The Inverse Problem of Analytic Interpolation with Degree Constraint

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***Abstract*— In [7], [6] a theory for degree constrained interpolation has been developed by exploring the minimizers of certain convex entropy functionals. In the present paper, we introduce and study relevant inverse problems. More specifically, we answer the following two questions. First, given a function f which satisfies specified interpolation conditions, when is it that f can be obtained as the minimizer of a suitably chosen entropy functional? Second, given a function g, when does there exist a suitably entropy functional so that the unique minimizer f which is subject to interpolation constraints also satisfies $|f| = |g|$ on the unit circle. The theory and answers to these questions suggest an approach to identifying interpolants of a given degree and of a given approximate shape.**

I. Introduction

The topic of this paper relates to the framework and the mathematics of modern robust control. Although a theoretical and computational approach which uses a state-space formalism exists, certain fundamental questions appear to be easier to deal with in the framework of analytic function theory. Indeed, the foundational work of George Zames in the early 1980's cast the basic robust control problem as an analytic interpolation one, where the interpolation constraints are there to ensure stability of the feedback scheme while a norm bound guarantees performance and robustness. This is the setting which motivated the question of characterizing analytic interpolants with a degree constraint (see [7], [6]). The theory in [7], [6] utilizes the functional form of minimizers of certain weighted entropy functionals. Conveniently, these minimizers are rational and of a certain generic McMillan degree (see also [12]).

Naturally, the quality of control depends on the frequency characteristics of the interpolants, which in turn depends heavily on suitable weight selection. The issue of how the choice of any weights and indices affects the final design is by no means unique here. It was R.E. Kalman [13] who, in the context of quadratic optimal control, first raised the question of what is it that characterizes optimal designs and further, how to describe all performance criteria for which a certain design is optimal. Following Kalman's example we pursue here the corresponding inverse problem for the theory of analytic interpolation with complexity constraint (as developed in [7], [6]).

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In Section II we establish notation as well as review basic facts in analytic interpolation and in complexity constrained interpolation. We only discuss interpolation on the unit disc $\mathbb{D} = \{z : |z| < 1\}$. In Section III we formulate the inverse problem and present our main results. In Section IV we present a method for degree reduction of interpolants, and in Section V we illustrate the application and insights gained by our main results as well as the degree reduction method of Section IV. Finally, in Appendix VII we collect the proofs of the various propositions.

II. Background

Given complex numbers $z_0, z_1, \ldots, z_n$ in $\mathbb{D}$ and complex numbers $w_0, w_1, \ldots, w_n$, the classical Pick interpolation problem asks for a function f in the *Schur class*

$$\mathcal{S} = \{f \in H_\infty(\mathbb{D}) \,:\, \|f\|_\infty \leq 1\}$$

which satisfies the interpolation condition

$$f(z_k) = w_k, \quad k = 0, 1, \ldots, n. \tag{1}$$

It is well-known (see, e.g., [8]) that such a function exists if and only if the Pick matrix

$$P = \left[\frac{1 - w_k \bar{w}_\ell}{1 - z_k \bar{z}_\ell}\right]_{k,\ell=0}^{n} \tag{2}$$

is positive semi-definite. The solution is unique if and only if P is singular, in which case, f is a Blaschke product of degree equal to the rank of P. In this paper, throughout, we assume that P is positive definite and hence, that there are infinitely many solutions to the Pick problem. A complete parameterization of all solutions was given by Nevanlinna (see e.g. [1]), and for this reason the subject is often referred to as Nevanlinna-Pick interpolation.

In engineering applications f usually represents the transfer function of a feedback control system or of a filter, and therefore the McMillan degree of f is of significant interest. Thus, it is naturally to require that f be rational and of bounded degree. Such a constraint completely changes the nature of the underlying mathematical problem.

The Nevanlinna-Pick theory provides a single solution, the so-called *central solution*, which is rational and of a generic degree equal to n. It provides no insight and no help in determining any other possible solutions of the same degree. This central solution is also referred to as the *maximum-entropy solution* because it maximizes the functional

$$\int_{\mathbb{T}} \log(1 - |f|^2) dm$$

subject to (1), where $\mathbb{T} = \{z : |z| = 1\}$ is the unit circle and m is the normalized Lebesgue measure on $\mathbb{T}$. Determining extremals for such an entropy functional leads to a set of linear equations (canonical equation) which can be expressed and solved in considerable generality in state space form [15].

Following [7], [12], we consider the more general entropy functional

$$\mathbb{K}_\Psi : \mathcal{S} \to \mathbb{R} \cup \infty, \quad \mathbb{K}_\Psi(f) = -\int_{\mathbb{T}} \Psi \log(1 - |f|^2) dm,$$

where Ψ is a function that generally only needs to be integrable and positive on $\mathbb{T}$. We study how the minimizers of

$$\min \mathbb{K}_\Psi(f) \;\; \text{s.t.} \;\; f(z_k) = w_k, \; k = 0, \ldots, n, \; f \in \mathcal{S}, \quad (3)$$

depend on the weighting function Ψ and then determine when an interpolant f is attainable as a minimizer of (3) for a suitable choice of Ψ. One particularly interesting case, as we will see below, is when $\Psi = |\sigma|^2$ and σ belongs to the class of rational functions with poles at the conjugate inverses of the interpolation points.

Let

$$\phi = \prod_{k=0}^{n} \frac{z_k - z}{1 - \bar{z}_k z}$$

and let $U : f(z) \to z f(z)$ denote the standard shift operator on H_2. Then ϕH_2 is a shift invariant subspace, i.e. $f \in \phi H_2$ imply that $U(f) = zf \in \phi H_2$. Denote by $\mathcal{K}$ the co-invariant subspace $H_2 \ominus \phi H_2$. Then $\mathcal{K}$ is invariant under U^*, where U^* denotes the adjoint of U. Let $\mathcal{K}_0$ denote the set of outer functions in $\mathcal{K}$ that are positive in the origin. The following result is taken from [7].

Theorem 1: Suppose that the Pick matrix (2) is positive definite, and let σ be an arbitrary function in $\mathcal{K}_0$. Then there exists a unique pair of elements $(a, b) \in \mathcal{K}_0 \times \mathcal{K}$ such that

(i) $f = b/a \in H^\infty$ with $\|f\|_\infty \le 1$

(ii) $f(z_k) = w_k, \quad k = 0, 1, \ldots, n$, and

(iii) $|a|^2 - |b|^2 = |\sigma|^2$ a.e. on $\mathbb{T}$.

Conversely, any pair $(a, b) \in \mathcal{K}_0 \times \mathcal{K}$ satisfying (i) and (ii) determines, via (iii), a unique $\sigma \in \mathcal{K}_0$. Moreover, setting $\Psi = |\sigma|^2$, the optimization problem

$$\min \mathbb{K}_\Psi(f) \;\; \text{s.t.} \;\; f(z_k) = w_k, \; k = 0, \ldots, n$$

has a unique solution f that is precisely the unique $f \in \mathcal{S}$ satisfying conditions (i), (ii) and (iii).

In the above, $\mathcal{K}$ consists of all rational functions $\rho(z)/\tau(z)$, where ρ is a polynomial of degree at most n with all its roots outside the unit disc $\mathbb{D}$ and

$$\tau(z) = \prod_{k=0}^{n} (1 - \bar{z}_k z).$$

The n roots of the polynomial $z^n \rho(z^{-1})$ are referred to as *spectral zeros*.

Theorem 1 has two parts: The first part implies that the interpolants of degree at most n are completely parameterized in terms of the spectral zeros; i.e., there is bijection between the class of pairs (b, a) so that $f = b/a$ is an interpolant of degree at most n, and the set of n points in the unit disc. Given an arbitrary choice of spectral zeros, the second part provides a convex optimization problem, the unique solution of which is precisely the corresponding interpolant. If an interpolation condition is specified at 0, i.e., $z_0 = 0$, then $\sigma \equiv 1 \in \mathcal{K}_0$, and the central solution is the one for which $\Psi \equiv 1$ and $\rho = \tau$. The corresponding spectral zeros are at the conjugate inverse (mirror image in unit circle) of $\{z_k\}_{k=1}^n$.

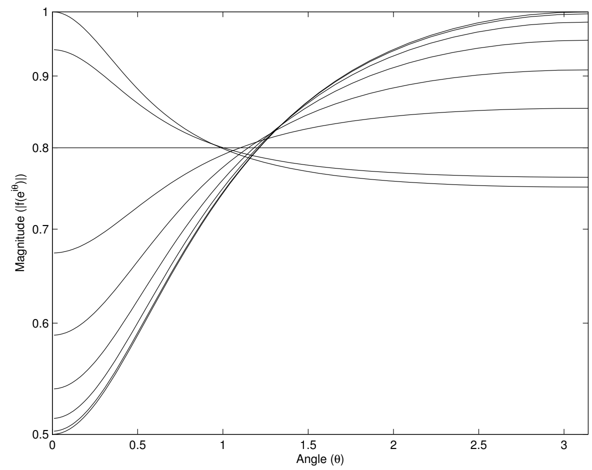

Fig. 1. *Possible shapes of degree 1 interpolants*

The theorem is stated in [7], allowing for considerably more general interpolation conditions than (ii). The special case where $\phi = z^n$ is analogous to the so-called covariance extension problem with degree constraints, which is stated for Charathéodory functions rather than Schur functions. The background to the derivation of Theorem 1 has a long history. The existence part of the parameterization was first proved in the covariance extension case in [9], [10] the uniqueness part (as well as well-posedness) in [5]. The optimization approach was initiated in [3] (also, see the extended version [4]) and further developed in, e.g., [6], [2], [11].

To motivate the basic inverse problem let us consider a simple toy example in sensitivity shaping taken from [12], which leads to a problem with $(z_0, w_0) = (0, 0.4)$ and $(z_1, w_1) = (0.5, 0)$.[1] The class of all interpolants of degree at most one are parameterized by one spectral zero. Figure 1 depicts the modulus of the interpolant f as the parameter ranges over the interval $(-1, 1)$. We want to chose the shape that best satisfies additional design specifications.

An inspection of Figure 1, shows that certain prespecified shapes $|f|$ are possible to achieve with an interpolant of degree one, others are not. In the next section we will consider whether a particular shape is attainable for an entropy minimizer. If it is attainable, we address the question of how to determine the corresponding entropy functional to which it is the minimizer.

[1]Since $\|S\|_\infty \le \frac{5}{2}$ in [12], we consider $f = \frac{2S}{5}$.

III. The Inverse Problem

Let g be a prespecified function on $\mathbb{T}$. When does there exist a positive function Ψ on $\mathbb{T}$ such that the unique solution f to (3) satisfies

$$|f(e^{i\theta})| = |g(e^{i\theta})|, \quad \theta \in (-\pi, \pi]? \tag{4}$$

Without loss of generality we may assume that g is an outer function an that, moreover, $g \in \mathcal{S}$. Before addressing (4) we will consider the following relaxed version:

$$|f(e^{i\theta})| \leq |g(e^{i\theta})|, \quad \theta \in (-\pi, \pi]. \tag{5}$$

This is in the form of a typical design specification for e.g., sensitivity shaping in a control problem. We seek "weighting" functions g for which there exists analytic functions $f \in \mathcal{S}$ satisfying (5) and (1). Given g, such f exists if and only if

$$\{f \in \mathcal{S} : \|fg^{-1}\|_\infty \leq 1, f(z_k) = w_k, k = 0, 1, \ldots, n\}$$

is nonempty, or equivalently, by setting $\chi = fg^{-1}$, if and only if

$$\{\chi \in \mathcal{S} : \chi(z_k) = w_k g(z_k)^{-1}, k = 0, 1, \ldots, n\} \tag{6}$$

is nonempty. The family (6) is nonempty if and only if the associated Pick-matrix is positive semi-definite

$$P(g) := \left[\frac{1 - w_k g(z_k)^{-1}\overline{w_l g(z_l)^{-1}}}{1 - z_k \overline{z_l}}\right]_{k,l=0}^{n}.$$

Clearly, the family (6) is a singleton if $P(g)$ is positive semi-definite and singular.

From this easy fact, it follows that a necessary condition for (3) to have a solution satisfying (4) is that $P(g)$ is positive semidefinite. However, if the matrix $P(g)$ is strictly positive definite, there exist interpolants $\hat{f}$ such that $|\hat{f}| < |g|$ in $\mathbb{T}$ and the design specifications (5) may be satisfied with strict inequality. Therefore, a minimizing interpolant f cannot satisfy (4), since $|\hat{f}| < |f|$ on $\mathbb{T}$ implies that $\mathbb{K}_\Psi(\hat{f}) < \mathbb{K}_\Psi(f)$ and contradicts the claim that f is the minimizer. Therefore the following proposition holds.

Proposition 2: Let f be the solution of the optimization problem (3), and suppose that g is the outer part of f (i.e. g is outer and $|f| = |g|$ a.e. on $\mathbb{T}$). Then

$$\left[\frac{1 - w_k g(z_k)^{-1}\overline{w_l g(z_l)^{-1}}}{1 - z_k \overline{z_l}}\right]_{k,l=0}^{n} \tag{7}$$

is positive semidefinite and singular.

Proof: The proof was outlined in the argument leading to the proposition. ∎

If g satisfies (7) and in addition it is rational with no zeros on $\mathbb{T}$, then it is possible to construct an entropy functional so that the outer part of the minimizer is precisely g. This is our main theorem.

Theorem 3: Let $g \in \mathcal{S}$ be a rational function with zeros and poles outside the closed unit circle $\bar{\mathbb{D}}$. A pair (Ψ, f) (of functions on $\mathbb{T}$) exists such that

1) Ψ is positive,
2) f is the solution of (3), and
3) $|f| = |g|$ on $\mathbb{T}$

if and only if $P(g)$ is positive semidefinite and singular. Furthermore, $\Psi = |\sigma p|^2$, where $\sigma \in \mathcal{K}$ and $\deg p \leq \deg g$.

Proof: See the appendix. ∎

The positivity of (7) is closely related to the number of zeros in the interpolant, or equivalently the degree of the inner factor of the interpolant (see Lemma 2). This observation can used to determine whether an interpolant is the minimizer of (3).

Theorem 4: Let $f \in \mathcal{S}$ be an rational function satisfying (1) and let f have at most n zeros in $\bar{\mathbb{D}}$. Then there exists a function Ψ such that f is the unique solution of (3).

Proof: See Appendix VII. ∎

Conversely, if f has more than n zeros in $\mathbb{D}$ then it is does not arise as a solution to (3) as state below.

Theorem 5: Let f be a function satisfying (1) and let f have more than n zeros in $\mathbb{D}$. Then there does not exist a positive function Ψ on $\mathbb{T}$ so that f is the solution of (3).

Proof: See Appendix VII. ∎

IV. An Application: Degree Reduction

The focus in [7], [6] has been on parametrizing interpolants of low degree. In this section we combine this theory with the solution of the inverse problem to obtain a method for degree reduction of interpolants.

Let f be an interpolant satisfying the conditions of Theorem 4. By Theorem 4, there is a functions Ψ such that

$$f = \arg\min \mathbb{K}_\Psi(f) \text{ s.t. } f(z_k) = w_k, k = 0, 1, \ldots, n, f \in \mathcal{S}.$$

If we choose $\hat{\Psi}$ close to Ψ, the corresponding solution $\hat{f}$ is also close to f. This is a consequence of the continuity of the minimizer on Ψ as stated below.

Theorem 6: The mapping $C(\mathbb{T})_+ \to H_2$ given by $\Psi \to f$, where f is the minimizer of (3), is continuous.

Proof: Let $\Psi_\ell \to \Psi$ in ∞-norm. Let f be the minimizer of (3) and let f_ℓ be the minimizers of $\mathbb{K}_{\Psi_\ell}(f_\ell)$ subject to $\{f_\ell(z_k) = w_k,\ k = 0, 1, \ldots, n$ and $f_\ell \in \mathcal{S}\}$. Then there exists a sequence ϵ_ℓ such that

$$(1 - \epsilon_\ell)\Psi \leq \Psi_\ell \leq (1 + \epsilon_\ell)\Psi,\ \epsilon_\ell \to 0 \text{ as } \ell \to \infty.$$

Since f_ℓ and f are entropy minimizers we have

$$\begin{aligned}
\mathbb{K}_\Psi(f) &\leq \mathbb{K}_\Psi(f_\ell) \leq \frac{1}{1-\epsilon_\ell}\mathbb{K}_{\Psi_\ell}(f_\ell) \\
&\leq \frac{1}{1-\epsilon_\ell}\mathbb{K}_{\Psi_\ell}(f) \\
&\leq \frac{1+\epsilon_\ell}{1-\epsilon_\ell}\mathbb{K}_\Psi(f) \to \mathbb{K}_\Psi(f) \text{ as } \ell \to \infty.
\end{aligned}$$

Hence $\mathbb{K}_\Psi(f_\ell) \to \mathbb{K}_\Psi(f)$. By [14, Theorem 4] we have $f_\ell \to f$ in H_2, which concludes the proof. ∎

From Theorem 1 we know that if $\hat{\Psi} = |\hat{\sigma}|^2$, $\hat{\sigma} \in \mathcal{K}$, then the degree of the corresponding minimizer $\hat{f}$ is less or equal to n. Therefore, by choosing $\hat{\Psi}$ in this class, we force the degree of the interpolant to be bounded by n.

Our interest is in selecting interpolants as minimizers of entropy functionals while "shaping" the minimizer indirectly by a suitable choice the weighting function. Initially we may begin with a desired shape $|g|$ and obtain an interpolant f and Ψ as before. Clearly, there is no guarantee that the dimension of f will be acceptable. Yet, we may focus on the parameter Ψ of the optimization problem in order to develop an efficient search for interpolants of a given dimension and approximate shape. Thus, the current framework suggests approximating Ψ by $\hat{\Psi}$ in a suitable class (e.g., with fixed poles). Experience suggests that the dynamic range of Ψ is what affects the minimizer the most, and thereby leads us to consider the approximation $\hat{\Psi}$ in the suitable class and so that $\|\log\Psi - \log\hat{\Psi}\|_\infty$ is minimal, or so that $\|1-\frac{\hat{\Psi}}{\Psi}\|_\infty$ is minimal instead. The latter optimization problem is linear in the coefficients of the numerator of $\hat{\Psi}$ and can be solved quite efficiently. In fact, a methodology using quasi-convex optimization is given in [16] (cf. also [17]).

In cases where there is no acceptable "shape" with an interpolant of degree n, we may conveniently consider $\hat{\Psi}$ having additional poles, i.e. $\hat{\Psi} = |\hat{\sigma}p|$, where $\sigma \in \mathcal{K}$ and p is a rational function with $\deg p \leq m$. From Theorem 1 and Lemma 1, the degree of $\hat{f}$ is bounded by $n+m$.

Integrating the above insights and arguments, we conclude with the following algorithm for shaping interpolants while allowing a certain control on their degree:

1) Let g be a desired shape satisfying the requirements of Theorem 3 and such that $P(g)$ is positive definite and singular.
2) In accordance with Theorem 3, we construct the pair (Ψ, f) so that f is the minimizer of $\mathbb{K}_\Psi(f)$ subject to (1) and so that $|f| = |g|$ on $\mathbb{T}$.
3) Let $\hat{\Psi}$ minimize $\|1-\frac{\hat{\Psi}}{\Psi}\|_\infty$, subject to $\hat{\Psi} = |\hat{\sigma}p|^2$ where $\hat{\sigma} \in \mathcal{K}, \deg p \leq m$.
4) Let $\hat{f}$ be the minimizer of (3) using $\hat{\Psi}$. Then $\hat{f}$ is an interpolant with degree bounded by $n+m$.

V. Case Study

Consider the following interpolation problem

$$f(-0.9) = 0.4, f(0) = 0.4, f(0.8) = 0, \text{ and } f \in \mathcal{S}. \quad (8)$$

Suppose further, design specifications suggest a desirable shape for the interpolant given as

$$|f(z)| = k\left|\frac{z-1.2}{z-7}\right| \text{ on } \mathbb{T}$$

where k a constant of a "small" value. For $k = 3.0197$ and

$$g = k\frac{z-1.2}{z-7},$$

$P(g)$ is positive semidefinite and singular. Therefore, g satisfies the requirements of Theorem 3 and, following the steps in the proof of Theorem 3, we may construct a pair (Ψ, f) such that $|f| = |g|$ on $\mathbb{T}$ and

$$f = \arg\min \mathbb{K}_\Psi(f) \text{ subject to (8)}.$$

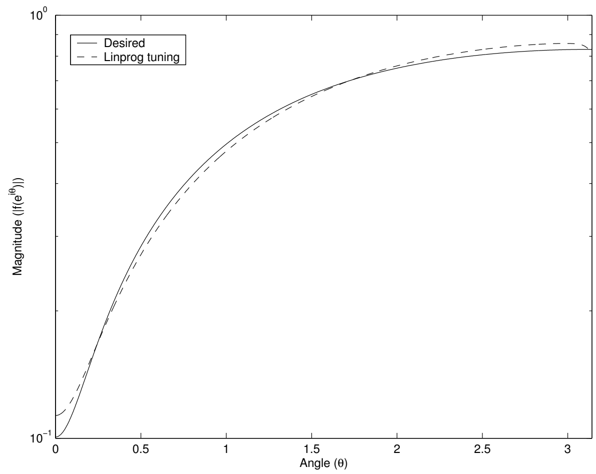

Fig. 2. *The interpolants*

Indeed, these functions turn out to be

$$\begin{aligned} f &= \frac{0.4314z^3 - 0.4461z^2 - 0.4192z + 0.4000}{0.1104z^3 - 0.7964z^2 + 0.0230z + 1.000}, \\ \Psi &= \left|\frac{(z+0.1450)(z+0.9659)}{(z+0.9000)(z-0.8333)}\right|^2, \end{aligned}$$

and their magnitude on the circle drawn with solid lines in Figure 2 and Figure 3.

We follow the method in Section IV in seeking an interpolant $\hat{f}$ "close" f and of degree 2. We approximate Ψ by the minimizer of

$$\left\|1 - \frac{\hat{\Psi}}{\Psi}\right\|_\infty \text{ s.t. } \hat{\Psi} = |\hat{\sigma}|^2, \hat{\sigma} \in \mathcal{K}, \quad (9)$$

(discretizing on a grid of 2000 points). Since $\hat{\Psi} = \frac{d}{|\tau|^2}$, where d is a trigonometric polynomial of degree 2, (9) can be solved using linear programming. The solution is

$$\hat{\Psi} = \left|\frac{(z+0.2467)(z+0.9591)}{(z+0.9000)(z-0.8000)}\right|^2.$$

Finally, using $\hat{\Psi}$ in (3) in the place of Ψ, the minimizing interpolant turns out to be

$$\hat{f} = \frac{0.5182z^2 + 0.0855z - 0.4000}{0.0805z^2 - 0.8798z - 1.0000}.$$

The magnitude of $\hat{f}$ and Ψ are shown with dashed lines in Figure 2 and Figure 3, respectively.

This example illustrates the method of Section IV for identifying interpolants of an approximate shape and of lower degree.

VI. Concluding Remarks

In the first part of the paper we consider the general problem of analytic interpolation with degree constraint and introduce the relevant inverse problem. It is known that interpolants of bounded degree arise as minimizers of suitable weighted entropy functionals [7], [6], and thus, we characterize optimizers and relevant weights that produce

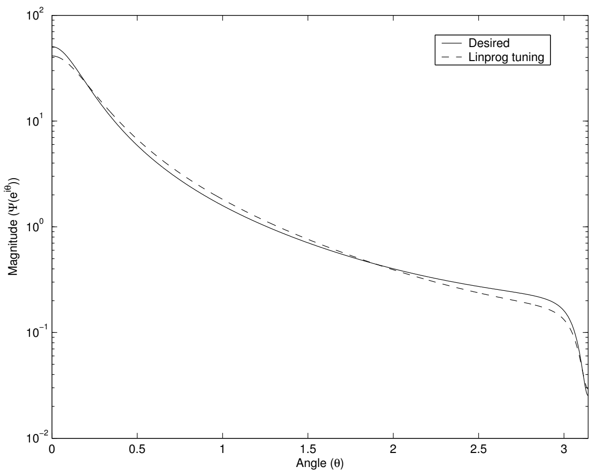

Fig. 3. *The weights*

those needed functionals. In the second part of the paper, we integrate insights and conclusions of the earlier theory to suggest a method for seeking interpolants of an approximate shape (dictated by design specifications) and of bounded degree. We illustrate the use of the method by a simple representative example.

VII. Appendix: Proofs and Lemmas

Lemma 1: Let f be the minimizer of (3), $\Lambda \subset \mathbb{D}$ contain a finite number of points, $B_\Lambda := \prod_{\lambda\in\Lambda} \frac{\lambda - z}{1-\bar{\lambda}z}$ and let $\hat{f}$ be the minimizer of

$$\begin{aligned} \mathbb{K}_\Psi(f) \quad \text{s.t.} \quad & f(z_k) = w_k B_\Lambda(z_k),\ k = 0,\ldots,n, \\ \text{and} \quad & f(\lambda) = 0,\ \lambda\in\Lambda. \end{aligned} \tag{10}$$

Further assume that $\{z_k\}_{k=0}^n \cap \Lambda = \emptyset$. Then $\hat{f} = B_\Lambda f$.

Proof: Clearly $\mathbb{K}_\Psi(f) = \mathbb{K}_\Psi(B_\Lambda f)$, and since $B_\Lambda f$ satisfies the interpolation conditions in (10), we have that $\mathbb{K}_\Psi(f) \geq \mathbb{K}_\Psi(\hat{f})$. Assume that $\mathbb{K}_\Psi(f) > \mathbb{K}_\Psi(\hat{f})$, but then $\hat{f}/B_\Lambda$ satisfies the conditions in (3) and $\mathbb{K}_\Psi(f) > \mathbb{K}_\Psi(\hat{f}/B_\Lambda)$, which contradicts that f is the solution of (3). This gives $\mathbb{K}_\Psi(f) = \mathbb{K}_\Psi(\hat{f})$. Since the minimizer is unique, $\hat{f} = B_\Lambda f$. ■

Remark 1: The assumption that $\{z_k\}_{k=0}^n \cap \Lambda = \emptyset$ as well as the notation suggestion z_k's as being discrete are only for simplicity and can easily be removed.

Proof: of Theorem 4: Let the set Γ be the set of zeros of f in $\bar{\mathbb{D}}^c$ (the complement of $\bar{\mathbb{D}}$, including the ones at infinity and counted with multiplicity), and let Λ be the reflection of Γ with respect to the unit citcle. Then $\hat{f} = B_\Lambda f$ satisfies

$$\hat{f}(z_k) = w_k B_\Lambda(z_k),\ k = 0,\ldots,n, \ \text{ and } \hat{f}(\lambda) = 0,\ \lambda\in\Lambda, \tag{11}$$

with the obvious modification for derivative conditions in the case of interpolation points with multiplicity. Since $\deg \hat{f} \leq m + n$, where m is the number of elements in Λ, $\hat{f}$ is the minimizer of $\min \mathbb{K}_\Psi(\hat{f})$ subject to (11). Here $\Psi = aa^* - bb^*$ where $\hat{f} = \frac{b}{a}$ and b and a belong to the co-invariant subspace

$$\hat{\mathcal{K}} = H_2 \ominus \hat{\phi} H_2 \text{ with } \hat{\phi} = \prod_{k=0}^n \frac{z_k - z}{1 - \bar{z}_k z} \times \prod_{\lambda\in\Lambda} \frac{\lambda - z}{1-\bar{\lambda} z}.$$

Note that $\Psi = |p\sigma|^2$, where $\sigma \in \mathcal{K}$ and $\deg p \leq |\Lambda|$. By Lemma 1 f is the solution of (3). ■

We now turn to the proof of our main theorem.

Proof: of Theorem 3:
($\Rightarrow$) Sufficiency follows from Proposition 2.
($\Leftarrow$) Since the matrix in (7) is nonnegative definite and singular, there is a unique f satisfying $\|fg^{-1}\| \leq 1$ and (1). Then $f = g\varphi$ where φ is inner and of degree $\leq n$. Since f is rational with at most n zeros in $\bar{\mathbb{D}}$, by Theorem 4 there exists a functions Ψ such that f is the minimizer of (3). Since f has at most $\deg g$ zeros outside $\bar{\mathbb{D}}$, it follows (see proof of Theorem 4) that $\Psi = |\sigma p|^2$, where $\sigma \in \mathcal{K}$ and $\deg p \leq \deg g$. ■

Lemma 2: Let φ be the H_∞-norm minimizing interpolant satisfying (1). Then φ is the unique function on the form αB_Λ, $\alpha \in \mathbb{C}$, and B_Λ a blaschke product of degree $\leq n$ satisfying (1).

Proof: First note that φ is of the form αB_Λ where B_Λ is a blaschke product of degree $\leq n$ (see e.g. [8]). To show uniqueness, assume there exists an other solution $\hat{\varphi} = \hat{\alpha} B_{\hat{\Lambda}}$ of degree $\leq n$. Clearly $|\hat{\alpha}| > |\alpha|$. By Theorem 1 $\hat{\varphi}$ is the minimizer of $\mathbb{K}_{\hat{\Psi}}(f)$ subject to $f(z_k) = w_k$, $k = 0,\ldots,n$, for some $\hat{\Psi}$. This is a contradiction, since φ satisfies the interpolation constraints and $\mathbb{K}_{\hat{\Psi}}(\hat{\varphi}) > \mathbb{K}_{\hat{\Psi}}(\varphi)$. Hence the solution is unique. ■

Proof: of Theorem 5: Factor f as $f = Bg$, where B is the Blaschke product that contains the zeros in $\mathbb{D}$. Let φ be the H_∞-norm minimizing interpolant satisfying $\varphi(z_k) = B(z_k)$, $k = 0,\ldots,n$. Since $\deg B > n$, Lemma 2 implies that $|\varphi| < |B|$. Since φg satisfies the interpolation conditions and $|\varphi g| < |f|$, f cannot be a solution of (3). ■

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