Synchronization of nonlinear delayed semi-Markov jump neural networks via distributed delayed impulsive control

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Abstract

This paper is concerned with synchronization problem for nonlinear delayed semi-Markov jump neural networks (s-MJNN) via distributed delayed impulsive control. By using stochastic Lyapunov functions together with Razumikhin technique, some sufficient conditions for synchronization for a class of nonlinear delayed s-MJNN via distributed delayed impulsive control are developed. Finally, two numerical examples are given to show the effectiveness and advantages of the proposed techniques.

Keywords: Synchronization; Semi-Markov jump neural networks (s-MJNN); Distributed delayed impulsive control; Razumikhin technique

1. Introduction

Over the past few decades, neural networks have received considerable interest due to their potential applications in science and engineering, such as pattern recognition [1], image and signal processing [2, 3], optimization problems [4] and so on. It is well-known that the transition of neural networks nodes from one state to another is a stochastic process, which can be regarded as a Markov process. For example, the packet loss in neural networks can be described by Markov process. The time between two successive transitions is called *sojourn time*, and the sojourn time of Markov jump systems obeys a memoryless exponential distribution (or geometric distribution), and therefore, the transition probabilities (TPs) of Markov jump systems (MJSs) are constants.

In practice, however, TPs of neural networks nodes may have time-varying characteristics, that is, the sojourn time of transition of neural networks obeys a memory distribution. For instance, [5] demonstrated that there are some variable repair rates and failure rates in complex manufacturing systems. Besides, packet loss occurs when network systems transmit information, and sometimes packet loss rates may not be constants. In general, a stochastic process with sojourn time following a memory distribution is called a semi-Markov process. Neural networks with a semi-Markov process are referred to as *semi-Markov jump neural networks* (s-MJNN).

In this paper we study synchronization of s-MJNN. The basic problem is to control a system called the *response system* so that its trajectories converge to the same values as another system called the *drive system*, although the systems start at different initial conditions. There is quite a large literature on the synchronization of s-MJNN (see, e.g., [6–9] and the references therein). For instance, in [6], Wei et al. discussed stability analysis and stabilization problems for stochastic synchronization of s-MJNN with time-varying delays by constructing a semi-Markovian Lyapunov-Krasovskii functional and a new integral inequality. In [8], Zhang et al. investigated the stochastic synchronization of neutral-type semi-Markovian jump neural networks with partial mode-dependent additive time-varying delays via event-triggered control. In [9], Qi et al. derived some sufficient conditions for finite-time synchronization of delayed semi-Markov switching neural networks with quantized measurement via a feedback controller. Moreover, since the neurons will transit information to other neurons, and the switching speed

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of actuations is limited, time delays are inevitable in neural networks. In fact, the effects of time delays may make a stable system unstable or make a system show unpredictable behavior such as oscillations, divergence and so on (see, e.g., [10–12] and the references therein).

Given the basic behavior of neural networks, synchronization is one of the most significant issues in such systems. Impulsive control has been proved to be an effective and powerful method to achieve synchronization of neural networks (see, e.g., [13–16] and the references therein). Generally, it takes time for controllers to process samples and transmit information to actuators, so it is necessary to consider time delays in the impulsive controllers. For example, as mentioned in [17], when patients are injected with insulin, it takes time for insulin to be transported from injection depot to the interstitial compartment and inhibit hepatic glucose production, which can be seen as time delays in impulsive control, that is, delayed impulsive control, these time delays cause the open-loop control to take longer time to lower glucose concentration level.

In recent years systems with delayed impulses have attracted extensive attentions (see, e.g., [18–23] and the references therein). However, these results only consider discrete-time delays in the impulsive control. In fact, distributed delays are applied to a large number of practical systems, such as Susceptible-Infected-Removed (SIR) epidemic model [24], predator-prey model [25], feeding systems and combustion chambers in a liquid monopropellant rocket motor with pressure feeding [26, 27] and so on. Lately, distributed delayed impulsive control was studied by [28–30]. As pointed out in [28], distributed delayed impulsive control means the jumps of systems states depend on the accumulation (or average) of the system states over a history time period. In [28], Liu and Zhang proposed a distributed delayed impulsive control to stabilize general nonlinear systems with time delays, a exponential stability criterion was obtained by using the Lyapunov-Razumikhin method. In [29], Liu et al. designed a distributed delayed impulsive consensus protocol to investigate the consensus of networked multi-agent systems with distributed delays. In [30], Xu et al. studied the synchronization of chaotic neural networks with time delays via distributed delayed impulsive control. However, the above results have some limitations. For instance, the results of [28] impose both upper and lower bounds on impulsive intervals, and the distributed delays in impulsive controllers are not allowed to be greater than the time delays in systems. In [29], the distributed delays in impulsive controllers are related to the time delays in systems. In [30], the delay in neural networks is constant rather than time-varying. Therefore, the research on distributed delayed impulsive control needs further implementation and improvement.

Motivated by the above discussion, in this paper, we address the synchronization problem for a class of nonlinear delayed s-MJNN via distributed delayed impulsive control. The main contributions of this paper are highlighted as follows: (i) by using stochastic Lyapunov functions together with Razumikhin technique, some sufficient conditions are obtained for synchronization of nonlinear delayed s-MJNN via distributed delayed impulsive control; (ii) the results obtained generalize the results in [28–30] by considering time-varying delays, removing the lower bounds of impulsive intervals and the relationship between the distributed delays in impulsive controllers and the time delays in systems; (iii) the results obtained show that distributed delayed impulses do contribute to the synchronization of nonlinear delayed s-MJNN.

The rest of this paper is organized as follows. In Section 2 the basic problem is formulated, and definitions and notations are introduced. In Section 3 the main results are presented. In particular, some synchronization criteria for a class of nonlinear delayed s-MJNN via distributed delayed impulsive control are presented. In Section 4, some numerical examples are given to demonstrate the effectiveness and superiority of the proposed results. Finally, conclusions are drawn in Section 5.

Notation: In this paper, $\mathbb{R}^n(\mathbb{R}^m_+)$ and $\mathbb{R}^{n\times m}$ denote, respectively, the m-dimensional (nonnegative) Euclidean space and the set of all $n \times m$ real matrices; $\|\cdot\|$ represents the Euclidean vector norm, where $x = (x_1, \ldots, x_n)^T$; the notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite); $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and minimum eigenvalue of A; A^T represents the transpose of A; I_n is the identity matrix with n-dimension; the symmetric terms below the main diagonal of a symmetric matrix are denoted by $*$; for the notation $(\Omega, \mathcal{F}, P)$, Ω represents the sample space, $\mathcal{F}$ is the σ-algebra of subsets of the sample space and P is the probability measure on $\mathcal{F}$; $\mathbb{R}$ denotes the set of real numbers; $\mathbb{R}^+$ denotes the set of nonnegative real numbers; $\mathbb{Z}^+$ denotes the set of positive integer numbers; for $\tau > 0$, let $PC([-\tau, 0], \mathbb{R}^n)$ denote the set of piecewise continuous function $\psi : [-\tau, 0] \to \mathbb{R}^n$ with norm $\|\psi\|_\tau = \sup_{-\tau\leq\theta\leq 0} \|\psi(\theta)\|$; Denote by $L^p_{\mathcal{F}_t}([-\tau, 0], \mathbb{R}^n)$ the family of all $\mathcal{F}_t$ measurable, $PC([-\tau, 0], \mathbb{R}^n)$-valued stochastic variables ϕ such that $\|\phi\|^p_{\mathbb{E}} = \sup_{-\tau\leq\theta\leq 0} \mathbb{E}\{\|\phi(\theta)\|^p\} < \infty$, $\mathbb{E}\{\cdot\,|(\psi, r_0)\}$ represents the mathematical expectation.

2. Problem formulation

Consider the following nonlinear delayed semi-Markov jump neural networks defined on a complete probability space $(\Omega, \mathcal{F}, P)$:

$$\begin{cases} \dot{x}(t) = -C(r(t))x(t) + A(r(t))f(x(t)) + B(r(t))f(x(t - \xi(t))) + J, \\ x(t_0 + \theta) = \varphi(\theta),\ \theta \in [-\xi, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector of the neural networks. $\{r(t), h\}_{t\geq 0} = \{r_m, h_m\}_{m\in\mathbb{Z}^+}$ is characterized by a continuous-time semi-Markov process that takes values in a finite set $\Psi = \{1, 2, \ldots, S\}, S \in \mathbb{Z}^+$, and governs the switching among S system modes, $\{r_m\}_{m\in\mathbb{Z}^+}$is the index of system mode at the mth transition, taking values in Ψ, and $\{h_m\}_{m\in\mathbb{Z}^+}$ is the sojourn-time of mode r_{m-1} between the $(m-1)th$ transition and mth transition, taking values in $\mathbb{R}^+$. For convenience, for each $r(t) = i \in \Psi$, we have $C(r(t)) = C_i, A(r(t)) = A_i$ and $B(r(t)) = B_i \in \mathbb{R}^{n\times n}$, $C_i = \mathrm{diag}(c_{i1}, \ldots, c_{in})$ satisfying $c_{ij} > 0, j = 1, 2, \ldots, n$. The matrices A_i and B_i denote the connection weight matrix and the delayed weight matrix, respectively. $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))]^T \in \mathbb{R}^n$ represents the neuron activation function. $\xi(t)$ refers to a time-varying delay satisfying $0 \leq \xi(t) \leq \xi < \infty$, where $\xi \in \mathbb{R}^+$ denotes the upper bounds of time-delay. J is an external input. $\varphi(\cdot) \in PC([-\xi, 0], \mathbb{R}^n)$ is the initial value. The evolution of semi-Markov process is governed by the following TPs [31]:

$$\begin{cases} Pr\{r_{m+1} = j, h_{m+1} \leq h + \epsilon | r_m = i, h_{m+1} > h\} = \lambda_{ij}(h)\epsilon + o(\epsilon), & i \neq j, \\ Pr\{r_{m+1} = j, h_{m+1} > h + \epsilon | r_m = i, h_{m+1} > h\} = 1 + \lambda_{ii}(h)\epsilon + o(\epsilon), & i = j, \end{cases} \tag{2}$$

where h denotes the sojourn-time that indicates the time duration between two successive mode transitions; $o(\epsilon)$ is the little-o notation defined as $\lim_{\epsilon\mapsto 0}(o(\epsilon)/\epsilon) = 0$, and $\lambda_{ij}(h) \geq 0$, for $j \neq i$, denotes the transition rate from mode i at time t to mode j at time $t + \epsilon$, and $\lambda_{ii}(h) = -\sum_{j=1,j\neq i}^{S} \lambda_{ij}(h)$. Thus, we get the generator matrix $\Lambda(h) = (\lambda_{ij}(h))_{S\times S}, h \geq 0$, which governs the evolution of semi-Markov process $\{r(t), t \geq 0\}$. For any $\varphi(\cdot) \in PC([-\xi, 0], \mathbb{R}^n)$, we assume that the function $f(x(t))$ satisfies all necessary conditions to ensure that neural network (1) admits a unique solution $x(t, \varphi, r_0)$ which exists in a maximal interval $[t_0 - \xi, \infty)$. Set $x(t) = x(t, \varphi, r_0)$.

If the probability density function (PDF) of sojourn time h in mode i is $g_i(h)$, by [32], then the cumulative distribution function (CDF) of sojourn time h in mode i is

$$G_i(h) = Pr\{h_{m+1} \leq h | r_m = i\} = \int_0^h g_i(s)ds, \forall i \in \Psi, \forall h > 0,$$

the transition rate of the system transition from mode i is

$$\lambda_i(h) = \lim_{\epsilon\to 0} \frac{G_i(h + \epsilon) - G_i(h)}{\epsilon(1 - G_i(h))}.$$

Let neural network (1) be the drive system and the corresponding response system be given by

$$\begin{cases} \dot{y}(t) = -C(r(t))y(t) + A(r(t))f(y(t)) + B(r(t))f(y(t - \xi(t))) + u(r(t)) + J, \\ y(t_0 + \theta) = \phi(\theta),\ \theta \in [-\tau, 0], \end{cases} \tag{3}$$

where $\phi(\cdot) \in PC([-\tau, 0], \mathbb{R}^n), \tau = \max\{\xi, d\},\ u(r(t))$ is the impulsive control input. Except for the control the reponse system (3) has the same dynamics as the drive system (1) but different initial condition.

Next a distributed delayed impulsive controller will be designed such that response system (3) can be synchronized to drive system (1). The distributed delayed impulsive control input is given as follows:

$$u(r(t)) = \sum_{k=1}^{\infty}\Big(D(r(t))e(t^-) + K(r(t))\int_{t-d_k}^{t} e(s)ds - e(t)\Big)\delta(t-t_k), k \in \mathbb{Z}^+, \tag{4}$$

where $e(t) = y(t) - x(t)$, $d_k(k \in \mathbb{Z}^+)$ denote the distributed delays in impulsive control input satisfying $0 \le d_k \le d < \infty$, $\delta(\cdot)$ is the delta function, for each $r(t) = i \in \Psi$, $D(r(t)) = D_i, K(r(t)) = K_i \in \mathbb{R}^{n\times n}$, D_i and K_i are known real matrices. Particularly, $e(t) = D(r(t))e(t^-) + K(r(t))\int_{t-d_k}^{t} e(s)ds$ $(k \in \mathbb{Z}^+)$ for $t = t_k$, $t_0 = 0, t_k \in \mathbb{R}^+$ for $k \in \mathbb{Z}^+$, $t_1 < t_2 < \ldots < t_k < \ldots$, with $\lim_{k\to\infty} t_k = +\infty$. $e(t^+) = \lim_{\epsilon\to 0^+} e(t+\epsilon)$ and $e(t^-) = \lim_{\epsilon\to 0^+} e(t-\epsilon)$, we assume that $e(t^+) = e(t)$. Then the error system is obtained as follows:

$$\begin{cases} \dot{e}(t) = -C(r(t))e(t) + A(r(t))F(e(t)) + B(r(t))F(e(t-\xi(t))),\ t \neq t_k, \\ e(t_k) = D(r(t_k))e(t_k^-) + K(r(t_k))\displaystyle\int_{t_k-d_k}^{t_k} e(s)ds,\ k \in \mathbb{Z}^+, \\ e(t_0+\theta) = \psi(\theta),\ \theta \in [-\tau, 0], \end{cases} \tag{5}$$

where $e(t) \in \mathbb{R}^n, F(e(\cdot)) = f(y(\cdot)) - f(x(\cdot)) \in \mathbb{R}^n, \psi(\theta) = \phi(\theta) - \varphi(\theta), \psi(\cdot) \in PC([-\tau, 0], \mathbb{R}^n)$.

Let $\tilde{A}$ be the infinitesimal generator, then according to the definition of $\tilde{A}$, see e.g. [33]

$$\tilde{A}V(e(t), t, r(t)) = \lim_{\epsilon\to 0} \frac{\mathbb{E}\{V(e(t+\epsilon), t+\epsilon, r(t+\epsilon))|(e(t), r(t))\} - V(e(t), t, r(t))}{\epsilon},$$

here $V(\cdot)$ denotes the Lyapunov function, ϵ is a small positive number.

Assumption 1. *Assume that each activation function $f_j(t), t \in \mathbb{R}(j = 1, 2, \ldots, n)$ is continuous, bounded, and there exist constants l_{j1}, l_{j2} such that*

$$l_{j1} \le \frac{f_j(a) - f_j(b)}{a-b} \le l_{j2}, j = 1, 2, \ldots, n,$$

where $a, b \in \mathbb{R}$, and $a \neq b$.

In this paper, the following definition and a proposition are needed for the derivation of the main results.

Definition 1. *The trivial solution of system (5) is said to be p-th moment exponentially stable, if there exist constants $\sigma > 0, M_0 > 0$ such that*

$$\mathbb{E}\Big\{\|e(t,\psi)\|^p\Big\} \le M_0\|\psi\|_{\mathbb{E}}^p e^{-\sigma(t-t_0)},\ \forall t \ge t_0,$$

for any initial data $\psi \in PC([-\tau, 0], \mathbb{R}^n)$. Furthermore, if $p = 2$, then neural network (1) is said to be mean-square exponentially synchronized to response system (3).

Proposition 1. *Suppose there exist some constants $c_1 > 0, c_2 > 0, \eta > 0, q > 0, k_1 \ge 0, k_2 \ge 0$ and $\varrho > 0$ such that*
(i) $c_1\|e(t)\|^p \le V(e(t), t, i) \le c_2\|e(t)\|^p$ for all $(e(t), t, i) \in \mathbb{R}^n \times [-\tau, \infty) \times \Psi$;
(ii) $\mathbb{E}\{\tilde{A}V(e(t), t, i)\} < q\mathbb{E}\{V(e(t), t, i)\}$ for all $t \in [t_{k-1}, t_k)$ when $\mathbb{E}\{V(e(t+\theta), t+\theta, i)\} \le \varrho e^{\eta\tau}\mathbb{E}\{V(e(t), t, i)\}$ for all $\theta \in [-\tau, 0]$, where $\varrho \ge e^{(\eta+q)(t_k-t_{k-1})}, k \in \mathbb{Z}^+$;
(iii) $\mathbb{E}\{V(e(t_k), t_k, i)\} \le k_1\mathbb{E}\{V(e(t_k^-), t_k, i)\} + k_2\int_{t_k-d_k}^{t_k} \mathbb{E}\{V(e(s), s, j)\}ds, i, j \in \Psi$;
(iv) $(k_1 + k_2 d_k e^{\eta d_k})\varrho \le 1, k \in \mathbb{Z}^+$.
Then the trivial solution of system (5) is p-th moment exponentially stable.

The proof of Proposition 1 is presented in Appendix A.

Proposition 1 presents a sufficient criterion for the exponential stability of system (5). In Razumikhin technique condition (ii), $q > 0$ indicates that original impulse-free system (5) may be unstable. Condition (iii) means the relationship among $e(t_k)$, $e(t_k^-)$ and $\int_{t_k-d_k}^{t_k} e(s)ds, k \in \mathbb{Z}^+$, the state of system at t_k depends not only on the state of system at impulsive instant t_k^- but also on the integral of system states from $t_k - d_k$ to t_k^-. Moreover, condition (iii) and condition (iv) imply that distributed delayed impulses do contribute to the stability of system (5). Therefore, Proposition 1 shows that an unstable system can be exponentially stabilized under distributed delayed impulsive control.

Corollary 1. *Assume that conditions (i)-(iii) hold and there exist constants $\eta > 0, k_1 \geq 0, k_2 > 0$ and ϱ such that $\eta \leq \frac{\ln(1-k_1\varrho)-\ln k_2 d\varrho}{d}$. Then the trivial solution of system (5) is p-th moment exponentially stable.*

As mentioned in [34, 35], MJSs can be regarded as semi-Markov jump systems if TPs are unknown, moreover, semi-Markov jump systems also can reduce to MJSs if the transition rate $\lambda_{ij}(h)$ becomes $\lambda_{ij}, i, j \in \Psi$, which is a constant. Recently, there are many interesting results on stability of semi-Markov jump systems (see, e.g., [33, 35–37] and the references therein). However, the above mentioned papers have some restrictions. For instance, in [33, 36], the transition rate $\lambda_{ij}(h)$ has upper and lower bounds. In the present paper, $\lambda_{ij}(h)$ can be unbounded for some $i, j \in \Psi$, and therefore our results may be less conservative than the results in [33, 36]. In addition, [35] did not consider delayed impulses and the results of [35] impose both upper and lower bounds on impulsive intervals. The paper [37] did not consider time delays in systems and impulsive control. Compared to [35, 37], in our paper a distributed delayed impulsive control is considered, and impulsive intervals only have upper bounds. Therefore, our results generalize the results in [35, 37].

3. Main results

In this section, some synchronization criteria for nonlinear delayed s-MJNN (1) with distributed delayed impulsive control will be developed.

Theorem 1. *Suppose Assumption 1 holds. Then, given constants $\eta > 0, \varrho > 0$, if there exist constants $\alpha \in \mathbb{R}, \beta > 0, k_1 \geq 0, k_2 \geq 0$, symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}, i \in \Psi$, and a positive diagonal matrix $U = diag\{u_1, u_2, \ldots, u_n\}$ such that*

$$\begin{bmatrix} -C_i^T P_i - P_i C_i + \sum_{j=1}^{S} \bar{\lambda}_{ij} P_j - \alpha P_i - U\hat{L} & P_i A_i + U\check{L} & \boldsymbol{0}_{n\times n} & P_i B_i \\ * & -U & \boldsymbol{0}_{n\times n} & \boldsymbol{0}_{n\times n} \\ * & * & -U\hat{L} - \beta P_l & U\check{L} \\ * & * & * & -U \end{bmatrix} < 0, \ i, l \in \Psi, \tag{6}$$

$$2D_i^T P_i D_i - k_1 P_i \leq 0, 2K_i^T P_i K_i - \frac{k_2}{d} P_z \leq 0, \ i, z \in \Psi, \tag{7}$$

$$k_1\varrho + k_2 d_k \varrho e^{\eta d_k} \leq 1, k \in \mathbb{Z}^+, \tag{8}$$

$$(\eta + \alpha + \beta\varrho \exp(\eta\tau))(t_k - t_{k-1}) - \ln \varrho \leq 0, k \in \mathbb{Z}^+, \tag{9}$$

where $\hat{L} = diag(l_{11}l_{12}, l_{21}l_{22}, \ldots, l_{n1}l_{n2}), \check{L} = diag\Big(\frac{l_{11}+l_{12}}{2}, \frac{l_{21}+l_{22}}{2}, \ldots, \frac{l_{n1}+l_{n2}}{2}\Big), \bar{\lambda}_{ij} = \mathbb{E}\{\lambda_{ij}(h)\} = \int_0^\infty \lambda_{ij}(h) g_i(h) dh$ with the PDF $g_i(h)$ of sojourn-time h at mode i, d_k is given in inequality (4), then the neural network (1) is mean-square exponentially synchronized to response system (3).

The proof of Theorem 1 is presented in Appendix B.

Remark 1. *In inequality (9) of Theorem 1, $\alpha + \beta\varrho e^{\eta\tau}$ is equal to q of Proposition 1, which describes the rate of change of function $V(t)$ over each impulsive interval $[t_{k-1}, t_k), k \in \mathbb{Z}^+$.*

More generally, if the control gain matrices D_i and $K_i, i \in \Psi$ are unknown, then we have the following theorem.

Theorem 2. *Suppose Assumption 1 holds. Then, given constants $\eta > 0, \varrho > 0$, if there exist constants $\alpha \in \mathbb{R}, \beta > 0, k_1 \geq 0, k_2 \geq 0$, symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, a positive diagonal matrix $U = diag\{u_1, u_2, \ldots, u_n\}$, matrices $G_i \in \mathbb{R}^{n\times n}$ and $H_i \in \mathbb{R}^{n\times n} (i \in \Psi)$ such that*

$$\begin{bmatrix} -C_i^T P_i - P_i C_i + \sum_{j=1}^{S} \bar{\lambda}_{ij} P_j - \alpha P_i - U\hat{L} & P_i A_i + U\check{L} & \boldsymbol{0}_{n\times n} & P_i B_i \\ * & -U & \boldsymbol{0}_{n\times n} & \boldsymbol{0}_{n\times n} \\ * & * & -U\hat{L} - \beta P_l & U\check{L} \\ * & * & * & -U \end{bmatrix} < 0, \ i, l \in \Psi, \tag{10}$$

$$\begin{bmatrix} -k_1 P_i & \sqrt{2} G_i \\ * & -P_i \end{bmatrix} < 0, \quad \begin{bmatrix} -\frac{k_2}{d} P_z & \sqrt{2} H_i \\ * & -P_i \end{bmatrix} < 0, \ i, z \in \Psi, \tag{11}$$

$$k_1 \varrho + k_2 d_k \varrho e^{\eta d_k} \leq 1, k \in \mathbb{Z}^+,$$

$$(\eta + \alpha + \beta \varrho \exp(\eta\tau))(t_k - t_{k-1}) - \ln \varrho \leq 0, k \in \mathbb{Z}^+,$$

where $\hat{L} = diag(l_{11}l_{12}, l_{21}l_{22}, \ldots, l_{n1}l_{n2})$, $\check{L} = diag\Big(\frac{l_{11}+l_{12}}{2}, \frac{l_{21}+l_{22}}{2}, \ldots, \frac{l_{n1}+l_{n2}}{2}\Big)$, $\bar{\lambda}_{ij} = \mathbb{E}\{\lambda_{ij}(h)\} = \int_0^\infty \lambda_{ij}(h) g_i(h) dh$ with the PDF $g_i(h)$ of sojourn-time h at mode i, then neural network (1) is mean-square exponentially synchronized to response system (3) under the control gain matrices $D_i = P_i^{-1} G_i^T$ and $K_i = P_i^{-1} H_i^T$.

The proof of Theorem 2 is presented in Appendix C.

Remark 2. *In particular, in the above theorems, if $k_1 = 0$, it follows from inequality (7) that we have $D_i = \boldsymbol{0}_{n\times n} (i \in \Psi)$, that is, the control (4) is impulsive control with distributed delays only. Similarly, if $k_2 = 0$, then we have $K_i = \boldsymbol{0}_{n\times n} (i \in \Psi)$, the control (4) is impulsive control without time delays.*

There are a few results on distributed delayed impulsive control (see, e.g., [28–30] and the references therein), which however have some limitations. For instance, the results of [28] impose both upper and lower bounds on impulsive intervals, and the distributed delays in impulsive controllers are not greater than the time delays in systems, that is, $r_n \leq \tau$. In [29], the distributed delays in the impulsive controllers are related to the time delays in systems. In [30], the delay in the neural networks is constant rather than time-varying. Moreover, we obtain the same distributed delayed impulsive control as in [30] if the semi-Markov process case is not considered and $D(r(t)) = \mathbf{0}_{n\times n}$. Compared with [28–30], we have the following advantages: 1) impulsive intervals do not have a lower bound; 2) the distributed delays in impulsive controllers are independent of the time delays in systems; 3) time-varying delays are considered in neural networks.

4. Simulation examples

To illustrate the effectiveness and superiority of the given results, some examples are given in this section.

Example 1. *Consider neural network (1) with three operation modes and the following system data:*

$$C_1 = \begin{bmatrix} 1.2 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1.1 & 0 \\ 0 & 1.2 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1.3 & 0 \\ 0 & 1.4 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} 2 & -0.1 \\ -5 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -0.1 \\ -6 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0.2 & -0.1 \\ -5 & 3 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -2.6 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1.3 & -0.1 \\ -0.1 & -2.5 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -1.4 & -0.1 \\ -0.2 & -2.1 \end{bmatrix},$$

$f(x(t)) = (\tanh(x_1(t)), \tanh(x_2(t)))^T \in R^2, \Psi = \{1, 2, 3\}$, time-varying delay $\xi(t) = 1 + 0.3\sin(t), \xi = 1.3$.

Consider the distributed delayed impulsive control with the following data:

$$D_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.2 & 0.01 \\ 0.1 & 0.2 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 0.2 & 0.1 \\ 0.01 & 0.2 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 0.2 & 0.01 \\ 0.1 & 0.2 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.2 & 0.01 \\ 0.1 & 0.2 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0.2 & 0.1 \\ 0.01 & 0.2 \end{bmatrix},$$

the impulsive intervals $t_k - t_{k-1} = 0.07$, distributed delays $d_k = 0.8, k \in \mathbb{Z}^+$.

We assume that the sojourn time h is subject to the Weibull distribution and exponential distribution respectively. Especially, when $i = 1$, the sojourn time $h \sim$ Weibull(4, 2), i.e., the probability density function (PDF) $g_1(h) = \frac{1}{8}he^{-1/16h^2}$; when $i = 2$, the sojourn time $h \sim$ Weibull(2, 2), i.e., the probability density function (PDF) $g_2(h) = \frac{1}{2}he^{-1/4h^2}$; when $i = 3$, the sojourn time $h \sim$ Exp(2), i.e., the probability density function (PDF) $g_3(h) = \frac{1}{2}e^{-1/2h}$, the generator matrix is denoted by

$$\Lambda(h) = \begin{bmatrix} \lambda_{11}(h) & \lambda_{12}(h) & \lambda_{13}(h) \\ \lambda_{21}(h) & \lambda_{22}(h) & \lambda_{23}(h) \\ \lambda_{31}(h) & \lambda_{32}(h) & \lambda_{33}(h) \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}h & \frac{1}{8}h & \frac{1}{8}h \\ \frac{1}{2}h & -h & \frac{1}{2}h \\ \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}. \tag{12}$$

Correspondingly, we can calculate the mathematical expectation of the transition rate $\lambda_{12}(h)$ as

$$\mathscr{E}\{\lambda_{12}(h)\} = \int_0^\infty \tfrac{1}{8}hg_1(h)dh = \int_0^\infty \tfrac{1}{64}h^2e^{-1/16h^2}dh = 0.4431.$$

With similar calculations, we have the mathematical expectation of the transition rates that

$$\mathscr{E}\{\Lambda(h)\} = \begin{bmatrix} -0.8862 & 0.4431 & 0.4431 \\ 0.8862 & -1.7724 & 0.8862 \\ 0.5 & 0.5 & -1 \end{bmatrix}.$$

It is easy to check that $\hat{L} = \mathbf{0}_{2\times 2}, \check{L} = 0.5I_2$. Choose $\eta = 0.1, \varrho = 2$. Then, using the Matlab LMI toolbox, one of the feasible solutions of Theorem 1 is $\alpha = 6.3685, \beta = 1.0476, k_1 = 0.0585, k_2 = 0.1904$,

$$P_1 = \begin{bmatrix} 0.1276 & 0.0126 \\ 0.0126 & 0.0905 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.1433 & 0.0000 \\ 0.0000 & 0.1437 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 0.1243 & 0.0089 \\ 0.0089 & 0.0956 \end{bmatrix}, \quad U = \begin{bmatrix} 0.8627 & 0 \\ 0 & 0.8627 \end{bmatrix},$$

and thus it follows from Theorem 1 that neural network (1) is mean-square exponentially synchronized to response system (3).

The following algorithm shows how to find a feasible solution for system (5) and design an appropriate impulsive sequence.

Algorithm 1

Step 1. Set $\eta := 0.1,\ \varrho := 0.1$.

Step 2. Using LMI toolbox in Matlab to find $\alpha, \beta, k_1, k_2, P_i, i \in \Psi$, and U for LMIs (6), (7) and (8).

Step 3. Using inequality (9) to determine the impulsive sequence.

Step 4. If a feasible solution is found, then stop, else set $\eta := \eta + 0.1,\ \varrho := \varrho + 0.1$ and go to Step 2.

Choose the initial condition $\varphi(\theta) = [0.6, 0.7]^T, \psi(\theta) = [1, 1.5]^T, \theta \in [-1.3, 0]$, and $J = 0$. With the distributed delayed impulsive control input $u(r(t)) = 0$, Figure 1 shows that response system (3) is not synchronized to drive *system (1). When the distributed delayed impulsive control input $u(r(t))$ is given by (4), response system (3) is synchronized to drive system (1), see Figure 2. Example 1 demonstrates that impulses do contribute to the synchronization of drive system (1) and response system (3).*

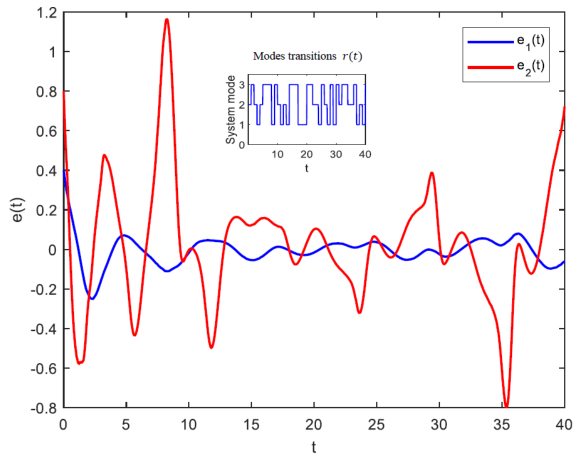

Figure 1: The error dynamics of impulse-free system (5) in Example 1

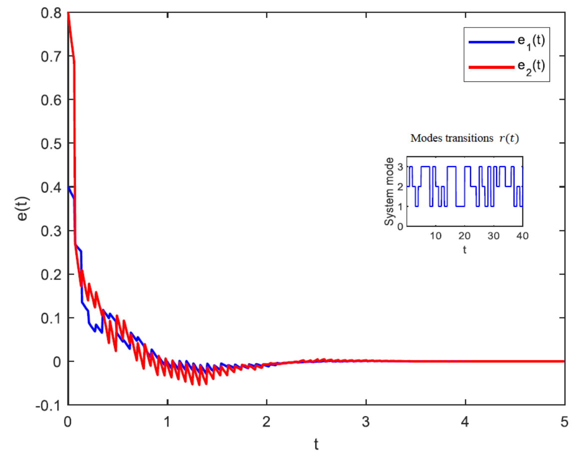

Figure 2: The error dynamics of system (5) with impulses in Example 1

Example 2. *Consider neural network (1) with two operation modes and the following system data:*

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} 3 & -0.1 \\ -4 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -0.2 \\ -5 & 3 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -2 & -0.1 \\ -0.2 & -2.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1.5 & -0.1 \\ -0.1 & -2 \end{bmatrix},$$

$f(x(t)) = (\tanh(x_1(t)), \tanh(x_2(t)))^T \in R^2, \Psi = \{1, 2\}$, *time-varying delay* $\xi(t) = 1 + 0.3\sin(t), \xi = 1.3$.

We assume that the sojourn time h is subject to the Weibull distribution and exponential distribution respectively. Especially, when $i = 1$, the sojourn time $h \sim$ Weibull(1, 2), i.e., the probability density function (PDF) $g_1(h) = 2he^{-h^2}$; when $i = 2$, the sojourn time $h \sim$ Weibull(2, 2), i.e., the probability density function (PDF) $g_2(h) = \frac{1}{2}he^{-1/4h^2}$, the generator matrix is denoted by

$$\Lambda(h) = \begin{bmatrix} \lambda_{11}(h) & \lambda_{12}(h) \\ \lambda_{21}(h) & \lambda_{22}(h) \end{bmatrix} = \begin{bmatrix} -2h & 2h \\ \frac{1}{2}h & -\frac{1}{2}h \end{bmatrix}. \tag{13}$$

Correspondingly, we can calculate the mathematical expectation of the transition rate $\lambda_{12}(h)$ as

$$\mathscr{E}\{\lambda_{12}(h)\} = \int_0^\infty 2hg_1(h)dh = \int_0^\infty 4h^2 e^{-h^2} dh = 0.4431.$$

With similar calculations, we have the mathematical expectation of transition rates that

$$\mathscr{E}\{\Lambda(h)\} = \begin{bmatrix} -1.7725 & 1.7725 \\ 0.8862 & -0.8862 \end{bmatrix}.$$

It is easy to check that $\hat{L} = \boldsymbol{0}_{2\times 2}, \check{L} = 0.5I_2$. Choose $\eta = 0.1, \varrho = 2$, the impulsive intervals $t_k - t_{k-1} = 0.07$, distributed delays $d_k = 0.8, k \in \mathbb{Z}^+$. One of the feasible solutions of Theorem 2 is $\alpha = 4.6056, \beta = 0.2594, k_1 =$

$0.1486, k_2 = 0.1791,$

$$P_1 = \begin{bmatrix} 0.6020 & 0 \\ 0 & 0.6020 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.5869 & 0 \\ 0 & 0.5869 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0.6885 & 0 \\ 0 & 0.6885 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.6615 & 0 \\ 0 & 0.6615 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0.7294 & 0 \\ 0 & 0.7294 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0.8037 & 0 \\ 0 & 0.8037 \end{bmatrix}, \quad U = \begin{bmatrix} 8.3158 & 0 \\ 0 & 8.3157 \end{bmatrix}.$$

Then the control gain matrices is obtained as follows:

$$D_1 = \begin{bmatrix} 1.1436 & 0 \\ 0 & 1.1436 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1.1270 & 0 \\ 0 & 1.1270 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 1.2115 & 0 \\ 0 & 1.2115 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1.3693 & 0 \\ 0 & 1.3693 \end{bmatrix},$$

thus it follows from Theorem 2 that neural network (1) is mean-square exponentially synchronized to response system (3).

Choose the initial condition $\varphi(\theta) = [0.2, 0.3]^T, \psi(\theta) = [0.3, 0.4]^T, \theta \in [-1.3, 0]$, *and* $J = 0$. *Figure 3 shows that response system (3) isn't synchronized to drive system (1) without distributed delayed impulsive control. As shown in Figure 4, response system (3) is synchronized to the drive system (1) under distributed delayed impulsive control.*

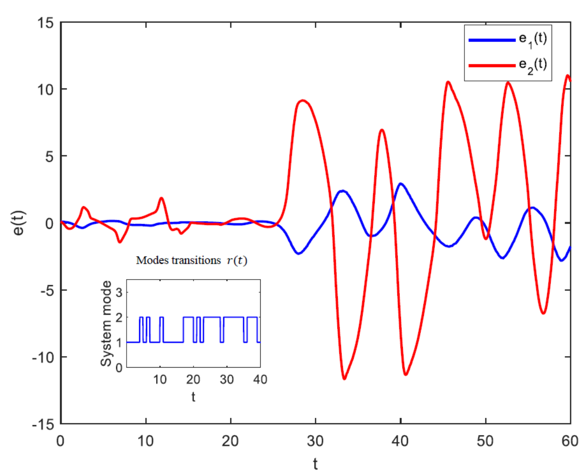

Figure 3: The error dynamics of impulse-free system (5) in Example 2

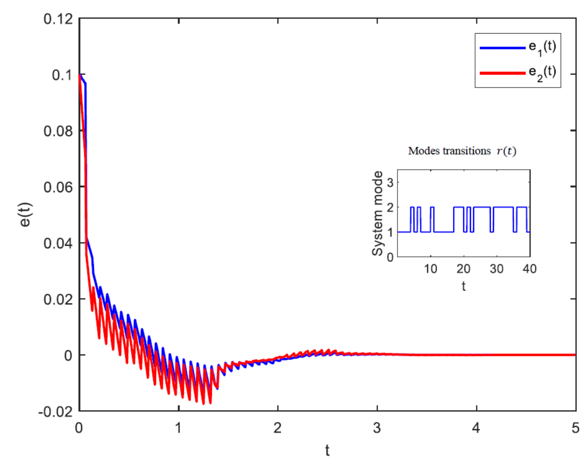

Figure 4: The error dynamics of system (5) with impulses in Example 2

5. Conclusion

In this paper, we have investigated the synchronization problem for nonlinear delayed s-MJNN via distributed delayed impulsive control. By using stochastic Lyapunov functions together with Razumikhin technique, combined with some LMIs, some synchronization criteria for a class of nonlinear delayed s-MJNN are derived to ensure that response system and derive system are exponentially synchronized. The results obtained show that distributed delayed impulses do contribute to synchronization of nonlinear delayed s-MJNN. In particular, our results generalize the results in [28–30]. Finally, some simulation examples have been given to illustrate the effectiveness and superiority of the proposed results. The limitation of our results is that the stability criteria are sufficient conditions rather than sufficient and necessary conditions for synchronization of the neural networks. A possible direction for future work is to obtain some robust stability criteria for delayed semi-Markov jump systems.

Appendix A

Proof of Proposition 1: For convenience, set $V(t) = V(e(t), t, i), i \in \Psi$, and let

$$W(t) = \begin{cases} V(t)e^{\eta(t-t_0)}, \ t \geq t_0, \\ V(t), \ t \in [t_0 - \tau, t_0). \end{cases} \tag{14}$$

We claim that

$$\mathbb{E}\{W(t)\} \leq \varrho \mathbb{E}\{\overline{V}(t_0)\}, \ t \geq t_0, \tag{15}$$

where $\overline{V}(t_0) := \sup\{V(s), t_0 - \tau \leq s \leq t_0\}$.

Firstly, we will prove

$$\mathbb{E}\{W(t)\} \leq \varrho \mathbb{E}\{\overline{V}(t_0)\}, \ t \in [t_0 - \tau, t_1).$$

(a) When $t \in [t_0 - \tau, t_0]$, it is obvious that (15) holds.

(b) Next, we will prove that (15) holds for $t \in (t_0, t_1)$.

If (15) is not true for $t \in (t_0, t_1)$, then there exists a $\acute{t} \in (t_0, t_1)$ such that $\mathbb{E}\{W(\acute{t})\} > \varrho \mathbb{E}\{\overline{V}(t_0)\}$. Due to the fact that $\mathbb{E}\{W(t_0)\} = \mathbb{E}\{V(t_0)\} < \varrho \mathbb{E}\{\overline{V}(t_0)\}$, set $t^* = \inf\{t \in (t_0, t_1), \mathbb{E}\{W(t)\} \geq \varrho \mathbb{E}\{\overline{V}(t_0)\}\}$ so that $\mathbb{E}\{W(t^*)\} = \varrho \mathbb{E}\{\overline{V}(t_0)\}$ and $\mathbb{E}\{W(t)\} < \varrho \mathbb{E}\{\overline{V}(t_0)\}, t \in (t_0, t^*)$. Furthermore, it follows from the definition of t^* and the continuity of $\mathbb{E}\{W(t)\}$ that there exists a $t_* = \sup\{t \in [t_0, t^*), \mathbb{E}\{W(t)\} \leq \mathbb{E}\{\overline{V}(t_0)\}\}$ so that $\mathbb{E}\{W(t_*)\} = \mathbb{E}\{\overline{V}(t_0)\}$ and $\mathbb{E}\{W(t)\} \geq \mathbb{E}\{\overline{V}(t_0)\}, t \in [t_*, t^*]$. It can be deduced that for $\theta \in [-\tau, 0]$ and $t \in [t_*, t^*]$, and therefore $\mathbb{E}\{W(t+\theta)\} \leq \varrho \mathbb{E}\{\overline{V}(t_0)\} \leq \varrho \mathbb{E}\{W(t)\}$ holds, which implies $\mathbb{E}\{V(t+\theta)\} \leq \varrho e^{-\eta\theta} \mathbb{E}\{V(t)\} \leq \varrho e^{\eta\tau} \mathbb{E}\{V(t)\}$. Then from condition (ii) we obtain $\mathbb{E}\{\tilde{A}V(t)\} < q\mathbb{E}\{V(t)\}$ for $t \in [t_*, t^*]$.

When $t \neq t_k, k \in \mathbb{Z}^+, r(t) = i$, it follows from (14) that

$$\begin{aligned} \mathbb{E}\{\tilde{A}W(t)\} &= \mathbb{E}\{\eta e^{\eta(t-t_0)}V(t)\} + e^{\eta(t-t_0)}\mathbb{E}\{\tilde{A}V(t)\} \\ &< \eta \mathbb{E}\{W(t)\} + e^{\eta(t-t_0)} q \mathbb{E}\{V(t)\} \\ &= (\eta + q)\mathbb{E}\{W(t)\}, t \in [t_*, t^*]. \end{aligned} \tag{16}$$

Integrate both sides of (16) from t_* to t^*. By using Dynkin's formula and the Gronwall-Bellman lemma, it follows from condition (ii) that

$$\begin{aligned} \mathbb{E}\{W(t^*)\} &< \mathbb{E}\{W(t_*)\} e^{(\eta+q)(t^*-t_*)} \\ &= \mathbb{E}\{\overline{V}(t_0)\} e^{(\eta+q)(t^*-t_*)} \\ &\leq \varrho \mathbb{E}\{\overline{V}(t_0)\}, \end{aligned}$$

which is a contradiction. Thus (15) holds for $t \in [t_0, t_1)$.

Now we assume that (15) holds for $t \in [t_{l-1}, t_l)$, for some $l \in \mathbb{Z}^+$. Next we claim that (15) holds for $t \in [t_l, t_{l+1})$. When $t = t_l$, it follows from condition (iii) that

$$\mathbb{E}\{V(t_l)\} \leq k_1 \mathbb{E}\{V(t_l^-)\} + k_2 \int_{t_l - d_l}^{t_l} \mathbb{E}\{V(s)\} ds.$$

There are two cases:

1) If $t_0 - \tau \leq t_l - d_l \leq t_0$, then it follows from (14) that

$$\begin{aligned} \mathbb{E}\{W(t_l)\} &\leq k_1 \mathbb{E}\{W(t_l^-)\} + k_2 e^{\eta(t_l - t_0)} \int_{t_l - d_l}^{t_0} \mathbb{E}\{V(s)\} ds + k_2 \int_{t_0}^{t_l} e^{\eta(t_l - t_0)} \mathbb{E}\{V(s)\} ds \\ &= k_1 \mathbb{E}\{W(t_l^-)\} + k_2 e^{\eta(t_l - t_0)} \int_{t_l - d_l}^{t_0} \mathbb{E}\{W(s)\} ds + k_2 \int_{t_0}^{t_l} e^{\eta(t_l - s)} \mathbb{E}\{W(s)\} ds \\ &\leq k_1 \mathbb{E}\{W(t_l^-)\} + k_2 \int_{t_l - d_l}^{t_l} e^{\eta(t_l - s)} \mathbb{E}\{W(s)\} ds \end{aligned}$$

$$
\begin{aligned}
&\leq k_1 \varrho \mathbb{E}\{\overline{V}(t_0)\} + k_2 d_l e^{\eta d_l} \varrho \mathbb{E}\{\overline{V}(t_0)\} \\
&\leq (k_1 + k_2 d_l e^{\eta d_l}) \varrho \mathbb{E}\{\overline{V}(t_0)\}.
\end{aligned} \tag{17}
$$

2) If $t_0 \leq t_l - d_l \leq t_l$, then it follows from (14) that

$$
\begin{aligned}
\mathbb{E}\{W(t_l)\} &\leq k_1 \mathbb{E}\{W(t_l^-)\} + k_2 \int_{t_l - d_l}^{t_l} e^{\eta(t_l - s)} \mathbb{E}\{W(s)\} ds \\
&\leq k_1 \varrho \mathbb{E}\{\overline{V}(t_0)\} + k_2 d_l e^{\eta d_l} \varrho \mathbb{E}\{\overline{V}(t_0)\} \\
&\leq (k_1 + k_2 d_l e^{\eta d_l}) \varrho \mathbb{E}\{\overline{V}(t_0)\}.
\end{aligned} \tag{18}
$$

Hence, it follows from condition (iv) that $\mathbb{E}\{W(t_l)\} \leq \mathbb{E}\{\overline{V}(t_0)\}$.

Suppose (15) is false for $t \in (t_l, t_{l+1})$. Then there exists a $\grave{t} \in (t_l, t_{l+1})$ such that $\mathbb{E}\{W(\grave{t})\} > \varrho \mathbb{E}\{\overline{V}(t_0)\}$. Set $\tilde{t} = \inf\{t \in (t_l, t_{l+1}), \mathbb{E}\{W(t)\} \geq \varrho \mathbb{E}\{\overline{V}(t_0)\}\}$. Then we have $\mathbb{E}\{W(\tilde{t})\} = \varrho \mathbb{E}\{\overline{V}(t_0)\}$ and $\mathbb{E}\{W(t)\} < \varrho \mathbb{E}\{\overline{V}(t_0)\}, t \in (t_l, \tilde{t})$. Furthermore, set $\bar{t} = \sup\{t \in [t_l, \tilde{t}), \mathbb{E}\{W(t)\} \leq \mathbb{E}\{\overline{V}(t_0)\}\}$, then we have $\mathbb{E}\{W(\bar{t})\} = \mathbb{E}\{\overline{V}(t_0)\}$ and $\mathbb{E}\{W(t)\} \geq \mathbb{E}\{\overline{V}(t_0)\}, t \in [\bar{t}, \tilde{t}]$. Then it can be deducted that for $\theta \in [-\tau, 0]$ and $t \in [\bar{t}, \tilde{t}]$, we have $\mathbb{E}\{W(t+\theta)\} \leq \varrho \mathbb{E}\{\overline{V}(t_0)\} \leq \varrho \mathbb{E}\{W(t)\}$ holds, which implies $\mathbb{E}\{V(t+\theta)\} \leq \varrho e^{-\eta\theta} \mathbb{E}\{V(t)\} \leq \varrho e^{\eta\tau} \mathbb{E}\{V(t)\}$. Then from condition (ii) we obtain $\mathbb{E}\{\tilde{A}V(t)\} < q\mathbb{E}\{V(t)\}$ for $t \in [\bar{t}, \tilde{t}]$.

When $t \neq t_k, k \in \mathbb{Z}^+, r(t) = i$, it follows from (14) that

$$
\begin{aligned}
\mathbb{E}\{\tilde{A}W(t)\} &= \mathbb{E}\{\eta e^{\eta(t-t_0)} V(t)\} + \mathbb{E}\{e^{\eta(t-t_0)} \tilde{A}V(t)\} \\
&< \eta \mathbb{E}\{W(t)\} + \mathbb{E}\{e^{\eta(t-t_0)} qV(t)\} \\
&= (\eta + q)\mathbb{E}\{W(t)\}, t \in [\bar{t}, \tilde{t}].
\end{aligned} \tag{19}
$$

Integrate both sides of (19) from $\bar{t}$ to $\tilde{t}$, by using Dynkin's formula and the Gronwall-Bellman lemma, it follows from condition (ii) that

$$
\begin{aligned}
\mathbb{E}\{W(\tilde{t})\} &< \mathbb{E}\{W(\bar{t})\} e^{(\eta+q)(\tilde{t}-\bar{t})} \\
&= \mathbb{E}\{\overline{V}(t_0)\} e^{(\eta+q)(\tilde{t}-\bar{t})} \\
&\leq \varrho \mathbb{E}\{\overline{V}(t_0)\},
\end{aligned}
$$

which is a contradiction. Thus (15) holds for $t \in [t_l, t_{l+1})$.

By the principle of mathematical induction, we have proved that (15) holds for $t \geq t_0$, then we have $\mathbb{E}\{V(t)\} \leq \varrho \mathbb{E}\{\overline{V}(t_0)\} e^{-\eta(t-t_0)}$, which implies that the trivial solution of system (5) is p-th moment exponentially stable.

Appendix B

Lemma 1. *[38]. Let $\Phi \in \mathbb{R}^{n \times n}$ be a positive definite matrix and $M \in \mathbb{R}^{n \times n}$ be a symmetric matrix, then for any $x \in \mathbb{R}^n$, the following inequality holds :*

$$
\lambda_{\min}(\Phi^{-1}M) x^T \Phi x \leq x^T M x \leq \lambda_{\max}(\Phi^{-1}M) x^T \Phi x.
$$

Lemma 2. *Let X and Y be any n-dimensional real vectors, and let P be an $n \times n$ positive semidefinite matrix. Then, the following matrix inequality holds:*

$$
2X^T PY \leq X^T PX + Y^T PY.
$$

Lemma 3. *[39]. Let $M \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix and constants $b > a$, then for any vector function $\omega : [a, b] \to \mathbb{R}^n$, the following inequality is established.*

$$
\left(\int_a^b \omega(s) ds \right)^T M \left(\int_a^b \omega(s) ds \right) \leq (b - a) \left(\int_a^b \omega^T(s) M \omega(s) ds \right).
$$

Lemma 4. *(Schur Complement). The linear matrix inequality*

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} < 0,$$

where $Q = Q^T, R = R^T$, is equivalent to either of the following conditions:
1) $Q < 0, R - S^T Q^{-1} S < 0$,
2) $R < 0, Q - S R^{-1} S^T < 0$.

Proof of Theorem 1: Choose the following stochastic Lyapunov function

$$V(e(t), t, r(t)) = e^T(t) P_{r(t)} e(t).$$

For simplicity, set $V(t) = V(e(t), t, r(t))$.

For any $t \neq t_k, k \in \mathbb{Z}^+$, noting that $r(t)$ takes values in $\Psi = \{1, 2, ..., S\}$ and $r(t) = r_m, m \in \mathbb{Z}^+$, we suppose that $r(t) = i \in \Psi$ and apply the law of total probability and conditional expectation to obtain

$$\begin{aligned}
\tilde{A}V(t) = &\lim_{\epsilon \to 0} \frac{1}{\epsilon} \Bigg[\mathbb{E} \Bigg\{ \sum_{j=1, j\neq i}^{S} Pr\{r_{m+1} = j, h_{m+1} \leq h + \epsilon | r_m = i, h_{m+1} > h\} e^T(t+\epsilon) P_j e(t+\epsilon) \\
&+ Pr\{r_{m+1} = i, h_{m+1} > h + \epsilon | r_m = i, h_{m+1} > h\} e^T(t+\epsilon) P_i e(t+\epsilon) \Bigg\} - e^T(t) P_i e(t) \Bigg] \\
= &\lim_{\epsilon \to 0} \frac{1}{\epsilon} \Bigg[\mathbb{E} \Bigg\{ \sum_{j=1, j\neq i}^{S} \frac{Pr\{r_{m+1} = j, r_m = i\}}{Pr\{r_m = i\}} \frac{Pr\{h < h_{m+1} < h + \epsilon | r_{m+1} = j, r_m = i\}}{Pr\{h_{m+1} > h | r_m = i\}} \\
&\times e^T(t+\epsilon) P_j e(t+\epsilon) + \frac{Pr\{h_{m+1} > h + \epsilon | r_m = i\}}{Pr\{h_{m+1} > h | r_m = i\}} e^T(t+\epsilon) P_i e(t+\epsilon) \Bigg\} - e^T(t) P_i e(t) \Bigg] \\
= &\lim_{\epsilon \to 0} \frac{1}{\epsilon} \Bigg[\mathbb{E} \Bigg\{ \sum_{j=1, j\neq i}^{S} \frac{q_{ij}(G_i(h+\epsilon) - G_i(h))}{1 - G_i(h)} e^T(t+\epsilon) P_j e(t+\epsilon) \\
&+ \frac{1 - G_i(h+\epsilon)}{1 - G_i(h)} e^T(t+\epsilon) P_i e(t+\epsilon) \Bigg\} - e^T(t) P_i e(t) \Bigg],
\end{aligned} \tag{20}$$

where $G_i(h)$ is the CDF of sojourn-time h when the system stays in mode i, and $q_{ij} = \frac{Pr\{r_{m+1}=j, r_m=i\}}{Pr\{r_m=i\}} = Pr\{r_{m+1} = j | r_m = i\}$ is the probability intensity of the system transition from mode i to mode j. Given a small ϵ, take the Taylor series with respect to ϵ at 0 as follows:

$$e(t+\epsilon) = e(t) + \epsilon \dot{e}(t) + o(\epsilon) = (\epsilon \Upsilon_i + \tilde{I}) \varsigma(t) + o(\epsilon), \tag{21}$$

where

$$\begin{cases} \varsigma(t) = [e^T(t), F^T(e(t)), F^T(e(t - \xi(t)))]^T, \\ \Upsilon_i = [-C_i, A_i, B_i], \\ \tilde{I} = [I_n, \mathbf{0}_{n\times n}, \mathbf{0}_{n\times n}]. \end{cases}$$

It follows from inequalities (20) and (21) that

$$\begin{aligned}
\tilde{A}V(t) = &\lim_{\epsilon \to 0} \frac{1}{\epsilon} \Bigg[\mathbb{E} \Bigg\{ \sum_{j=1, j\neq i}^{S} \frac{q_{ij}(G_i(h+\epsilon) - G_i(h))}{1 - G_i(h)} \varsigma^T(t) (\epsilon \Upsilon_i + \tilde{I})^T P_j (\epsilon \Upsilon_i + \tilde{I}) \varsigma(t) \\
&+ \frac{1 - G_i(h+\epsilon)}{1 - G_i(h)} \varsigma^T(t) (\epsilon \Upsilon_i + \tilde{I})^T P_i (\epsilon \Upsilon_i + \tilde{I}) \varsigma(t) \Bigg\} - e^T(t) P_i e(t) \Bigg], \; t \neq t_k, k \in \mathbb{Z}^+.
\end{aligned}$$

Considering the condition that $\lim_{\epsilon\to 0}\frac{G_i(h+\epsilon)-G_i(h)}{1-G_i(h)}=0$, one has

$$\begin{aligned}\tilde{A}V(t)=\lim_{\epsilon\to 0}\mathbb{E}\Bigg\{\varsigma^T(t)\Bigg[&\frac{1-G_i(h+\epsilon)}{1-G_i(h)}(\Upsilon_i^T P_i\tilde{I}+\tilde{I}^T P_i\Upsilon_i)\\&+\tilde{I}^T\Bigg(\sum_{j=1,j\neq i}^{S}\frac{q_{ij}(G_i(h+\epsilon)-G_i(h))}{\epsilon(1-G_i(h))}P_j+\frac{G_i(h)-G_i(h+\epsilon)}{\epsilon(1-G_i(h))}P_i\Bigg)\tilde{I}\Bigg]\varsigma(t)\Bigg\}, t\neq t_k, k\in\mathbb{Z}^+.\end{aligned}$$

Using the property of the CDF, we have

$$\lim_{\epsilon\to 0}\frac{1-G_i(h+\epsilon)}{1-G_i(h)}=1,\ \ \lim_{\epsilon\to 0}\frac{G_i(h+\epsilon)-G_i(h)}{\epsilon(1-G_i(h))}=\lambda_i(h),$$

where $\lambda_i(h)$ refers to the transition rate of the system transition from mode i.

Define $\lambda_{ij}(h)=q_{ij}\lambda_i(h),\ i\neq j$ and $\lambda_{ii}(h)=-\sum_{j=1,j\neq i}^{S}\lambda_{ij}(h)$, similar to [31], one has

$$\begin{aligned}\tilde{A}V(t)&=\varsigma^T(t)\Bigg[(\Upsilon_i^T P_i\tilde{I}+\tilde{I}^T P_i\Upsilon_i)+\tilde{I}^T\Bigg(\sum_{j=1}^{S}\bar{\lambda}_{ij}P_j\Bigg)\tilde{I}\Bigg]\varsigma(t)\\&=\varsigma^T(t)\begin{bmatrix}-C_i^T P_i-P_iC_i+\sum_{j=1}^{S}\bar{\lambda}_{ij}P_j & P_iA_i & P_iB_i\\ A_i^TP_i & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\\ B_i^TP_i & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\end{bmatrix}\varsigma(t),\end{aligned}$$

where $\varsigma(t)=[e^T(t),F^T(e(t)),F^T(e(t-\xi(t)))]^T,\ \bar{\lambda}_{ij}=\mathbb{E}\{\lambda_{ij}(h)\}=\int_0^\infty\lambda_{ij}(h)g_i(h)dh$.

It follows from Assumption 1 that

$$[F_j(e_j(t-\xi(t)))-l_{j1}e_j(t-\xi(t))]^T[l_{j2}e_j(t-\xi(t))-F_j(e_j(t-\xi(t)))]\geq 0,\ j=1,2,\ldots,n.$$

Therefore

$$\begin{aligned}0&\leq\sum_{j=1}^{n}u_j[F_j(e_j(t-\xi(t)))-l_{j1}e_j(t-\xi(t))]^T[l_{j2}e_j(t-\xi(t))-F_j(e_j(t-\xi(t)))]\\&=-e^T(t-\xi(t))U\hat{L}e(t-\xi(t))+2e^T(t-\xi(t))U\check{L}F(e(t-\xi(t)))\\&\quad-F^T(e(t-\xi(t)))UF(e(t-\xi(t))).\end{aligned}\tag{22}$$

From inequalities (5) and (22), for any $t\neq t_k, k\in\mathbb{Z}^+$, taking $r(t)=i$ and $r(t-\xi(t))=l\in\Psi$, we have

$$\begin{aligned}\tilde{A}V(t)&\leq\tilde{\omega}^T(t)\begin{bmatrix}-C_i^TP_i-P_iC_i+\sum_{j=1}^{S}\bar{\lambda}_{ij}P_j-U\hat{L} & P_iA_i+U\check{L} & \mathbf{0}_{n\times n} & P_iB_i\\ * & -U & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\\ * & * & -U\hat{L} & U\check{L}\\ * & * & * & -U\end{bmatrix}\tilde{\omega}(t)\\&=\tilde{\omega}^T(t)\begin{bmatrix}-C_i^TP_i-P_iC_i+\sum_{j=1}^{S}\bar{\lambda}_{ij}P_j-\alpha P_i-U\hat{L} & P_iA_i+U\check{L} & \mathbf{0}_{n\times n} & P_iB_i\\ * & -U & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n}\\ * & * & -U\hat{L}-\beta P_l & U\check{L}\\ * & * & * & -U\end{bmatrix}\tilde{\omega}(t)\\&\quad+\alpha e^T(t)P_ie(t)+\beta e^T(t-\xi(t))P_le(t-\xi(t))\\&<\alpha e^T(t)P_ie(t)+\beta e^T(t-\xi(t))P_le(t-\xi(t)),\end{aligned}$$

where $\tilde{\omega}(t) = [e^T(t), F^T(e(t)), e^T(t-\xi(t)), F^T(e(t-\xi(t)))]^T$. Hence

$$\tilde{A}V(t) < \alpha V(t) + \beta V(t-\xi(t)). \tag{23}$$

Taking expectation $\mathbb{E}\{\cdot\}$ on both sides of (23) leads to

$$\mathbb{E}\{\tilde{A}V(t)\} < \alpha \mathbb{E}\{V(t)\} + \beta \mathbb{E}\{V(t-\xi(t))\}.$$

For all $t \in [t_{k-1}, t_k), k \in \mathbb{Z}^+, \theta \in [-\tau, 0]$, if $\mathbb{E}\{V(t+\theta)\} \leq \varrho e^{\eta\tau}\mathbb{E}\{V(t)\}$, then we have

$$\begin{aligned}\mathbb{E}\{\tilde{A}V(t)\} &< \alpha \mathbb{E}\{V(t)\} + \beta \mathbb{E}\{V(t-\xi(t))\} \\ &\leq \alpha \mathbb{E}\{V(t)\} + \beta \varrho e^{\eta\tau}\mathbb{E}\{V(t)\} \\ &\leq (\alpha + \beta\varrho e^{\eta\tau})\mathbb{E}\{V(t)\}.\end{aligned} \tag{24}$$

For any $t = t_k, k \in \mathbb{Z}^+, r(t_k) = i$, it follows from Lemma 2, Lemma 3 and (5) that

$$\begin{aligned}V(t_k) &= e^T(t_k)P_i e(t_k) \\ &= \Big[D_i e(t_k^-) + K_i \int_{t_k-d_k}^{t_k} e(s)ds\Big]^T P_i \Big[D_i e(t_k^-) + K_i \int_{t_k-d_k}^{t_k} e(s)ds\Big] \\ &= e^T(t_k^-)D_i^T P_i D_i e(t_k^-) + 2e^T(t_k^-)D_i^T P_i K_i \int_{t_k-d_k}^{t_k} e(s)ds \\ &\quad + \Big(K_i \int_{t_k-d_k}^{t_k} e(s)ds\Big)^T P_i \Big(K_i \int_{t_k-d_k}^{t_k} e(s)ds\Big) \\ &\leq 2e^T(t_k^-)D_i^T P_i D_i e(t_k^-) + 2\Big(\int_{t_k-d_k}^{t_k} e(s)ds\Big)^T K_i^T P_i K_i \Big(\int_{t_k-d_k}^{t_k} e(s)ds\Big) \\ &\leq k_1 e^T(t_k^-)P_i e(t_k^-) + k_2\Big(\int_{t_k-d_k}^{t_k} e^T(s)P(r(s))e(s)ds\Big) \\ &= k_1 V(t_k^-) + k_2 \int_{t_k-d_k}^{t_k} V(s)ds.\end{aligned} \tag{25}$$

Taking expectations $\mathbb{E}\{\cdot\}$ in the above inequality, we have that

$$\mathbb{E}\{V(t_k)\} \leq k_1\mathbb{E}\{V(t_k^-)\} + k_2 \int_{t_k-d_k}^{t_k} \mathbb{E}\{V(s)\}ds, \;\; k \in \mathbb{Z}^+. \tag{26}$$

Thus by Proposition 1, we obtain

$$\mathbb{E}\{V(t)\} \leq \varrho \mathbb{E}\{\overline{V}(t_0)\}e^{-\eta(t-t_0)},$$

and therefore it follows from Lemma 1 that

$$\mathbb{E}\Big\{\|e(t,\psi)\|^2\Big\} \leq \frac{\varrho \max_{i\in\Psi}\{\lambda_{\max}(P_i)\}\|\psi\|_{\mathbb{E}}^2}{\min_{i\in\Psi}\{\lambda_{\min}(P_i)\}} e^{-\eta(t-t_0)}, \;\; \forall t \geq t_0,$$

which implies that neural network (1) is mean-square exponentially synchronized to response system (3). The proof of Theorem 1 is completed.

Appendix C

Proof of Theorem 2: It follows from Lemma 4 and (11) that $-k_1P_i + 2D_i^T P_i D_i < 0, -\frac{k_2}{d}P_z + 2K_i^T P_i K_i < 0,\; i, z \in \Psi$, and thus inequality (7) holds. Then by Theorem 1, neural network (1) is mean-square exponentially synchronized to response system (3) under the control gain matrices $D_i = P_i^{-1}G_i^T$ and $K_i = P_i^{-1}H_i^T$, $i \in \Psi$.

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