$, let σ be the outer spectral factor of Ψ (i.e., σ is outer and $|\sigma|^2 = \Psi$ on $\mathbb{T}$), and let $f = \varphi(\Psi)$ and $f_r = \varphi(\Psi_r)$ be the corresponding minimizers. Then

$$\|\sigma(f - f_r)\|_2^2 \leq 2(e^{2\epsilon} - 1)\mathbb{K}_\Psi(f). \tag{22}$$

In the sequel we restrict our attention to weights which are positive and bounded. The following corollary is immediate.

Corollary 9: Let Ψ, Ψ_k, $k = 1, 2, \ldots$, be positive bounded functions on $\mathbb{T}$, and let $f = \varphi(\Psi)$ and $f_k = \varphi(\Psi_k)$ be the corresponding minimizers. If $d(\Psi, \Psi_k) \to 0$ as $k \to \infty$, then $\|f - f_k\|_2 \to 0$.

The statement shows that φ is continuous in above sense. The following further consequence of Proposition 8 will be used next and proven in the Appendix.

Corollary 10: Under the conditions of Corollary 9 and with $f = b/a$ and $f_k = b_k/a_k$ with $b, b_k \in \mathcal{K}$ and a, a_k outer as in Theorem 3, if $d(\Psi, \Psi_k) \to 0$ as $k \to \infty$, then $b_k \to b$ coefficient-wise.

When the H_∞-norm is used for the range space, continuity fails as shown in the following example.

Example 1: Consider a problem with one interpolation condition: $f(0) = 1/2$. Given a weight Ψ, the minimizer f satisfies $|f|^2 = (1 + \Psi/|b|^2)^{-1}$ where $b \in \mathcal{K}$ (Theorem 3), which implies that b is constant. Since $n = 0$, the minimizer f cannot have any zeros (Corollary 5). Hence f is outer and b is chosen so that the interpolation constraint is satisfied. If $\Psi \equiv 1$, then $f \equiv 1/2$. Define

$$\Psi_\epsilon(e^{i\theta}) = \begin{cases} 1 & \text{for} \quad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 1 + \epsilon & \text{for} \quad \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \end{cases}$$

and let $f_\epsilon = b_\epsilon/a_\epsilon = \varphi(\Psi_\epsilon)$, with $b_\epsilon \in \mathcal{K}$ and a_ϵ outer, as in Theorem 3. By Corollary 10, $b_\epsilon \to b$ as $\epsilon \to 0$. But since $|f_\epsilon|^2$ is discontinuous at $\theta = \pi/2$ for $\epsilon > 0$, the phase of the spectral factor is unbounded [1, p. 147] (cf. example in [1, pp. 145–158]). Therefore, as $\epsilon \to 0$, f_ϵ fails to converge in the H_∞-sense.

We now discuss the continuity of the modulus of the interpolant as a function of Ψ. If we disregard the phase, it turns out that the "shape" $|f|$ is continuous in a stronger sense, namely in the L_∞-sense. To see this consider $\Psi,\ \Psi_k,\ f,\ f_k, k = 1, 2, \ldots$, as in Corollary 9, and let $f = b/a$ and $f_k = b_k/a_k$ with $b, b_k \in \mathcal{K}$ and a, a_k outer as in Theorem 3. Then, by Corollary 10, $b_k \to b$ (coefficient-wise as $\mathcal{K}$ is finite-dimensional). Hence, if $d(\Psi, \Psi_k) \to 0$ as $k \to \infty$, then

$$|f_k|^2 = \left(1 + \left|\frac{\Psi_k}{b_k}\right|^2\right)^{-1} \to \left(1 + \left|\frac{\Psi}{b}\right|^2\right)^{-1} = |f|^2$$

uniformly.

B. Approximation of Interpolants

The continuity of φ allows for approximating interpolants in a way that exploits the correspondence between minimizers and weights. Given an interpolant f we would like to find a degree-r approximating interpolant f_r of f, where $r \geq n$. From the inverse problem there is a set $\varphi^{-1}(f)$ of admissible weights Ψ for which a given f is the minimizer of (4). Our first task is to find a pair (Ψ, Ψ_r) for which $\Psi \in \varphi^{-1}(f)$ and $\Psi_r = |\sigma_r|^2$, with $\sigma_r \in \mathcal{K}_{r-n}$, so that their logarithmic distance is minimal. That is, we consider the problem

$$\min_{\Psi,\Psi_r} \Big\{ \| \log(\Psi) - \log(\Psi_r)\|_\infty \mid \Psi \in \varphi^{-1}(f) \quad \text{and} \\ \Psi_r = |\sigma_r|^2 \text{ with } \sigma_r \in \mathcal{K}_{r-n} \Big\}. \tag{23}$$

This can be reformulated as a quasi-convex optimization problem and solved efficiently. By Corollary 4 the degree of the interpolant $f_r = \varphi(\Psi_r)$ is bounded by r, and, based on the quality of approximation obtained via quasi-convex optimization, an explicit bound on the approximation error $f - f_r$ can be obtained from Proposition 8. Thus, the basic idea is to replace the hard nonconvex problem of approximating f by another interpolating function f_r of degree at most r, by the simpler quasi-convex problem to approximate a $\Psi \in \varphi^{-1}(f)$ by a $\Psi_r = |\sigma_r|^2$ with $\sigma_r \in \mathcal{K}_{r-n}$.

The theory presented so far suggests a computational procedure in several steps, which we now summarize. Typically, the general problem is to find an analytic function F, of a desired shape, which satisfies $\|F\|_\infty \leq \gamma$ as well as the interpolation conditions

$$F(z_k) = W_k \quad \text{for} \quad k = 0, 1 \ldots, n. \tag{24}$$

Equivalently, for the given γ, we set $f = \frac{1}{\gamma}F$ and seek an $f \in \mathcal{S}$ such that

$$f(z_k) = w_k := \frac{1}{\gamma} W_k, \quad k = 0, 1, \ldots, n. \tag{25}$$

Thus, achieving a desired shape for either F or f are equivalent problems.

Step 1 Find an interpolant f, or F as above, having the desired shape, but without restricting its degree. We begin with a family of outer functions $\{g_\alpha\}$ having desirable shapes, monotonically decreasing pointwise on $\mathbb{T}$ as a function of α, and we select the largest value for α for which an interpolant exists with modulus bounded by $|g_\alpha|$. More specifically, for a fixed value for γ, we solve the optimization problem

$$\max_\alpha \{\alpha \mid (25) \text{ and } |f| \leq |g_\alpha|\}.$$

The maximal value of α is attained when $\mathbf{Pick}(g_\alpha)$ is positive semidefinite and singular (Proposition7). At the end of this step we require that $\log(1 - |f|^2) \in L_1$ (or, equivalently, that $\log(\gamma^2 - |F|^2) \in L_1$). Relaxation of γ will insure that this condition holds. Then, by Proposition 7, there is a Ψ such that $f := \varphi(\Psi)$ satisfies $|f(z)| = |g_{\alpha_{\max}}(z)|$ for all $z \in \mathbb{T}$. The functional form of $|g_\alpha|$ depends on the application. In the first of our motivating examples we take

$$g_\alpha = \frac{w}{\alpha}$$

which leads to a standard H_∞ optimization problem. In the second example g_α is specified by

$$|g_\alpha|^2 = \frac{1 - \alpha^2 |w|^2}{\gamma^2 \alpha^2 |w|^2}.$$

This choice makes $\alpha|w|$ a frequency-dependent robustness radius (see (14) and also [44]).

Step 2 For a given $r \geq n$, find an approximation f_r of f of degree at most r which satisfies the same interpolation conditions. To this end, find functions Ψ and Ψ_r that solve the optimization problem (23). This is a quasi-convex optimization problem. In fact, $\|\log(\Psi) - \log(\Psi_r)\|_\infty \leq \epsilon$ if and only if

$$1 - e^\epsilon \leq 1 - \frac{\Psi_r(z)}{\Psi(z)} \leq 1 - e^{-\epsilon} \quad \text{for all } z \in \mathbb{T}. \tag{26}$$

Equation (26) defines an infinite set of linear constraints on the pseudo-polynomials representing the nominator and denominator, respectively, of Ψ_r/Ψ.To see this, first note that by (16) $\Psi \in \varphi^{-1}(f)$ is of the form

$$\Psi = (|f|^{-2} - 1)|b|^2 \quad \text{where} \quad b \in \mathcal{K},\ bf^{-1}\text{outer}.$$

Now, if ψ is the inner part of f, $b = \hat{b}\psi$, where $\hat{b}$ is a polynomial of degree $n - \deg\psi$. Moreover, by (15), $\Psi_r = |q/(\tau\pi)|^2$, and therefore

$$\frac{\Psi_r(z)}{\Psi(z)} = \left|\frac{qh}{p}\right|^2$$

where $p := \hat{b}\pi$ and q are polynomials of degrees $r - \deg\psi$ and r, respectively, and $|h|^2 := |\tau\psi|^{-2}(|f|^{-2} - 1)^{-1}$ is fixed. Since the sublevel set of the trigonometric polynomials $|p|^2$ and $|q|^2$ satisfying (26) is convex for each $\epsilon > 0$, the problem is quasiconvex [6] and can be solved as problem (21) in [30]. Then $\Psi_r = |q/(\tau\pi)|^2$ is obtained from the optimal p and q, where π is any factor of p of appropriate dimension. However, in general the choice of π is unique. In fact, the design in Step 1 generically leads to interpolants with n unstable zeros, in which case $\deg\psi = n$ and $\hat{b}$ is a constant, so that we may take $\pi = p$.

Step 3 Next we find $f_r = \varphi(\Psi_r)$, i.e., we solve the optimization problem

$$\min_{f_r \in \mathcal{S}} \{\mathbb{K}_{\Psi_r}(f_r) \mid f_r(z_k) = w_k, \quad k = 0, \ldots, n\} \tag{27}$$

for the unique solution f_r. In Step 2 we have determined the weight Ψ_r as an approximation of Ψ, and therefore f_r will also be an approximation of to f, for which the bound (22) holds. Furthermore, since $\Psi_r = |\sigma_r|^2$ with $\sigma_r \in \mathcal{K}_{r-n}$, the degree of f_r is bounded by r (Corollary 4). Finally, we renormalize the interpolant

$$F_r := \gamma f_r$$

to obtain the approximant which solves the original interpolation problem (24).

Remark 2: The family $\{g_\alpha\}$ needs to be chosen in a judicious manner in order to insure good approximation in Step 2. For example, large variations or discontinuities in the values of $\log|g_\alpha|$ may severely inflate the error bounds in the quasi-convex optimization (23).

Remark 3: Step 2 is similar to the one used in [41], where, however, there is another rationale for choosing which Ψ to approximate. Compare also with [38], [40] where the ellipsoid algorithm and linear matrix inequalities are used to solve these types of problems.

Remark 4: In Step 3, for computational purposes, (27) may be transformed into an equivalent problem of the form in Theorem 1 whenever σ_r is rational without poles on $\mathbb{T}$. Utilizing [27, Lemma 1], (27) is transformed into

$$\min_{\hat{f}\in\mathcal{S}}\{\mathbb{K}_{\Psi_r}(\hat{f}) \mid \hat{f}(z_k)=w_kB(z_k),\ k=0,\ldots,n, \\ \hat{f}(\lambda)=0, \lambda\in\Lambda\} \tag{28}$$

where $B(z)=\prod_{\lambda\in\Lambda}(\lambda-z)/(1-\bar{\lambda}z)$ and

$$\Lambda=\{\lambda \mid \lambda \text{ is a pole of } \sigma_r^*\tau^*\}.$$

Then $\sigma_r\in\hat{\mathcal{K}}$, where $\hat{\mathcal{K}}$ is the coinvariant subspace of (28) and the minimizers of (27) and (28) satisfy $\hat{f}=Bf_r$ [27, p. 563], [28], [31]. Since $\sigma_r\in\hat{\mathcal{K}}$, we utilize (2.16), (2.17) in [13] to transform (28) into an interpolation problem with positive real domain and find the minimizer using the software in [36].

VI. Examples Revisited

We now return to the two examples from Section III. In both examples, the underlying mathematical problem is an analytic interpolation problem where a desired shape is sought for the interpolant. These are addressed using the procedure outlined in Section V.

A. *Sensitivity Minimization (Continued)*

In this example, we consider the sensitivity function $S=(1-PC)^{-1}$ of the feedback system with plant

$$P(z)=\frac{1}{z-2}.$$

Since P has one unstable pole at 2 and an unstable zero at ∞, we require that the sensitivity function satisfies

$$S(\infty)=1 \quad \text{and} \quad S(2)=0.$$

We further require that the specifications (6) are met. The relaxed bound on the infinity-norm of S is $\|S\|_\infty\le\gamma:=5/2$, and we therefore define the function

$$f(z)=\gamma^{-1}S(z^{-1})$$

which is normalized so that f satisfies $\|f\|_\infty\le 1$ and is analytic in $\mathbb{D}$. The constraints on S can be directly translated into constraints on f

$$f(0)=0.4, \qquad f(0.5)=0,$$

and

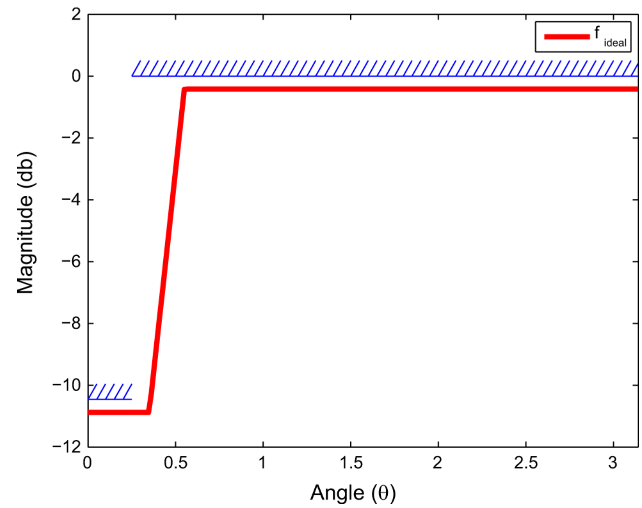

Fig. 2. Ideal function: $f_{\text{ideal}}=\gamma^{-1}S_{\text{ideal}}^*$.

$$|f(e^{i\theta})|\le\epsilon\gamma^{-1} \quad \text{on } (0,\theta_c), \\ |f(e^{i\theta})|\le 1 \quad \text{on } (\theta_c,\pi)$$

where, as before, $\gamma=2.5$, $\epsilon=0.75$ and $\theta_c=0.25$. We begin with a particular interpolant f_{ideal} that meets the criteria without regard to any constraint on the degree, shown in Fig. 2, which we then approximate using the theory in Section V-B.

We determine f_{ideal} as follows. We first define an outer function w with the property that $\log|w(e^{i\theta})|$ is piecewise linear in $\theta\in(0,\pi)$ and passes through the points

$$(0,\epsilon\gamma^{-1}),\ (\theta_c+0.1,\epsilon\gamma^{-1}),\ (\theta_c+0.3,1),\ \text{and } (\pi,1).$$

This is our desired shape. Then we set $g=w/\alpha$ and scale $\alpha>0$ so that there exists a minimizer of (4) which satisfies $|f|=|g|$ on $\mathbb{T}$. By Proposition 7, α is specified by the requirement that $\mathbf{Pick}(g)$ is positive semidefinite and singular. In this case, $\alpha=1.0498>1$, and hence $|f_{\text{ideal}}|=|g|$ is consistent with the requirement $\|f_{\text{ideal}}\|_\infty\le 1$. It is clear that neither g nor f_{ideal} is a rational function, but, unlike g, f_{ideal} is an interpolant.

Next we approximate f_{ideal} by an interpolant of small degree. We first characterize the inverse image of f_{ideal} under the map (21), which according to Proposition 6 is given by

$$\varphi^{-1}(f_{\text{ideal}})=\{\Psi:\Psi=(|f_{\text{ideal}}|^{-2}-1)|b|^2, \\ b\in\mathcal{K},\ bf_{\text{ideal}}^{-1}\ \text{outer}\}.$$

In the present case, $|b|^2$ is a positive constant. Hence, $\varphi^{-1}(f_{\text{ideal}})$ contains a single element, modulo scaling, and we choose $\Psi_{\text{ideal}}=(|f_{\text{ideal}}|^{-2}-1)$. As described in Section IV-C we let f_r be the approximant of degree less or equal to r obtained from

$$f_r=\arg\min\left\{\mathbb{K}_{|\sigma_r|^2}(f_r) \;\middle|\; f_r\in\mathcal{S} \text{ and } (1) \text{ holds}\right\}$$

where σ_r is the solution of the quasi-convex optimization

$$\min_{\sigma_r}\left\{\left\|\log(|\sigma_r|^2)-\log(\Psi_{\text{ideal}})\right\|_\infty \;\middle|\; \text{for } \sigma_r\in\mathcal{K}_{r-n}\right\}.$$

Finally, by scaling $S_r(z):=\gamma f_r(z^{-1})$ and $S_{\text{ideal}}(z):=\gamma f_{\text{ideal}}(z^{-1})$, we obtain admissible sensitivity functions.

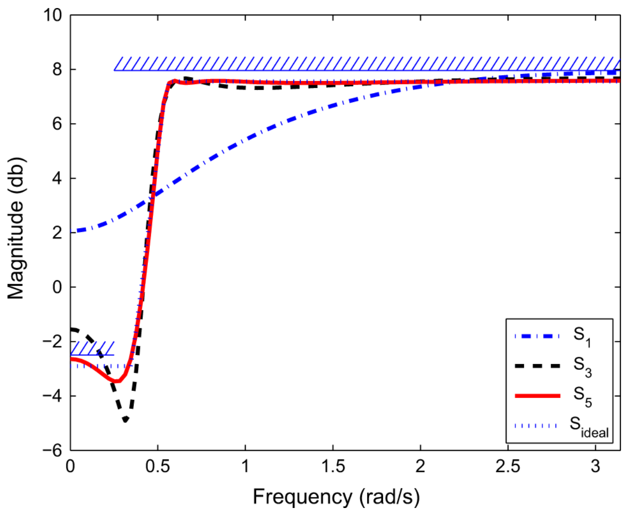

Fig. 3. Approximations of degree 1, 3, and 5.

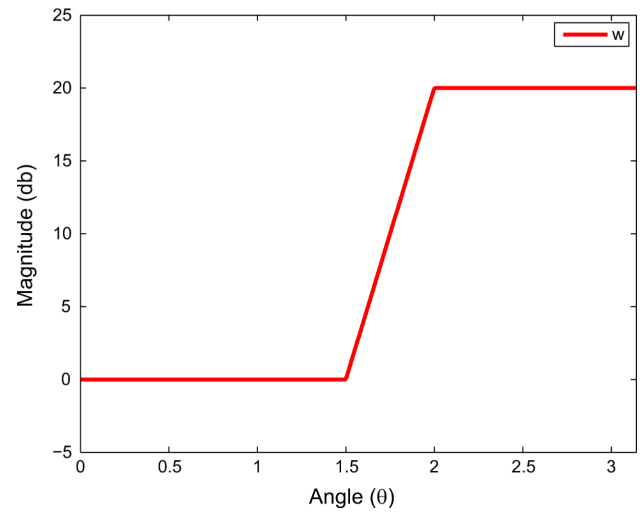

Fig. 4. Frequency-dependent robustness shape w.

We compute S_r for $r = 1, 3, 5$ (as shown in the footnote [1]) and display the magnitudes of S_1, S_3 and S_5 in Fig. 3. Neither the degree-one nor the degree-three approximant of S_{ideal} satisfies the design specifications, wheras S_5 does. It is interesting to note that even though f_{ideal} is infinite-dimensional it is possible to find satisfactory low-dimensional approximants.

It is also interesting to note that when the bound $\|S\|_\infty \leq \gamma$ is removed and only stability is required, as in [30], the approximation is better in the low-frequency band $(0, \theta_c + 0.1)$, but worse in the high-frequency band $(\theta_c + 0.3, \pi)$. It seems as if approximation with a bound puts more emphasis on the region where the interpolant is close to the bound (i.e., $|S| \approx \gamma$) at the expense of the region where $|S_{\text{ideal}}| \ll \gamma$.

B. Frequency-Dependent Robustness Margin (Continued)

We consider a continuous-time plant having one integrator, a slow unstable pole, and a time-lag, modeled via $\frac{s-1}{s+1}\frac{1}{s(s-0.1)}$. We base our design on its discrete-time counterpart

$$P_0(z) = \frac{0.08772z^3 - 0.08772z^2 - 0.4386z - 0.2632}{z^3 - 2.439z^2 + 1.807z - 0.3684}$$

obtained via the Möbius transform

$$s \to z = \frac{2+s}{2-s}$$

and restrict our analysis to the discrete-time domain.

The design objective is encapsulated in the choice of a weight w, chosen as in Section III to increase robustness to high-frequency modeling uncertainty and a bound $\|F\|_\infty \leq \gamma$ which ensures that the overall robustness radius is at least $(\gamma^2+1)^{-1/2}$. The selected "nominal weight" w is shown in Fig. 4. Furthermore, in this particular example, it is deemed appropriate to allow $\|F\| \leq \gamma = 20$ and consider the problem of shaping the function $f(z) = \frac{1}{\gamma}F(z^{-1})$ subject to the interpolation conditions

$$f(-0.9000) = \frac{1}{3\gamma} \tag{29a}$$

$$f(0.0659 \pm 0.9295i) = \frac{1}{\gamma}(0.3829 \pm 0.3601i). \tag{29b}$$

that are obtained by evaluating F_0 at the roots of ϕ in (12). The maximal scaling parameter $\alpha_{\max}$ can be readily computed by first calculating the outer factor g_α of

$$|g_\alpha|^2 = \frac{1 - \alpha^2|w|^2}{\gamma^2\alpha^2|w|^2} \text{ on } \mathbb{T}$$

and then, finding the maximal value of α for which the Pick matrix $\mathbf{Pick}(g_\alpha)$ is positive semidefinite (Proposition 7). The Pick matrix $\mathbf{Pick}(g_\alpha)$, defined in (20), is computed from the interpolation data (29) and is independent of γ.

Let f_{ideal} be the unique interpolant which satisfies $|f_{\text{ideal}}| = |g_{\alpha_{\max}}|$, and denote the corresponding controller by C_{ideal}. Here, we require that $\log(1 - |f_{\text{ideal}}|^2) \in L_1$, which holds for the chosen value of γ. (The condition fails only for the minimal γ for which the interpolation problem is solvable as explained in Step 1, in Section V-B.) Since w is not rational, neither are f_{ideal} and C_{ideal}. Next we describe how to approximate f_{ideal} with an admissible interpolant of low degree which, accordingly, leads to a controller and closed-loop transfer functions of low degree.

Next we determine a degree-r approximant f_r of f_{ideal} for a choice of $r \geq n$. To this end we follow Step 2 in Section V-B to compute a solution of the quasi-convex optimization problem to find a $\sigma_r \in \mathcal{K}_{r-n}$ and a $\Psi \in \varphi^{-1}(f_{\text{ideal}})$ which minimize

$$\left\|\log(|\sigma_r|^2) - \log(\Psi)\right\|_\infty.$$

Then we follow Step 3 in Section V-B to determine

$$f_r = \arg\min\{\mathbb{K}_{|\sigma_r|^2}(f_r)\big|f_r \in \mathcal{S} \text{ and } (29)\}$$

[1] $S_1(z) = \frac{3.778z-1.889}{z-4.723}$, $S_3(z) = \frac{2.58z^3-6.534z^2+5.645z-1.511}{z^3-4.9z^2+7.142z-3.779}$, $S_5(z) = \frac{2.404z^5-11.05z^4+21.22z^3-20.92z^2+10.44z-2.029}{z^5-6.718z^4+17.65z^3-23.87z^2+16.77z-5.07}$.

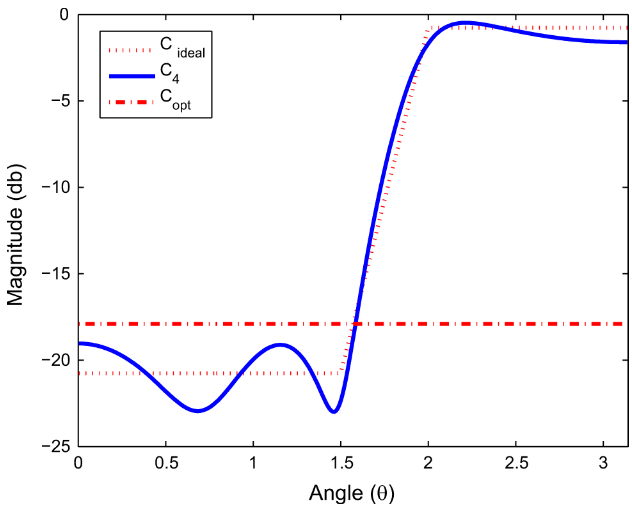

Fig. 5. Robustness radius obtained for the controllers C_{ideal}, C_4, and C_{opt}.

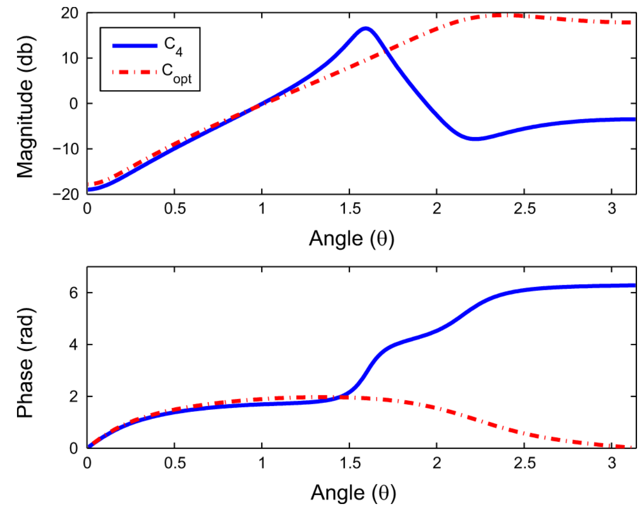

Fig. 6. Bode plots of controllers C_4, and C_{opt}.

and the corresponding controller C_r, for $r = 4$, by obtaining q from (12) and substituting into (8).

The uniform robustness radius for gap-metric uncertainty is maximal for an optimal choice of the controller C_{opt} and equals $b_{\mathrm{opt}}(P)$ ([23], [43]). The robustness radius is the inverse of the H_∞-norm of the "parallel projection" operator $\Pi_{P//C_{\mathrm{opt}}}$, and its value is shown in Fig. 5 plotted with a dash-dotted line (in logarithmic scale). On the other hand, the inverse of the maximal singular value of $\Pi_{P//C_{\mathrm{ideal}}}$, as a function of frequency is drawn with a dotted line, and represents a frequency-dependent robustness radius [43] having the shape of w. Both are now compared with the radius of the degree-four controller

$$C_4 = \frac{0.876z^4 - 0.190z^3 - 0.0669z^2 - 0.460z + 0.157}{z^4 + 0.1205z^3 + 1.389z^2 + 0.07538z + 0.2214}.$$

It is seen that there is a substantial improvement of robustness as compared to $b_{\mathrm{opt}}(P)$ in the high-frequency range for both the "ideal" design as well as for the low-degree approximation. Fig. 6 compares the gains of C_4 and C_{opt}. Similarly, Figs. 7 and 8 compare the loop-gains and the Nyquist plots, respectively, for the two cases. It is seen that some form of phase compensation is effected by C_4 around 1.6 rad/sec, as compared to C_{opt} so as to gain the sought advantage. Fig. 9 compares the gains of the four entries of the closed-loop transfer matrix $\Pi_{P//C}$. The figure shows that there is a slight improvement in the sensitivity function at the middle range at the expense of a slight degradation at low frequencies.

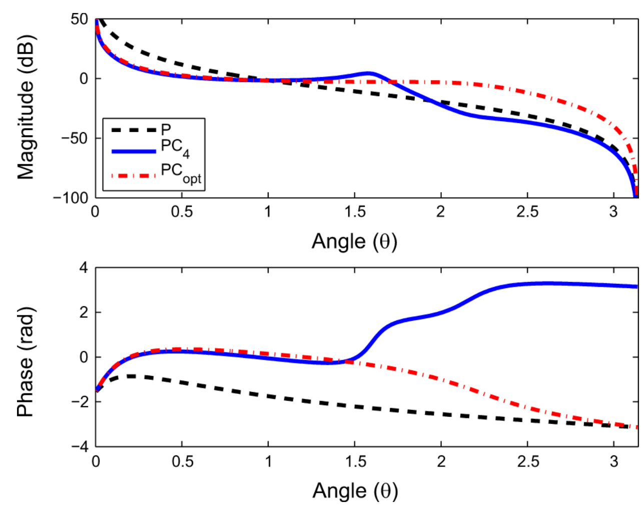

Fig. 7. Bode plots of P and of loop gains PC_4, and PC_{opt}.

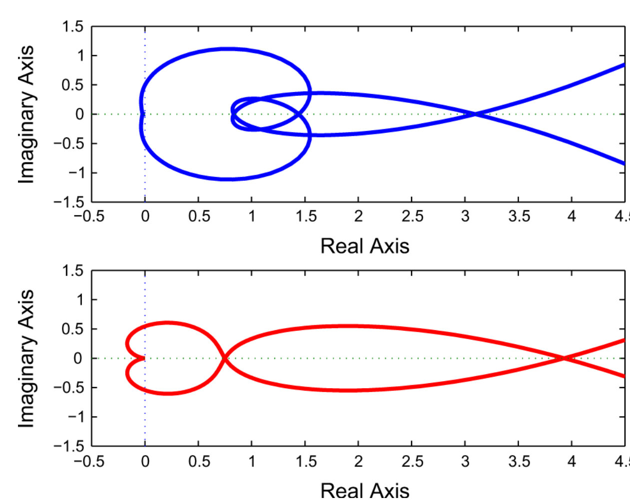

Fig. 8. Nyquist plots of the loop gains PC_4 (above) and PC_{opt} (below), respectively.

VII. Conclusion

The formulation of feedback control synthesis as an analytic interpolation problem has been at the heart of modern developments in robust control. Yet, many of the standard approaches often lead to designs of a large degree, due to degree inflation when introducing and absorbing "weights" into the controller. At various stages, alternative methodologies for dealing with control design under structural and dimensionality constraints were developed by several authors based primarily on suitable approximations and a linear matrix inequality formalism (see [3], [16], [24], [39]). In particular, a comparison between the viewpoint in Gahinet and Apkarian [16] and Skelton, Iwasaki, and Grigoriadis [39] and the viewpoint advocated in our work is provided in [22].

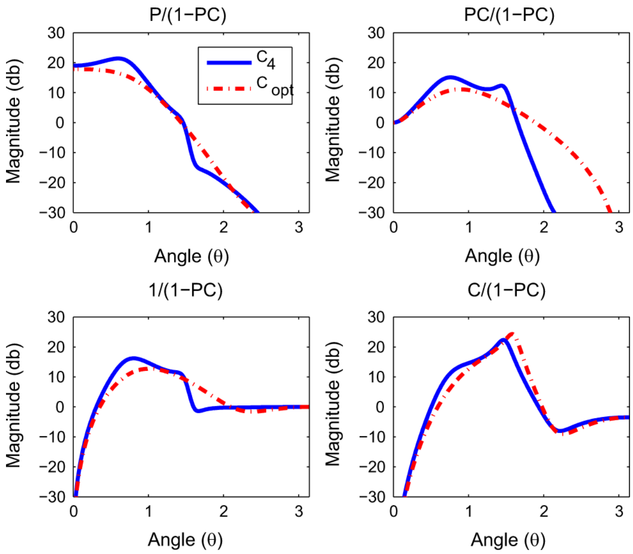

Fig. 9. Four closed-loop transfer functions of $\Pi_{P//C_4}$ and $\Pi_{P//C_{\rm opt}}$.

Our approach builds on the original H_∞-formulation of control synthesis as an analytic interpolation problem and on the recently discovered fact that, in contrast to H_∞-minimization, dimensionality and performance are inherited by the weighted-entropy minimization. In this setting, "weights" provide the means of shaping interpolants in a manner akin to H_∞ design. Thus, the advantage of the new methodology which involves entropy functionals stems from the fact that selection of weights within a specific class does not unduly penalize the degree of the design. However, the choice of weights is not immediate, as it is in the standard H_∞ paradigm [15]. The choice of weights that lead to acceptable controllers is, in itself, a non-convex optimization problem. Thus, one of the contributions of this paper is a relaxation of this non-convex problem into one which is quasi-convex, and thus solvable by standard methods. The methodology builds on a more fundamental question which forms a main theme of the paper, namely the characterization of all possible minimizers of weighted entropy functionals. The inverse problem of constructing weights for permissible minimizers is the basis for our design theory. In a more general context, the results of this paper provide a solution to the problem of determining which spectral zeros correspond to a certain desired shape of the interpolant.

This paper provides a considerable extension of the results presented in the conference paper [27]. The modified problem obtained by removing the *a priori* bound γ on the interpolants has been studied in [29]–[30]. In fact, by allowing the upper bound γ to tend to infinity, the entropy optimization problem becomes an H_2 optimization problem, and the interpolants are then parameterized in terms of poles rather than in terms of spectral zeros.

APPENDIX

A. Proof of Proposition 2

Given the controller parameterization (8), we have $C = U/V$, where $U = U_0 + M_0 q$ and $V = V_0 + N_0 q$. Since $U, V \in H_\infty$, the number of the distinct poles of U and V (counted with multiplicity, including poles at zero and at ∞) is at least as large as the degree of C. From (11) and (12) we have

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} \phi^* M_0 \\ \phi^* N_0 \end{pmatrix} F + \begin{pmatrix} -N_0^* \\ M_0^* \end{pmatrix}$$

and, since $M_0,\ N_0,\ \phi M_0^*,\ \phi N_0^* \in H_\infty$, any stable pole of U or V is a stable pole of F. However, U and V have no unstable poles, and therefore the degree of C is bounded by the degree of F, as claimed.

B. Proof of Theorem 3

The proof of Theorem 3 traces similar steps as that of Theorem 1 in [13]. Below we discuss modifications that are needed to handle the present situation. The proof is based on a series of lemmas, which run in parallel to those in [13], as we point out differences. For ease of reference we retain (modulo a sign change) the notation of [13], and for detailed proofs we refer to [26].

We start by showing that for any log-integrable weight Ψ, there exists a strictly contractive interpolant w with finite generalized entropy.

Lemma 11: Suppose that $\Psi \geq 0$ satisfies $\log \Psi \in L_1(\mathbb{T})$, and let P be the Pick matrix (2). Then, if $P > 0$, there exists a $w \in H_\infty$ satisfying (1), $\|w\|_\infty < 1$, and $\mathbb{K}_\Psi(w) < \infty$.

Proof: Let g_ϵ be the outer function such that $|g_\epsilon|^2 = \frac{1}{1+\epsilon+\epsilon\Psi}$, for $\epsilon > 0$. Since $g_\epsilon \to 1$ pointwise in $\mathbb{D}$ as $\epsilon \to 0$, and since $\mathbf{Pick}(1) = P > 0$, it is possible to choose ϵ small enough so that $\mathbf{Pick}(g_\epsilon) \geq 0$. Therefore, following the argument in Section IV-C, it is possible to find a w which satisfies (1) and for which $|w| \leq |g_\epsilon|$ on $\mathbb{T}$. The calculation

$$\begin{aligned}\mathbb{K}_\Psi(w) &\leq \mathbb{K}_\Psi(g_\epsilon)\\ &= \int_{\mathbb{T}} \Psi \log\left(\frac{1+\epsilon+\epsilon\Psi}{\epsilon+\epsilon\Psi}\right) dm\\ &\leq \int_{\mathbb{T}} \Psi \left(\frac{1+\epsilon+\epsilon\Psi}{\epsilon+\epsilon\Psi} - 1\right) dm\\ &\leq \frac{2\pi}{\epsilon}\end{aligned}$$

shows that $\mathbb{K}_\Psi(w) < \infty$. ■

With w fixed and satisfying the properties in Lemma 11, we define

$$X = \{v \mid \|w + \phi v\|_\infty \leq 1,\ \mathbb{K}_\Psi(w + \phi v) < \infty\}$$

where ϕ is the Blaschke product (5), and we consider the convex functional $F : X \to [0, \infty]$ given by

$$F(v) = -\int_{\mathbb{T}} |\sigma|^2 \log(1 - |w + \phi v|^2) dm \qquad (30)$$

where as before σ is outer and $|\sigma|^2 = \Psi$.

The statement of the following lemma is identical to Lemma 7 in [13], but the proof is adjusted to handle the case when $\Psi \notin L_1$.

Lemma 12: There exists a set $\mathcal{L}$ of continuous affine functionals on H_∞ of the form

$$\lambda(v) = \lambda_0 + \int_{\mathbb{T}} \Re e(hv)dm$$

with $\lambda_0 \in \mathbb{R}, h \in L^1$, such that

$$F(v) = \sup_{\lambda \in \mathcal{L}} \lambda(v) \text{ for every } v \in X. \tag{31}$$

Proof: Let $\mathcal{L}$ be the class of affine functionals on H_∞ defined via

$$\lambda_0 = -\int_{\mathbb{T}} \Psi_\epsilon \left(\log(1-(1-\epsilon)|w+\phi u|^2) + 2(1-\epsilon)\frac{|w+\phi u|^2 - \Re e((\overline{w+\phi u})w)}{1-(1-\epsilon)|w+\phi u|^2}\right) dm$$

and

$$h = \Psi_\epsilon \frac{2(1-\epsilon)(\overline{w+\phi u})\phi}{1-(1-\epsilon)|w+\phi u|^2}$$

where $\Psi_\epsilon := \min(|\sigma|^2, 1/\epsilon)$, for all $\epsilon \in (0,1)$ and $u \in X$. Equality (31) now follows in the same way as in the proof of Lemma 7 in [13]. The use of the family $\Psi_\epsilon, \epsilon \in (0,1)$, accounts for the situation where Ψ is not in L_1. ■

This leads to the main lemma, the proof of which follows verbatim from [13, p. 974], using Lemma 12.

Main Lemma 13: The optimization problem (4) has a unique minimizer on X.

Lemma 14: Let $f = w + \phi v$, where $w \in H_\infty$ is as in Lemma 11 and $v = \arg\min(F)$. Then

$$\frac{|\sigma f|^2}{1-|f|^2} \in L_1(\mathbb{T}). \tag{32}$$

Proof: By noting that $\int_{\mathbb{T}} |\sigma f|^2 dm \le \mathbb{K}_\Psi(f) < \infty$ and $\int_{\mathbb{T}} |\sigma w|^2 dm \le \mathbb{K}_\Psi(w) < \infty$, the proof follows along the same lines as Lemma 12 in [13]. ■

Condition (32) implies that $\int_{\mathbb{T}} \log \frac{|\sigma f|^2}{1-|f|^2} dm < +\infty$. Since $\sigma \in H_2$ and $f \in H_\infty$, $\log|\sigma f|^2 \in L_1(\mathbb{T})$. Now since $|f| \le 1$, $\int_{\mathbb{T}} \log \frac{|\sigma f|^2}{1-|f|^2} dm > -\infty$. It follows that $\log \frac{|\sigma f|^2}{1-|f|^2} \in L_1(\mathbb{T})$. Hence

$$\log(1-|f|^2) \in L_1(\mathbb{T}). \tag{33}$$

We conclude that $\log \frac{|\sigma|^2}{1-|f|^2} \in L_1(\mathbb{T})$ from which it follows that there is a unique outer factor a with $a(0) > 0$ such that

$$|a|^2 = \frac{|\sigma|^2}{1-|f|^2}.$$

Following [13], we define $b := fa$, which belongs to H_2 by Lemma 14. Then a and b satisfy

$$f = \frac{b}{a},$$
$$|\sigma|^2 = |a|^2 - |b|^2.$$

Next, using the following lemma we show that $b \in \mathcal{K}$.

Lemma 15: Let $f = w + \phi v$, where $w \in H_\infty$ is as above and $v = \arg\min(F)$, and let f_0 be the outer part of f. Then

$$\frac{|\sigma|^2 f_0 f^* \phi}{1-|f|^2} \in zH_1(\mathbb{D}).$$

Proof: In view of (33), there is an outer function g such that $|g|^2 = 1 - |f|^2$. Then for any $h \in H_\infty$ we have that $v + tg^2 f_0 h \in X$ for $t \in (-\epsilon, \epsilon)$ and ϵ sufficiently small (as in the proof of [13, Lemma 13]).

Again following the proof of Lemma 13 in [13], since $v = \arg\min(F)$, the derivative of F must be zero at v in the directions $g^2 f_0 h$ for arbitrary $h \in H_\infty$ and g the outer spectral factor of $1-|f|^2$; i.e.,

$$\int_{\mathbb{T}} |\sigma|^2 \frac{\Re e\{f^* \phi g^2 f_0 h\}}{1-|f|^2} dm = 0$$

for all $h \in H_\infty$. Therefore

$$|\sigma|^2 \frac{f^* \phi g^2 f_0}{1-|f|^2} \in zH_1(\mathbb{D}).$$

Since g is outer and $|\sigma|^2 \frac{f^* \phi f_0}{1-|f|^2} \in L_1$, Lemma 15 follows. ■

By Lemma 14, we have that $b = af = \frac{\sigma f}{g} \in H_2$. By Lemma 15, $\frac{b^* \sigma f_0 \phi}{g} = zq$ where $q \in H_1$, and hence $\frac{b^* \phi}{z} = \frac{qg}{\sigma f_0} \in H_2$. Since $b \in H_2$ and $\frac{b^*\phi}{z} \in H_2$ it follows that $b \in \mathcal{K}$. Finally, since $a = \frac{\sigma}{g}$, the minimizing interpolant f is of the form stated in Theorem 3.

We have thus established that the minimizer exists and satisfies (i), (ii), and (iii). It remains to prove that there is only one function $f \in \mathcal{S}$ satisfying (i), (ii), and (iii). This follows directly from the arguments in [13, p. 977] and by noting that

$$|\sigma|^2 \frac{|f_1^*(f_1-f_2)|}{1-|f_1|^2} \le \frac{|\sigma f_1|^2}{1-|f_1|^2} + \frac{|\sigma f_2|^2}{1-|f_2|^2} \in L_1$$

and hence that

$$|\sigma|^2 \frac{2\Re e\{f_1^*(f_1-f_2)\}}{1-|f_1|^2} \in L_1.$$

This concludes the proof of Theorem 3.

C. Proof of Proposition 8

We first prove a somewhat more general statement.

Lemma 16: Under the assumptions of Proposition 8

$$\int_{\mathbb{T}} \Psi \log\left(1 + \frac{|f-f_r|^2}{2}\right) dm \le (e^{2\epsilon}-1)\mathbb{K}_\Psi(f). \tag{34}$$

Proof: Since $d(\Psi, \Psi_r) = \epsilon$

$$e^{-\epsilon}\Psi(z) \le \Psi_r(z) \le e^{\epsilon}\Psi(z), \text{ for } z \in \mathbb{T}$$

and hence

$$\begin{aligned}\mathbb{K}_\Psi(f_r) &\le e^\epsilon \mathbb{K}_{\Psi_r}(f_r)\\ &\le e^\epsilon \mathbb{K}_{\Psi_r}(f) \le e^{2\epsilon}\mathbb{K}_\Psi(f).\end{aligned}$$

As in [26, p. 33]

$$\frac{1}{2}(\mathbb{K}_\Psi(f)+\mathbb{K}_\Psi(f_r)) \geq \mathbb{K}_\Psi\left(\frac{f+f_r}{2}\right) + \frac{1}{2}\int_{\mathbb{T}} \Psi \log\left(1+\frac{|f-f_r|^2}{2}\right) dm.$$

Then, since f is the minimizer of $\mathbb{K}_\Psi$, we have

$$\mathbb{K}_\Psi(f) \leq \mathbb{K}_\Psi\left(\frac{f+f_r}{2}\right)$$

and consequently

$$\int_{\mathbb{T}} \Psi \log\left(1+\frac{|f-f_r|^2}{2}\right) dm \leq \mathbb{K}_\Psi(f_r) - \mathbb{K}_\Psi(f) \leq (e^{2\epsilon}-1)\mathbb{K}_\Psi(f)$$

as claimed. ■

Returning to Proposition 8, since $\log(1+t) \geq t/2$ for $t \in [0,2]$

$$\int_{\mathbb{T}} \Psi \log\left(1+\frac{|f-f_r|^2}{2}\right) dm \geq \frac{1}{2}\int_{\mathbb{T}} \Psi |f-f_r|^2 dm.$$

Then (22) follows from Lemma 16. This concludes the proof of Proposition 8.

D. *Proof of Corollary 10*

From the above proof of Theorem 3, $b = f\sigma/g$ with σ the outer factor of Ψ and g the outer factor of $1-|f|^2$. Similarly $b_k = f_k\sigma_k/g_k$ where σ_k, g_k are outer, $|\sigma_k|^2 = \Psi_k$ and $|g_k|^2 = 1-|f_k|^2$. Since $d(\Psi,\Psi_k) \to 0$, $f_k \to f$ in H_2 (Corollary 9). Consequently, $1-|f_k|^2 \to 1-|f|^2$ in L_1 and hence $g_k \to g$ in H_2. This follows from [5, page 767], and the fact that $\{1-|f_k|^2\}_k$ are log-integrable. Now, since $b, b_k \in \mathcal{K}, b = \beta/\tau, b_k = \beta_k/\tau$ with $\beta, \beta_k \in \mathrm{Pol}(n)$, we conclude that $g_k\beta_k = f_k\sigma_k\tau$. The coefficientwise convergence $b_k \to b$ now follows, since $g_k \to g,\ f_k \to f$ and $\sigma_k \to \sigma$ and g is not identically zero.

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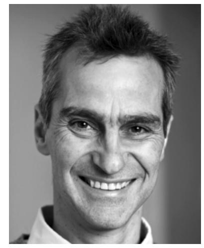

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