# ZEROS OF SPECTRAL FACTORS, THE GEOMETRY OF SPLITTING SUBSPACES, AND THE ALGEBRAIC RICCATI INEQUALITY*

ANDERS LINDQUIST †, GYÖRGY MICHALETZKY ‡, AND GIORGIO PICCI §

**Abstract.** In this paper we show how the zero dynamics of (not necessarily square) spectral factors relate to the splitting subspace geometry of stationary stochastic models and to the corresponding algebraic Riccati inequality. We introduce the notion of *output-induced subspace* of a minimal Markovian splitting subspace, which is the stochastic analogue of the *supremal output-nulling subspace* in geometric control theory. Through this concept the analysis can be made coordinate-free, and straightforward geometric methods can be applied. We show how the zero structure of the family of spectral factors relates to the geometric structure of the family of minimal Markovian splitting subspaces in the sense that the relationship between the zeros of different spectral factors is reflected in the partial ordering of minimal splitting subspaces. Finally, we generalize the well-known characterization of the the solutions of the algebraic Riccati equation in terms of Lagrangian subspaces invariant under the corresponding Hamiltonian to the larger solution set of the algebraic Riccati inequality.

**Key words.** Zero dynamics, Markovian splitting subspaces, minimal spectral factors, matrix Riccati inequality, algebraic Riccati equation, geometric control theory.

**AMS subject classifications.** 93E03, 93B27, 60G10.

## 1. Introduction.

By now it should be fairly well-known that there is a one-one correspondence between each two the following three fundamental areas of systems theory.

**(i)** *Minimal spectral factorization* of a rational (full-rank) $m \times m$ spectral density matrix $\Phi$. The problem is to find *all* (square *and* rectangular) rational functions

$$W(s) = C\,(sI - A)^{-1}\,B + D, \tag{1.1}$$

(where prime denotes transposition) with poles in the open left half plane, satisfying the factorization equation

$$W(s)W(-s)' = \Phi(s), \tag{1.2}$$

and being *minimal* in the sense that the McMillan degree of $W$ is exactly half of that of $\Phi$. The class of all such minimal spectral factors, each defined modulo right multiplication by a constant orthogonal matrix, will be denoted by $\mathcal{W}$. The subclass of *square* spectral factors will be denoted $\mathcal{W}_0$. Throughout this paper we shall always consider representations for which (A,B,C) is a minimal triplet and $\begin{bmatrix} B \\ D \end{bmatrix}$ has independent columns. This results in no loss of generality [16].

**(ii)** Finding all symmetric solutions of the *algebraic Riccati inequality*

$$\Lambda(P) \leq 0, \tag{1.3}$$

* This work was partially supported by the Swedish Board for Technical Development, the Göran Gustafsson Foundation, LADSEB-CNR, and OTKA under grant No 2042.

† Division of Optimization and Systems Theory, Royal Institute of Technology, 100 44 Stockholm, Sweden

‡ Department of Probability Theory and Statistics, Eötvös Loránd University, Budapest

§ Department of Electronics and Informatics, University of Padova, and LADSEB-CNR, Padova, Italy

where $\Lambda : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is given by

$$\Lambda(P) = AP + PA' + (\bar{C} - CP)' R^{-1}(\bar{C} - CP), \tag{1.4}$$

the matrices $A \in \mathbb{R}^{n\times n}, C, \bar{C} \in \mathbb{R}^{m\times n}$ and $R \in \mathbb{R}^{m\times m}$ being defined through a minimal realization

$$\Phi_+(s) = C(sI - A)^{-1}\bar{C}' + \frac{1}{2}R \tag{1.5}$$

of the positive real part $\Phi_+$ of the spectral density $\Phi$, i.e. of all rational matrix functions satisfying

$$\Phi(s) = \Phi_+(s) + \Phi_+(-s)' \tag{1.6}$$

$\Phi_+$ is the one having all its poles in the open left half plane. Here we assume that $R := \Phi(\infty) > 0$.

Let us denote by $\mathcal{P}$ the solution set of (1.3). Then each $P \in \mathcal{P}$ corresponds to a spectral factor (1.1) whose $B$- and $D$-matrices are determined by a full-rank matrix factorization of the type

$$\begin{bmatrix} B \\ D \end{bmatrix} [B', D'] = \begin{bmatrix} -AP - PA' & \bar{C}' - PC' \\ \bar{C} - CP & R \end{bmatrix} \tag{1.7}$$

Obviously the correspondence is one-one modulo trivial coordinate transformations ([1], [9]).

**(iii)** Describing all *minimal stochastic realizations* of an $m$-dimensional stationary-increments process $\{y(t); t \in \mathbb{R}\}$ having the (incremental) spectral density $\Phi$. Each stochastic realization is obtained by passing a suitable "white noise" through a filter

$$\xrightarrow{dw} \boxed{W} \xrightarrow{dy} \tag{1.8}$$

having an $m \times p$ minimal spectral factor as its transfer function, thus yielding a linear dynamical model

$$(\Sigma) \quad \begin{cases} dx = Axdt + Bdw \\ dy = Cxdt + Ddw \end{cases} \tag{1.9}$$

for $dy$, defined on the whole real line. More precisely, $w$ is a vector Wiener process on $\mathbb{R}$ of a dimension $p$ equal to the number of columns of $W$. The system $\Sigma$ is in statistical steady state so that the $n$-dimensional state process $x$ and the increments of the $m$-dimensional output process $y$ are jointly stationary. The model $\Sigma$ is a *minimal* stochastic realization in the sense that there is no other representation of $dy$ of type (1.9) with a state process with fewer components.

In regard to topic (iii), it is actually more natural to consider a coordinate-free representation by assigning to each model $\Sigma$ the $n$-dimensional space

$$X = \{a'x(0)\,|a \in \mathbb{R}^n\} \tag{1.10}$$

of random variables. This space is the subspace of an *ambient space* $H$ of the model (1.9), defined as the closure of the linear hull of the following random variables $\{w_i(t) - w_i(\tau);\ i = 1, 2, \dots,\ p; t, \tau \in \mathbb{R}\}$ in the topology of the inner product

$$< \xi, \eta >= E\{\xi\eta\}, \tag{1.11}$$

where $E\{\cdot\}$ stands for mathematical expectation. The ambient space $H$ is naturally equipped with the shift induced by $dw$, i.e. the strongly continuous group of unitary operators $\{U_t;\ t \in \mathbb{R}\}$ on $H$ such that $U_t\left[w_i(\tau) - w_i(\sigma)\right] = w_i(\tau + t) - w_i(\sigma + t)$ for all $i = 1, 2, \ldots, p$ and $t, \tau, \sigma \in \mathbb{R}$. All random variables of $\Sigma$ belong to $H$, and moreover the processes $x$ and $dy$ are stationary with respect to $\{U_t\}$, i.e. $U_t x_i(\tau) = x_i(\tau + t)$ for all $i = 1, 2, \ldots, n$ and $t, \tau \in \mathbb{R}$ and $U_t\left[y_i(\tau) - y_i(\sigma)\right] = y_i(\tau + t) - y_i(\sigma + t)$ for all $i = 1, 2, \ldots, m$ and $t, \tau, \sigma \in \mathbb{R}$. Minimality of $\Sigma$ corresponds to minimality of the subspace $X$ in the sense of subspace inclusion, and hence also in the sense of dimension [16].

Defining the *past* and *future output spaces* as

$$H^- = \text{closure}\left\{a'\left[y(t) - y(s)\right] | a \in \mathbb{R}^m, t, s \leq 0\right\}$$

and

$$H^+ = \text{closure}\left\{a'\left[y(t) - y(s)\right] | a \in \mathbb{R}^m, t, s \geq 0\right\}$$

respectively, it is easy to show and well-established in the literature ([15], [16], [6]) that each $X$, defined as in (1.10), is a *minimal Markovian splitting subspace* for $H^-$ and $H^+$, i.e., in particular renders $H^-$ and $H^+$ conditionally orthogonal given $X$. Moreover this property captures the concept of stochastic state space model of $dy$ in a coordinate-free way. Given any $X$ together with its ambient space $H$, equipped with a shift, we can construct the model $\Sigma$ modulo the choice of coordinates in the state space [16].

Modulo coordinate-transformations, there is a one-one correspondence between the family $\mathcal{X}$ of minimal Markovian splitting subspaces and the solution set $\mathcal{P}$ of the algebraic Riccati inequality (1.3) under which

$$P = E\left\{x(0)x(0)'\right\}, \tag{1.12}$$

is the state covariance. Under this correspondence the subset $\mathcal{P}_0 \subset \mathcal{P}$ of solutions of the *algebraic Riccati equation*

$$\Lambda(P) = 0, \tag{1.13}$$

corresponds to the subclass $\mathcal{X}_0 \subset \mathcal{X}$ of stochastic realizations such that [1]

$$X \subset H_0 := H^- \vee H^+ \tag{1.14}$$

i.e. *internal* realizations constructed by using only random quantities contained in the subspace

$$H_0 = \text{closure}\left\{a'\left[y(t) - y(s)\right] | a \in \mathbb{R}^n\right\}$$

spanned by the output. Under the correspondence mentioned above, $\mathcal{X}_0$ and $\mathcal{P}_0$ correspond to $\mathcal{W}_0 \subset \mathcal{W}$, the subclass of square spectral factors.

Although the structure of the solution set of the algebraic Riccati equation (1.13) is by now fairly well established ([27], [20], [26], [13]), it is fair to say that the structure of the complete solution set $\mathcal{P}$ of the algebraic Riccati inequality (1.3) is far less

[1] In the sequel, given two subspaces $A$ and $B$, we shall write $A \vee B$ to denote the closure of $\{\alpha + \beta \,|\, \alpha \in A, \beta \in B\}$. To stress that the sum is direct we write instead $A + B$ or, if it is an orthogonal direct sum $A \oplus B$.

understood, and, except for [10], [16], and [25], little seems to have appeared in the literature since the monograph [9]. We stress that the algebraic Riccati inequality and the set $\mathcal{P}$ are important in many areas of systems and control, including dissipative systems and $H^\infty$ control.

In this respect, one purpose of this paper is to provide new results on the structure of $\mathcal{P}$ and new concepts for the study and classification of this set based on the *zero structure* of the family $\mathcal{W}$ of minimal spectral factors $W$. The work reported here is a continuation and a deepening of the results presented in [16] and [19]. In particular it was shown in [16] that

1° The set $\mathcal{P}$ (which is bounded and convex) has *facets* each of which is uniquely defined by a pair of solutions of the algebraic Riccati equation (1.13). For each $P \in \mathcal{P}$ there is a minimal facet $[P_{0-}, P_{0+}]$ containing $P$, called the *tightest local frame* of $P$, defined as the set of all solutions $Q$ of the algebraic Riccati inequality (1.3) satisfying the relation $P_{0-} \leq Q \leq P_{0+}$, where

$$P_{0-} := \sup \{P_0 \in \mathcal{P}_0 \mid P_0 \leq P\}$$

$$P_{0+} := \inf \{P_0 \in \mathcal{P}_0 \mid P \leq P_0\}.$$

Here, for any $P_1, P_2 \in \mathcal{P}$, $P_1 \leq P_2$ means that $P_2 - P_1$ is nonnegative definite. The tightest bounds of $P$, $P_{0-}$ and $P_{0+}$, can be computed as the limit solutions of the matrix Riccati differential equation $\dot{\Pi} = \Lambda(\Pi)$, with initial condition $\Pi(0) = P$, as $t$ tends to $-\infty$ and $\infty$ respectively.

2° The *open tightest frame* $(P_{0-}, P_{0+})$ of $P \in \mathcal{P}$, consisting of all $Q \in [P_{0-}, P_{0+}]$ having $P_{0-}$ and $P_{0+}$ as tight bounds, can be characterized in terms of the *zeros* of the corresponding minimal spectral factor $W$. If $(W_{0-}, W_{0+})$ is the pair of square minimal spectral factors corresponding to $P_{0-}$ and $P_{0+}$, then the zeros of $W$ are precisely the common zeros of $W_{0-}$ and $W_{0+}$.

In this paper we greatly expand on the above characterization of facets and tight frames providing necessary and sufficient conditions in terms of zeros (or, better, the *zero dynamics*) of spectral factors. To this end, in Section 2, we first provide a geometric characterization of the zero dynamics in the stochastic framework (Theorem 2.10). In particular, we demonstrate how the zero structure of each $W$ can be recovered directly from the corresponding *output-induced subspace* $X \cap H_0$ and a related compressed shift. We introduce a dual control problem and show that its maximal output-nulling subspace consists of precisely those $a \in \mathbb{R}^n$ for which $a'x(0) \in X \cap H_0$ and that these $a$ are also the zero directions of $W$. In this way we not only provide the appropriate connection to geometric control theory ([3],[28]) but also obtain elegant coordinate-free proofs of the main theorems of Sections 2 and 3.

Next, in Section 3, we analyze the relation between partial ordering of minimal splitting subspaces and zeros, and characterize the ordering in terms of invariant subspaces for the zero dynamics and right-half-plane zeros. The results on ordering are very intuitive and are in agreement with some early observations of Anderson [2] and Robinson [23]. The characterizations in terms of invariant subspaces extend those known sofar for square spectral factors and the algebraic Riccati equation, as for example reported in the survey of Kucera [13].

Sofar all results are coordinate-free. Then, in Section 4, we introduce coordinates and translate the geometric characterizations of Sections 2 and 3 in terms of covariances and solutions of the algebraic Riccati inequality. Through this analysis we also obtain a natural generalization of the well-known characterization (Potter[22], MacFarlane[17]; also see [26]) of $\mathcal{P}$ in terms of the $n$-dimensional Lagrangian subspaces $\mathcal{L} \subset \mathbb{R}^{2n}$, invariant under multiplication by the Hamiltonian $\mathcal{H}$ corresponding to $\Phi$. In fact, in Section 5, we show that the $\mathcal{H}$-invariant isotropic subspaces $\mathcal{L}$ of dimension $k \leq n$ are in one-one correspondence to the facets of $\mathcal{P}$ whose elements $P$ have identical zero structure. Under this correspondence

$$\mathcal{L} = \begin{bmatrix} I \\ P \end{bmatrix} \mathcal{V}^*, \tag{1.15}$$

where $\mathcal{V}^* \subset \mathbb{R}^n$ is the space on which the zero dynamics of $W$ is defined and which corresponds in $\mathcal{X}$ to the output-induced subspace $X \cap H_0$ of $X$.

We make extensive cross reference between the three frameworks of $\mathcal{P}, \mathcal{X}$ and $\mathcal{W}$, and there are some very good reasons for this. The geometric splitting subspace theory provides a very natural setting also for analyzing the algebraic Riccati inequality. In fact, several geometric results which are linked to such concepts as *splitting* and *internal subspace* have less obvious counterparts in the $\mathcal{P}$-setting and could easily have been overlooked had it not been for the interaction with the geometry of splitting subspaces.

**2. Zero dynamics and splitting subspaces.** It is well-known by now that the poles of a spectral factor $W$ can be expressed in terms of the shift $\{U_t\}$ and the corresponding splitting subspace $X$ [16]. In fact the compressed forward shift on $X$,

$$U_t(X) := E^X U_t \,|_X \quad \text{for} \quad t \geq 0 \tag{2.1}$$

(where $E^X$ is the orthogonal projector onto $X$), is a strongly continuous and uniformly bounded semigroup so that

$$U_{t+\tau}(X) = U_t(X)U_\tau(X), \tag{2.2}$$

and therefore there is an operator $F : X \to X$ such that

$$U_t(X) = e^{Ft} \tag{2.3}$$

Then it can be shown that

$$\{\text{poles of } W\} = \sigma(F) \tag{2.4}$$

i.e. the poles of $W$ are precisely the eigenvalues of $F$. To see this, take $a \in \mathbb{R}^n$ and integrate (1.9) to obtain

$$a'x(t) = a'e^{At}x(0) + \int_0^t a'e^{A(t-s)}Bdw(s), \tag{2.5}$$

the last term of which is orthogonal to $X$. Consequently, $E^X U_t a'x(0) = a'e^{At}x(0)$, i.e. $e^{Ft}a'x(0) = a'e^{At}x(0)$, showing that $A'$ is in fact a matrix representation of $F$.

The basic question which we shall address in this section is the following. Is there an analogous geometric characterization of the *zeros* of $W$ in terms of $\{U_t\}$ and $X$? As we shall see, the answer to this question is yes.

To simplify matters, in this paper we shall make the blanket assumption that the spectral density $\Phi$ is *coercive*, i.e. $\Phi$ has no zeros on the imaginary axis or at infinity. In particular this implies that

$$R := \Phi(\infty) > 0 \tag{2.6}$$

so that all minimal spectral factors $W$ are of dimension $p \times m$ with $p \geq m$ and of full rank $m$ almost everywhere in the complex plane, and hence right invertible. Let

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{2.7}$$

be a minimal realization of $W$. Recall [7] that a complex number $\lambda$ is called a *right zero* of $W$ (or, equivalently, of the state-space system (2.7)) if, for some $\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \neq 0$, $u(t) = u_0 e^{\lambda t}$, $x(t) = x_0 e^{\lambda t}$ satisfy (2.7) while at the same time keeping the output $y(t)$ (identically) zero for all $t \in \mathbb{R}$.

It is well known and trivial to check that $\lambda \in \mathbb{C}$ is a right zero of $W$ if and only if there are nonzero solutions of [2]

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0. \tag{2.8}$$

More generally it can be shown [28] that constraining the dynamic variables $x$ and $u$ in (2.7) to yield an identically zero output $y \equiv 0$ requires confining, for all times $t \in \mathbb{R}$, the state $x(t)$ of the system (2.7) to a particular subspace $\mathcal{V}^* = \mathcal{V}^*(A, B, C, D) \subset \mathbb{R}^n$ called the *maximal output nulling subspace* of the system (2.7). The inputs $u$ which keep $x(t)$ in $\mathcal{V}^*$ for all $t \in \mathbb{R}$ can be generated by suitable linear state feedback laws

$$u = Kx + Lv \quad x \in \mathcal{V}^* \tag{2.9}$$

where $L$ is such that $\mathrm{Im} BL \subset \mathcal{V}^*$, $DL = 0$, and $v$ is an unconstrained input function. Any $K$ achieving this is called a *friend* of $\mathcal{V}^*$ [28]. It can be shown that $\mathcal{V}^*$ is actually the largest subspace $\mathcal{V} \subset \mathbb{R}^n$ for which there is a feedback matrix $K$ such that

$$(A + BK)\mathcal{V} \subset \mathcal{V} \subset \ker(C + DK). \tag{2.10}$$

It follows from the discussion above that all $x_0$ solving (2.8) belong to $\mathcal{V}^*(A, B, C, D)$. Conversely, the subspace $\mathcal{V}^*$ can be associated to the right zeros of (2.7) in the following sense. If $K$ is a friend of $\mathcal{V}^*$ and $u$ is generated by a feedback law (2.9), all solutions of

$$\dot{x} = (A + BK)x + BLv \quad x(0) \in \mathcal{V}^* \tag{2.11}$$

belong to $\mathcal{V}^*$ for all times t and all inputs $v$. Pick $\lambda_0$ in the spectrum of $(A+BK)|_{\mathcal{V}^*}$, let $x_0$ be the corresponding eigenvector, and set $u_0 := Kx_0$. Then it is trivial to check that $\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}$ solves (2.8) for $\lambda = \lambda_0$ and hence $\lambda_0$ is a right zero of (2.7). Those zeros which are reachable modes for the system (2.11) can actually be moved arbitrarily in the complex plane by a suitable choice of $v$. Those which are *not* reachable are fixed and are called *invariant zeros* of $W$. They are in fact even independent of the particular choice of the matrix $K$ [28]. The *maximal reachability subspace* $\mathcal{R}^*(A, B, C, D)$ of $W$ is precisely the maximal subspace of $\mathcal{V}^*$ which is reachable by inputs produced by feedback laws of the form (2.9). If $\mathcal{R}^*(A, B, C, D) = 0$, then all zeros are invariant.

[2] Note that there are infinitely many $\lambda$ for which the matrix in (2.8) has a nonzero kernel when $p > m$, and hence there are infinitely many right zeros in this case.

In our setup the spectral factors $W$ are most naturally viewed as operators acting on input functions from the left, and it is more appropriate to consider *left zeros* instead. These are defined simply as the right zeros of the transpose $W'$. Given a minimal realization of $W$ as in (2.7), a complex number $\lambda$ is then a left zero of $W$ if and only if there is a nonzero vector $\begin{bmatrix} z_0 \\ u_0 \end{bmatrix}$ which solves

$$\begin{bmatrix} A' - \lambda I & C' \\ B' & D' \end{bmatrix} \begin{bmatrix} z_0 \\ u_0 \end{bmatrix} = 0. \tag{2.12}$$

It is easy to show that the vectors $z_0$ solving (2.12) for some $\lambda$ form a subspace $\mathcal{V} \subset \mathbb{R}^n$ which is $(A', C')$-invariant and output-nulling. In fact, $\mathcal{V}$ is a subspace of the *maximal output nulling subspace* $\mathcal{V}^* := \mathcal{V}^*(A', C', B', D')$ of the *dual* system

$$(\Sigma') \quad \begin{cases} \dot{z} = A'z + C'u \\ v = B'z + D'u \end{cases} \tag{2.13}$$

corresponding to $W'$. We note that the maximal reachability subspace of $\Sigma'$, i.e. the subspace $\mathcal{R}^*(A', C', B', D')$ is just the zero space, since $W'$ is left invertible [12] (Theorem 3.36). In other words, the left zeros of $W$ are all invariant.

Now, since $\mathcal{R}^*(A', C', B', D') = 0$, it can be shown that there is a friend $K'$, whose restriction to $\mathcal{V}^*$ is unique, making $\mathcal{V}^*$ $(A' + C'K')$-invariant. The autonomous system

$$\dot{z}(t) = (A' + C'K')z(t) \quad z(0) \in \mathcal{V}^*, \tag{2.14}$$

with state space $\mathcal{V}^*$, will be called the (left) *zero dynamics* of $\Sigma$ (or of $W$) ([4], [21]). The eigenvalues of the feedback matrix $(A' + C'K')|_{\mathcal{V}^*}$ are the *(left) zeros* of $W$. As we have pointed out above, all left zeros are invariant. Clearly the invariant zeros of $W$ are the same from the left and from the right. There are, however, noninvariant right zeros of $W$ which are not left zeros (since, in general, $\mathcal{R}^*(A, B, C, D) \neq 0$). From now on we shall only consider left zeros and left zero dynamics and therefore we shall drop the attribute ”left”.

Note that the zero dynamics of $W$ is naturally defined only modulo similarity, i.e. modulo coordinate transformations in the state space of minimal realizations (1.1) of $W$. The vector space $\mathcal{V}^* := \mathcal{V}^*(A', C', B', D')$ will be called the space of *zero directions* of $W$.

For later reference we shall now explicitly compute the zero dynamics of $W$ for the special case under consideration. To this end, it is convenient to write the system (1.9) in standard form taking

$$\begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ R^{1/2} & 0 \end{bmatrix} \tag{2.15}$$

where $R = DD'$ and $R^{1/2}$ is the symmetric square root of $R$. This can be achieved by an orthogonal coordinate transformation in input space, which of course will not affect the zeros of the spectral factor $W$. Eliminating the noise $dw_1$ in

$$(\Sigma) \quad \begin{cases} dx = Axdt + B_1 dw_1 + B_2 dw_2 \\ dy = Cxdt + R^{1/2} dw_1 \end{cases} \tag{2.16}$$

produces a state representation

$$dx = \Gamma x dt + B_1 R^{-1/2} dy + B_2 dw_2 \tag{2.17}$$

in feedback form where $\Gamma$ is the feedback matrix

$$\Gamma = A - B_1 R^{-1/2} C. \tag{2.18}$$

Let us return to the dual control system (2.13). Then, setting the output $v$ equal to zero yields

$$\begin{cases} \dot{z} = A'z + C'u \\ 0 = B_1'z + R^{1/2}u \\ 0 = B_2'z \end{cases} \tag{2.19}$$

or, eliminating the control $u$,

$$\begin{cases} \dot{z} = \Gamma'z \\ B_2'z = 0. \end{cases} \tag{2.20}$$

Consequently, the maximal output-nulling subspace $\mathcal{V}^*$ is precisely

$$\mathcal{V}^* = \langle \Gamma \mid B_2 \rangle^{\perp} \tag{2.21}$$

i.e. the orthogonal complement of the reachability space

$$\langle \Gamma \mid B_2 \rangle = \operatorname{Im}(B_2, \Gamma B_2, \Gamma^2 B_2, \ldots) \tag{2.22}$$

in $\mathbb{R}^n$. Now, it follows from the discussion above that the invariant zeros of $W$ are precisely the eigenvalues of $\Gamma' |_{\mathcal{V}^*}$, for the maximal reachability space $\mathcal{R}^*$ of the autonomous dynamics (2.20) is zero. Consequently, $\Gamma' |_{\mathcal{V}^*}$ is the generator of the zero dynamics of $W$. In particular

$$\{\text{zeros of } W\} = \sigma \{\Gamma' |_{\mathcal{V}^*}\} \tag{2.23}$$

Next we turn to the stochastic version of this theory. For this we need the following definition.

DEFINITION 2.1. *Let $X$ be a Markovian splitting subspace. A subspace $Y \subset X$ is called output induced if*

(i) $Y \subset H_0$

(ii) $U_t Y \subset Y \vee H^+_{[0,t]}$ *for* $t \geq 0$,

*where $H^+_{[0,t]}$ is the subspace spanned by the output $dy$ on the finite interval $[0,t]$, i.e.*

$$H^+_{[0,t]} = \text{closure}\, \{a'\,[y(\tau) - y(s)]\, |a \in \mathbb{R}^m, \tau, s \in [0,t]\, \}\,.$$

(iii) $U_t Y \subset Y \vee H^-_{[t,0]}$ *for* $t \leq 0$,

*where $H^-_{[t,0]}$ is spanned by the output on $[t,0]$.*

The following proposition, the proof of which will be postponed to the Appendix, establishes the fact that an output-induced subspace is actually a stochastic counterpart of an $(A,B)$-invariant subspace in geometric control theory.

PROPOSITION 2.2. *Let $Y \subset X \cap H_0$ be output-induced. Then*

$$FY \subset Y \vee \operatorname{Im} N \tag{2.24}$$

*where the linear operators $F : X \to X$ and $N : \mathbb{R}^m \to X$ are defined by (2.3) and*

$$Na = \lim_{h\downarrow 0} \frac{1}{h} E^X a'[y(h) - y(0)] \tag{2.25}$$

*respectively.*

As we have already noted above, $F$ has the matrix representation $A'$ in the basis in $X$ consisting of the components of $x(0)$. Moreover, it was proven in [14] that

$$Cx(0) = \lim_{h\downarrow 0} \frac{1}{h} E^X [y(h) - y(0)] \tag{2.26}$$

and consequently $Na = a'Cx(0)$, i.e. $N$ has the matrix representation $C'$ in the basis $x(0)$. Therefore, condition (2.24) is equivalent to $(A', C')$-invariance of the representative of Y in the aforementioned coordinate system. To make this correspondence more precise we shall consider next the problem of finding the *maximal* output-induced subspace of a given minimal Markovian splitting subspace.

THEOREM 2.3. *Let $X$ be a minimal Markovian splitting subspace. Then there is a maximal output-induced subspace of $X$, namely $Y^* := X \cap H_0$. The subspace $Y^*$ is maximal in the sense that $Y \subset Y^*$ for any other output-induced subspace $Y$ of $X$.*

There is a close connection between the concept of maximal output-induced subspace of a minimal Markovian splitting subspace and the zero dynamics of the corresponding minimal spectral factor. This connection is best understood by regarding the realization (1.9).

LEMMA 2.4. *Let $X \in \mathcal{X}$ and let (1.9) be a corresponding minimal realization. Then*

$$X \cap H_0 = \{a'x(0) | \, a \in \mathcal{V}^*(A', C', B', D')\} .$$

*Proof.* First take $\xi \in X \cap H_0$. Then $\xi$ has a representation $\xi = a'x(0)$ where $a \in \mathbb{R}^n$. We shall prove that $a \in \mathcal{V}^* := \mathcal{V}^*(A', C', B', D')$. We immediately see that

$$\xi = a'x(0) = \int_{-\infty}^{0} a'e^{-At} B dw(t). \tag{2.27}$$

On the other hand, since $\xi \in H_0$, there is a representation

$$\xi = \int_{-\infty}^{\infty} \hat{u}(i\omega)' d\hat{y}(i\omega), \tag{2.28}$$

where $\hat{u}$ is a vector function on the imaginary axis which is $L_2$ with respect to the matrix measure $\frac{1}{2\pi}\Phi(i\omega)d\omega$ and $d\hat{y}$ is the spectral measure [24] of the process $dy$, i.e.

$$y(t) - y(s) = \int_{-\infty}^{\infty} \frac{e^{i\omega t} - e^{i\omega s}}{i\omega} d\hat{y}.$$

This spectral measure may be written

$$d\hat{y} = W d\hat{w}$$

in terms of the spectral factor (1.1), the transfer function of (1.9), and the spectral measure $d\hat{w}$ of the generating noise $dw$ of (1.9). Consequently,

$$\xi = \int_{-\infty}^{\infty} \hat{u}(i\omega)' W(i\omega) d\hat{w}(i\omega), \tag{2.29}$$

Where $\hat{f} := W'\hat{u}$ is an $L_2$ function on the imaginary axis with inverse Fourier transform

$$f(t) = \int_{-\infty}^{t} B'e^{A'(t-s)}C'u(s)ds + D'u(t), \tag{2.30}$$

where $u$ is the inverse Fourier transform of $\hat{u}$ in the $L_2$ sense. (To see that $\hat{u}$ is $L_2$ note that $\Phi(\infty)$ is nonsingular by assumption.) Then (2.29) may be written

$$\xi = \int_{-\infty}^{\infty} f(-t)'dw(t) \tag{2.31}$$

in the time domain [24] [15], and, in view of (2.27), we must have

$$f(t) = \begin{cases} B'e^{A't}a & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t \leq 0 \end{cases} \tag{2.32}$$

Hence, if we set

$$v(t) := B'\left\{e^{A't}(-a) + \int_{-\infty}^{t} e^{A'(t-s)}C'u(s)ds\right\} + D'u(t) \tag{2.33}$$

$$= B'\left\{e^{A't}[-a + \bar{z}(0)] + \int_{0}^{t} e^{A'(t-s)}C'u(s)ds\right\} + D'u(t), \tag{2.34}$$

where

$$\bar{z}(0) = \int_{-\infty}^{0} e^{-A's}C'u(s)ds,$$

it is seen from (2.32) that $v(t) = 0$ for $t \geq 0$, and hence $u$ is an output-nulling input for the dual control system

$$(\Sigma') \quad \begin{cases} \dot{z} = A'z + C'u \\ v = B'z + D'u \end{cases} \tag{2.35}$$

initiated at $z(0) = -a + \bar{z}(0)$. Therefore $-a + \bar{z}(0) \in \mathcal{V}^*$. On the other hand, (2.30) and (2.32) show that the output of $\Sigma'$ with control $u$ and initial condition $z(-\infty) = 0$ is identically zero on the negative real axis. Therefore the corresponding state trajectory

$$\bar{z}(t) = \int_{-\infty}^{t} e^{A'(t-s)}C'u(s)ds$$

belongs to $\mathcal{V}^*$ for $t \leq 0$. Hence, in particular, $\bar{z}(0) \in \mathcal{V}^*$, and consequently $a \in \mathcal{V}^*$ as claimed.

To prove the converse statement, we first note that the coercivity of $\Phi$ insures that $\Gamma' |_{\mathcal{V}^*}$ has no eigenvalues on the imaginary axis, since the a minimal spectral factor $W$ are also zeros of the spectral density $\Phi$. Therefore $\mathcal{V}^*$ can be decomposed into a direct sum

$$\mathcal{V}^* = \mathcal{V}^*_- + \mathcal{V}^*_+, \tag{2.36}$$

where $\mathcal{V}^*_-$ is the sum of the generalized eigenspaces corresponding to eigenvalues of $\Gamma' |_{\mathcal{V}^*}$ with negative real part and $\mathcal{V}^*_+$ is the corresponding subspace for eigenvalues with positive real parts. Both $\mathcal{V}_-^*$ and $\mathcal{V}_+^*$ are of course invariant for $\Gamma'$. We want to prove that, if $a \in \mathcal{V}^*$, then $a'x(0) \in X \cap H_0$. To this end, take $a \in \mathcal{V}^*$ and let $a = a_- + a_+$ where $a_- \in \mathcal{V}_-^*$ and $a_+ \in \mathcal{V}_+^*$. Since, in view of (2.21), $\mathcal{V}^* \perp \mathrm{Im}\, B_2$, (2.17) yields

$$d(a'x) = a'\Gamma x dt + a'B_1 R^{-1/2} dy \tag{2.37}$$

for any $a \in \mathcal{V}^*$. Therefore, by choosing a basis in $\mathcal{V}^*$ consistent with the direct sum decomposition (2.36) , (2.37) produces two equations relative to $\mathcal{V}_-^*$ and $\mathcal{V}_+^*$ which by $\Gamma$-invariance can be integrated separately on the negative and positive time axis respectively. It then follows that

$$a_-'x(0) = \int_{-\infty}^{0} a_-' e^{-\Gamma t} B_1 R^{-1/2} dy(t) \quad \text{for} \quad a_- \in \mathcal{V}_-^* \tag{2.38}$$

and

$$a_+'x(0) = \int_{0}^{\infty} a_+' e^{-\Gamma t} B_1 R^{-1/2} dy(t) \quad \text{for} \quad a_+ \in \mathcal{V}_+^* \tag{2.39}$$

and hence $a_-'x(0) \in X \cap H^-$ and $a_+'x(0) \in X \cap H^+$ so that $a'x(0) \in X \cap H_0$, proving the lemma. □

*Remark.* Note that the basic idea of this construction is that $\mathcal{V}^*$ acts dually in the model (1.9) as a maximal "exogenous-noise-nulling" subspace in the sense that multiplying (1.9) by an $a \in \mathcal{V}^*$ removes the influence of the noninternal components of the input noise $dw$. An alternative and perhaps more elegant way of seeing this is to consider the adjoint control system

$$(\Sigma^*) \quad \begin{cases} \dot{z} = -A'z + C'u \\ v = -B'z + D'u \end{cases} \tag{2.40}$$

with transfer function $W^*(s) = W(-s)$, instead of the dual system $\Sigma'$ defined by (2.13). Clearly $\Sigma^*$ and $\Sigma'$ have the same output-nulling subspaces $\mathcal{V}$, and, in particular, the same $\mathcal{V}^*$. (In fact, by a computation similar to the one given above for $\Sigma'$, we see that the generator of the zero dynamics of $\Sigma^*$ is $-\Gamma'|_{\mathcal{V}^*}$.) The study of linear functionals $a'x(0)$ of the state at time zero leads naturally to considering the adjoint system $\Sigma^*$. Given the stochastic system (1.9), differentiating the bilinear form $z'x$ yields

$$d(z'x) = z'dx + \dot{z}'xdt \tag{2.41}$$
$$= u'dy - v'dw \tag{2.42}$$

showing that the exogenous noise is blocked out if $z(0) \in \mathcal{V}^*$, i.e. $v = 0$. Then

$$d(z'x) = u'dy$$

can be integrated to establish that $z(0)'x(0) \in X \cap H_0$.

The same idea is used in the following proof.

*Proof of Theorem 2.3.* Let $\xi \in X \cap H_0$. Then, by Lemma 2.4, $\xi = a'x(0)$ where $a \in \mathcal{V}^*$. Consequently, integrating (2.17) and noting that $\mathcal{V}^* \perp \mathrm{Im} B_2$, we obtain

$$a'x(t) = a'e^{\Gamma t}x(0) + \int_0^t a'e^{\Gamma(t-s)} B_1 R^{-1/2} dy(s). \tag{2.43}$$

Since $\mathcal{V}^*$ is $\Gamma'$-invariant, $e^{\Gamma' t}a \in \mathcal{V}^*$ and hence the first term in the sum (2.43) belongs to $X \cap H_0$ (Lemma 2.4). Consequently, $X \cap H_0$ satisfies the conditions of Definition 2.1 and is thus output-induced. Since all output-induced subspaces are contained in $X \cap H_0$, it must be maximal. ∎

The fact that the zero dynamics of $W$ is autonomous is reflected in the following lemma, to be proved in the Appendix.

LEMMA 2.5. *Under the coercivity assumption above,* $X \cap H^+_{[0,t]} = 0$ *for* $t \geq 0$ *and* $X \cap H^-_{[t,0]} = 0$ *for* $t \leq 0$ *so that the vector sums are direct in (ii) and (iii) of Definition 2.1.*

In view of Lemma 2.5, an equivalent way of stating Theorem 2.3 is to say that

$$U_t \{X \cap H_0\} \subset X \cap H_0 + H^+_{[0,t]} \quad \text{for} \quad t \geq 0 \tag{2.44}$$

and

$$U_t \{X \cap H_0\} \subset X \cap H_0 + H^-_{[t,0]} \quad \text{for} \quad t \leq 0. \tag{2.45}$$

Note that the direct sum property in Lemma 2.5 is lost as $t \to \infty$, since $H^-$ and $H^+$ in general have nontrivial intersections with $X \cap H_0$, namely $X \cap H^-$ and $X \cap H^+$ respectively.

Now, in view of (2.44) and (2.45), there are *oblique* time-varying projectors

$$\pi_t : (X \cap H_0) + H^+_{[0,t]} \to X \cap H_0$$

and

$$\bar{\pi}_t : (X \cap H_0) + H^-_{[-t,0]} \to X \cap H_0$$

the first being the projection onto $X \cap H_0$ parallel to $H^+_{[0,t]}$ and the second projection onto $X \cap H_0$ parallel to $H^-_{[-t,0]}$. The projectors play the role of feedback in geometric control theory in confining the motion of the state to the subspace $X \cap H_0$. Accordingly, we form the compressed shift operators $V_t(X)$ and $\bar{V}_t(X)$ on $X \cap H_0$ by the relations

$$V_t(X)\xi = \pi_t U_t \xi \tag{2.46}$$

and

$$\bar{V}_t(X)\xi = \bar{\pi}_t U_t^* \xi. \tag{2.47}$$

LEMMA 2.6. *The families* $\{V_t(X);\ t \geq 0\}$ *and* $\{\bar{V}_t(X); t \geq 0\}$ *of linear operators are strongly continuous semigroups on* $X \cap H_0$.

*Proof.* Let $\xi \in X \cap H_0$ and form

$$V_t(X)V_s(X) = \pi_t U_t \pi_s U_s \xi \tag{2.48}$$

$$= \pi_{t+s} U_t \pi_s U_s \xi \tag{2.49}$$

$$= V_{t+s}(X)\xi - \pi_{t+s} U_t (1 - \pi_s) U_s \xi \tag{2.50}$$

where we have used the fact that $\pi_{t+s}|_{X \cap H_0 + H^+_{[0,t]}} = \pi_t$ for $s \geq 0$. But $(1 - \pi_s)U_s\xi \in H^+_{[0,t]}$ and hence

$$U_t(1 - \pi_s)U_s\xi \in H^+_{[0,t+s]}$$

and therefore the last term in (2.50) equals zero, establishing the semigroup property for $\{V_t(X);\ t \geq 0\}$. To prove strong continuity, note that, if $t \leq T$, $V_t(X)\xi = \pi_t U_t \xi = \pi_T U_t \xi$ which tends to $\xi$ as $t \to 0$. The rest follows from a symmetric argument. □

Consequently there are infinitesimal generators, i.e. operators $G, \bar{G} : X \cap H_0 \to X \cap H_0$ such that

$$V_t(X) = e^{Gt} \tag{2.51}$$

and

$$\bar{V}_t(X) = e^{\bar{G}t}. \tag{2.52}$$

Lemma 2.7. *For each $t \geq 0$,*

$$\bar{V}_t(X) = V_t(X)^{-1}, \tag{2.53}$$

*i.e., in particular,*

$$\bar{G} = -G. \tag{2.54}$$

*Proof.* Let $\xi \in X \cap H_0$. Then

$$\bar{V}_t(X)V_t(X)\xi = \bar{\pi}_t U_t^* \pi_t U_t \xi \tag{2.55}$$

$$= \xi - \bar{\pi}_t U_t^* (1 - \pi_t) U_t \xi \tag{2.56}$$

Since $(1 - \pi_t)U_t \xi \in H^+_{[0,t]}$, we have

$$U_t^*(1 - \pi_t)U_t \xi \in H^-_{[-t,0]},$$

and therefore the last term of (2.56) is zero. □

Consequently, we may define $V_t(X)$ also for negative $t$. In fact, setting

$$V_t(X) = \bar{V}_{-t}(X)$$

is equivalent to defining $V_t(X)$ for all $t \in \mathbb{R}$ by means of (2.46) with $\pi_{-t} = \bar{\pi}_t$ for $t \leq 0$. Hence the family of operators $\{V_t(X);\ t \in \mathbb{R}\}$ is actually a *group*.

The following proposition characterizes the output-induced subspaces of $X$ as the invariant subspaces for the group $\{V_t(X);\ t \in \mathbb{R}\}$.

Proposition 2.8. *The output-induced subspaces of $X$ are precisely the $G$-invariant subspaces of $X \cap H_0$.*

*Proof.* First suppose that $Y \subset X$ is output-induced. Then

$$U_t Y \subset Y + H^+_{[0,t]} \quad \text{for} \quad t \geq 0, \tag{2.57}$$

so applying the projection $\pi_t$ to both sides we see that $e^{Gt}Y \subset Y$. Conversely, suppose that $Y \subset X \cap H_0$ is $e^{Gt}$-invariant. From (2.44) we have that

$$U_t Y \subset X \cap H_0 + H^+_{[0,t]} \quad \text{for} \quad t \geq 0. \tag{2.58}$$

We want to show that $X \cap H_0$ in (2.58) can be exchanged for $Y$ so that (2.57) is obtained. However, this is obvious by applying the projector $\pi_t$ to (2.58) and noting that, by assumption, $e^{Gt}Y \subset Y$. Trivially, the corresponding statement for $t \leq 0$ follows from (2.45) by an analogous argument. □

We shall identify two particularly important $G$-invariant subspaces of $X \cap H_0$, namely the *past-output-induced* subspace $X \cap H^-$ and the the *future-output-induced* subspace $X \cap H^+$. In fact, suppose that $\xi \in X \cap H^-$ and $t \geq 0$. Then, $U_t^*\xi \in H^-$ and

$$(1 - \bar{\pi}_t)U_t^*\xi \in H^-_{[-t,0]} \subset H^-,$$

and hence

$$e^{-Gt}\xi = \bar{V}_t(X \cap H_0)\xi = \bar{\pi}_t U_t^*\xi \in X \cap H^-,$$

because the range of $\bar{\pi}_t$ is contained in $X$. Therefore $X \cap H^-$ is $G$-invariant. A symmetric argument shows that $X \cap H^+$ is also $G$-invariant. Consequently, by Proposition 2.8, $X \cap H^-$ and $X \cap H^+$ are output-induced subspaces of $X$.

Coercivity also implies that $H^- \cap H^+ = 0$ [14] so that the sum

$$H_0 = H^- + H^+ \tag{2.59}$$

is direct. The following lemma states in particular that the maximal output-induced subspace can be represented as a direct sum of $X \cap H^-$ and $X \cap H^+$.

LEMMA 2.9. *Let $H^-, H^+, H_0$ be defined as in Section 1, and let $X$ be a splitting subspace. Then*

$$X \cap H_0 = \big(X \cap H^-\big) + \big(X \cap H^+\big) \tag{2.60}$$

*where the sum is direct.*

For the proof let us first recall that a Markovian splitting subspace can be uniquely represented as the intersection

$$X = S \cap \bar{S} \tag{2.61}$$

of a pair $(S, \bar{S})$ of subspaces of the ambient subspace $H$ which satisfy

$$S \supset H^- \quad \text{and} \quad \bar{S} \supset H^+, \tag{2.62}$$

the invariance properties

$$U_t^* S \subset S \quad \text{and} \quad U_t\bar{S} \subset \bar{S} \quad \text{for all} \quad t \geq 0, \tag{2.63}$$

and intersect perpendicularly in the sense that

$$H = S^\perp \oplus X \oplus \bar{S}^\perp, \tag{2.64}$$

where $S^\perp$ and $\bar{S}^\perp$ are the orthogonal complements in $H$ of $S$ and $\bar{S}$ respectively (see, e.g., [16]). We shall write $X \sim (S, \bar{S})$ to refer to this representation. The class $X$ of minimal Markovian splitting subspaces consists precisely of the $X \sim \big(S, \bar{S}\big)$ which are *observable*, i.e.

$$\bar{S} = H^+ \vee S^\perp \tag{2.65}$$

and *constructible*, i.e.

$$S = H^- \vee \bar{S}^\perp \tag{2.66}$$

(see [16]).

*Proof of Lemma 2.9.* Since $X \cap H^- \subset X \cap H_0$ and $X \cap H^+ \subset X \cap H_0$, it trivially holds that

$$X \cap H_0 \supset \left(X \cap H^-\right) \vee \left(X \cap H^+\right) \tag{2.67}$$

Since $H_0 = H^- + H^+$ is a direct sum, then so is that of (2.67). Hence it just remains to show that the converse inclusion holds. To this end suppose that $\lambda \in X \cap H_0$. Since $\lambda \in H_0 = H^- + H^+$, there are unique $\alpha \in H^-$ and $\beta \in H^+$ such that

$$\lambda = \alpha + \beta$$

Then, since $\lambda \in X \subset \bar{S}$ and $\beta \in H^+ \subset \bar{S}$, we have $\alpha = \lambda - \beta \in \bar{S}$, and hence

$$\alpha \in \bar{S} \cap H^- = \bar{S} \cap S \cap H^- = X \cap H^-$$

Then $\beta = \lambda - \alpha \in X$, i.e., $\beta \in X \cap H^+$. This completes the proof of the lemma. □

*Remark.* The fact that $X \cap H^-$ and $X \cap H^+$ are output-induced can be seen from first principles using Lemma 2.9. In fact, from (2.61), we see that $X \cap H^- = \bar{S} \cap H^-$, and hence

$$U_t \left\{X \cap H^-\right\} \subset \bar{S} \cap \left(H^- + H^+_{[0,t]}\right) \tag{2.68}$$

$$= \bar{S} \cap H^- + H^+_{[0,t]}. \tag{2.69}$$

Here the first inclusion follows from the $U_t$-invariance (2.63) of $\bar{S}$, and the second equality from Lemma 2.9, noting that $\bar{S} \sim (H, \bar{S})$ is a splitting subspace and $\bar{S} \supset H^+_{[0,t]}$. This shows that $X \cap H^-$ satisfies Condition (ii) of Definition 2.1. A symmetric argument proves Condition (iii), while Condition (i) is trivially satisfied. Hence $X \cap H^-$ is output-induced. In the same way we show that $X \cap H^+$ is output-induced.

The following theorem is one of the main results of this paper, tying together the geometry of minimal Markovian splitting subspaces to the zero dynamics of minimal spectral factors.

THEOREM 2.10. *Let $X$ be a minimal Markovian splitting subspace and let $W$ be the corresponding spectral factor. Then the group $\{V_t(X);\ t \in \mathbb{R}\}$ acting on the maximal output-induced subspace $X \cap H_0$, of $X$, is isomorphic to the zero dynamics (2.14) of $W$ in the sense that the linear bijective map $T : \mathcal{V}^*(A', C', B', D') \to X \cap H_0$, defined by $Ta = a'x(0)$, makes the following diagram commutative.*

$$\begin{array}{ccc} X \cap H_0 & \xrightarrow{V_t(X)} & X \cap H_0 \\ \uparrow T & & \uparrow T \\ \mathcal{V}^* & \xrightarrow{e^{(A'+C'K')t}} & \mathcal{V}^* \end{array}$$

*In particular,*

$$\{\text{zeros of } W\} = \sigma(G), \tag{2.70}$$

*where $\sigma(G)$ is the spectrum of the infinitesimal generator of the group $\{V_t(X);\ t \in \mathbb{R}\}$. The restricted groups $V_t^-(X) := V_t(X)|_{X \cap H^-}$ and $V_t^+(X) := V_t(X)|_{X \cap H^+}$, $t \in \mathbb{R}$,* *describe the asymptotically stable and antistable zero dynamics of $W$, the respective generators*

$$G_s = G|_{X \cap H^-} \quad \text{and} \quad G_u = G|_{X \cap H^+}$$

*having spectra $\sigma(G_s)$ and $\sigma(G_u)$ coinciding with the zeros of $W$ with respectively negative and positive real parts.*

Hence, in particular, $\dim(X \cap H_0)$, $\dim(X \cap H^-)$ and $\dim(X \cap H^+)$ are respectively the total number of zeros of $W$, the number zeros in the open left half plane (*stable zeros*) and the number of zeros in the open right half plane (*antistable zeros*). (The last statement is actually a splitting subspace version of Theorem 4.1 in [11] (see also [1]) as we shall see in Section 4 upon introducing state covariances.) If $X$ is internal and $\dim X = n$, then there are exactly $n$ zeros. If $X \cap H_0 = 0$, there are no zeros.

We shall call $G$ the *generator of the zero dynamics* of $X$. Since in this paper we consider the special case when $R$ is nonsingular, we may, as we have already pointed out above, write the zero dynamics (2.14) as

$$\dot{z} = \Gamma' z \quad z \in \mathcal{V}^*, \tag{2.71}$$

where $\Gamma$ is defined by (2.18). By Lemma 2.4, the map $T$ in the commutative diagram of Theorem 2.10 assigns the value $a'x(0) \in X \cap H_0$ to each $a \in \mathcal{V}^*$, i.e.

$$T : a \to a'x(0). \tag{2.72}$$

*Proof.* Take $a \in \mathcal{V}^*$ so that $a'x(0) \in X \cap H_0$. Then (2.43) holds. From this sum with the first term in $X \cap H_0$ and the second in $H^+_{[0,t]}$ for $t \geq 0$, we obtain

$$\pi_t U_t a'x(0) = \pi_t a'x(t) = a'e^{\Gamma t}x(0)$$

for $t \geq 0$, i.e.

$$e^{Gt}a'x(0) = a'e^{\Gamma t}x(0).$$

Hence $G\,[a'x(0)] = a'\Gamma x(0)$, i.e. $GTa = T\Gamma' a$, proving the similarity

$$G = T\Gamma'|_{\mathcal{V}^*}T^{-1}. \tag{2.73}$$

Moreover, note that (2.38) and (2.39) imply that

$$T\mathcal{V}^*_- \subset X \cap H^- \quad \text{and} \quad T\mathcal{V}^*_+ \subset X \cap H^+.$$

However, since, by Lemma 2.9 and (2.36) the two vector sums

$$T\mathcal{V}^* = T\mathcal{V}^*_- + T\mathcal{V}^*_+$$

and

$$X \cap H_0 = X \cap H^- + X \cap H^+$$

are direct and $T\mathcal{V}^* = X \cap H_0$ (Lemma 2.4), it must hold that

$$T\mathcal{V}^*_- = X \cap H^- \quad \text{and} \quad T\mathcal{V}^*_+ = X \cap H^+. \tag{2.74}$$

Then, by retracing the first part of the proof with $\mathcal{V}^*$ replaced by $\mathcal{V}_-^*$ and $\mathcal{V}_+^*$, we establish the similarity relations

$$G_s = T\Gamma'|_{\mathcal{V}_-^*} T^{-1} \quad \text{and} \quad G_u = T\Gamma'|_{\mathcal{V}_+^*} T^{-1},$$

which clearly shows that $G_s$ is stable and $G_u$ is antistable. This completes the proof of the theorem. □

Next we shall derive some representation formulas for the restrictions of the group $\{V_t(X);\ t \in \mathbb{R}\}$ to the complementary invariant subspaces $X \cap H^-$ and $X \cap H^+$. These relations are connected to the generalization to the Riccati inequality of certain projection results concerning the algebraic Riccati equation due to Willems [27]. This will be discussed in Section 5.

Because of the direct sum decomposition (2.59), any $\eta \in H_0$, has a unique decomposition

$$\eta = \pi_-\eta + \pi_+\eta \tag{2.75}$$

where $\pi_- : H_0 \to H^-$ is the projection on $H^-$ along $H^+$ and $\pi_+ : H_0 \to H^+$ is the projection on $H^+$ along $H^-$.

LEMMA 2.11. *Let $t \geq 0$. Then, if $\xi \in X \cap H^-$, we have $\pi_- U_t \xi \in X \cap H^-$, and, dually, if $\xi \in X \cap H^+$, it follows that $\pi_+ U_t^* \xi \in X \cap H^+$. Moreover, the restrictions of $V_t(X)$ to the complementary invariant subspaces $X \cap H^-$ and $X \cap H^+$ coincide with the above compressed shifts $\pi_- U_t : X \cap H^- \to X \cap H^-$ and $\pi_+ U_t^* : X \cap H^+ \to X \cap H^+$ respectively, i.e.*

$$V_t^-(X) := V_t(X)|_{X\cap H^-} = \pi_- U_t\,|_{X\cap H^-}$$

*and*

$$V_{-t}^+(X) := V_{-t}(X)|_{X\cap H^+} = \pi_+ U_t^*\,|_{X\cap H^+}.$$

*Proof.* Let $t \geq 0$, and take $\xi \in X \cap H^-$. Since $X \cap H^-$ is output-induced (see, e.g., the remark before Theorem 2.10),

$$U_t \xi \in X \cap H^- + H^+_{[0,t]}.$$

Therefore, since $X \cap H^- \subset H^-$ and $H^+_{[0,t]} \subset H^+$, we have

$$\pi_- U_t \xi = \pi_t U_t \xi = V_t(X)\xi.$$

The $(\pi_- U_t)$-invariance of $X \cap H^-$ now follows from the $V_t(X)$-invariance. A symmetric result yields the corresponding result for $X \cap H^+$. □

**3. Zeros and ordering.** In this section we shall study the zero structure of the family of all minimal (analytic) spectral factors by using a partial ordering of the family $\mathcal{X}$ of all minimal Markovian splitting subspaces which are defined in some common probabilistic setting. Such a setting can be described by a sufficiently large common Hilbert space $\hat{H}$ containing $H_0$. It can be shown ([16]; Sections 5.2 and 5.3) that it suffices to take $\hat{H}$ to be of the form

$$\hat{H} = H_0 \oplus H\,(d\eta)\,,$$

where $d\eta$ is some $n$-dimensional Wiener process independent of $dy$, and $H(d\eta)$ is the space generated by the increments of the components of $\eta$. The Hilbert space $\hat{H}$ is endowed with a shift $\left\{\hat{U}_t; t \in \mathbb{R}\right\}$, namely the one induced by $(dy, d\eta)$, and the ambient space of each minimal $X$ in this setting is a doubly invariant subspace of $\hat{H}$ containing $H_0$.The shift $\{U_t\}$ corresponding to $X \in \mathcal{X}$ is just the restriction of $\left\{\hat{U}_t\right\}$ to its ambient space $H$. Recall that the ambient space $H$ has a representation $H(dw)$, where the Wiener process $dw$ may be identified with the driving noise of a minimal stochastic realization (1.9) corresponding to $X$.

In [16] we introduced a partial order of $\mathcal{X}$ defined as follows. Given two minimal Markovian splitting subspaces, $X_1$ and $X_2$, we say that $X_1 \leq X_2$ if

$$\left|\left|E^{X_1}\lambda\right|\right| \leq \left|\left|E^{X_2}\lambda\right|\right| \text{ for all } \lambda \in H^+$$

or, equivalently

$$\left|\left|E^{X_2}\lambda\right|\right| \leq \left|\left|E^{X_1}\lambda\right|\right| \text{ for all } \lambda \in H^-$$

With the above choice of Hilbert space $\hat{H}$, it can be shown that $\leq$ is a *bona fide* partial ordering relation of $\mathcal{X}$, i.e., in particular, $X_1 \leq X_2$ and $X_2 \leq X_1$ imply that $X_1 = X_2$. Moreover, $\mathcal{X}$ has a maximal and a minimal element, $X_+$ and $X_-$, in this ordering, i.e.

$$X_- \leq X \leq X_+ \tag{3.1}$$

for each $X \in \mathcal{X}$, where $X_- := E^{H^-}H^+$ and $X_+ := E^{H^+}H^-$ are respectively the forward and the backward predictor spaces. Clearly both $X_-$ and $X_+$ belong to $\mathcal{X}_0$.

As it can be seen from (3.1), any $X \in \mathcal{X}$ is bounded from below and from above by elements in $\mathcal{X}_0$, namely by $X_-$ and $X_+$ respectively. In this context, a relevant question is whether these internal bounds could be tightened. In [16] it was shown that, for each $X \in \mathcal{X}$, there are unique $X_{0-}$, $X_{0+} \in \mathcal{X}_0$ so that

$$X_1 \leq X_{0-} \leq X \leq X_{0+} \leq X_2$$

for all $X_1, X_2 \in \mathcal{X}_0$ such that $X_1 \leq X \leq X_2$. In other words

$$X_{0-} = \max\{X_0 \in \mathcal{X}_0 \mid X_0 \leq X\}$$

$$X_{0+} = \min\{X_0 \in \mathcal{X}_0 \mid X \leq X_0\}$$

are unique, and we call them the *tightest internal bounds* of $X$.

At several instances below we shall consider a restriction of some linear operator to an invariant subspace. Whenever such a restriction occurs, the invariance is automatically implied and will not be stated explicitly.

LEMMA 3.1. *Let $X_1, X_2 \in \mathcal{X}$ and suppose that $X_1 \leq X_2$. Then,*

(i) $X_1 \cap H^+ \subset X_2 \cap H^+$ and $X_2 \cap H^- \subset X_1 \cap H^-$
(ii) $V_t^-(X_1)|_{X_2\cap H^-} = V_t^-(X_2)$ $\quad t \in \mathbb{R}$
(iii) $V_t^+(X_2)|_{X_1\cap H^+} = V_t^+(X_1)$ $\quad t \in \mathbb{R}$

*Proof.* (i): Recall that if $X \sim (S, \bar{S})$ is a minimal Markovian splitting subspace then the corresponding tightest lower internal bound $X_{0-} \sim (S_{0-}, \bar{S}_{0-})$ has the property that $S_{0-} = S \cap H_0$ (Theorem 6.11 in [16]). Now, if $X_1 \leq X_2$, then, with self-explanatory notations, $(X_1)_{0-} \leq X_1 \leq X_2$, and consequently $(X_1)_{0-} \leq (X_2)_{0-}$, or, equivalently, $S_1 \cap H_0 \subset S_2 \cap H_0$, which implies that $S_1 \cap H^+ \subset S_2 \cap H^+$. But, in view of (2.61) and (2.62), this is equivalent to $X_1 \cap H^+ \subset X_2 \cap H^+$. A symmetric argument yields $X_2 \cap H^- \subset X_1 \cap H^-$.

(ii): First take $t \geq 0$. Then, by Lemma 2.11,

$$V_t^-(X) = \pi_- U_t \,|_{X \cap H^-}$$

for any $X \in \mathcal{X}$, where $\pi_- : H_0 \to H^-$ is the oblique projection parallel to $H^+$. Therefore, since $X_2 \cap H^- \subset X_1 \cap H^-$ and these spaces are both invariant for the compressed shift $\pi_- U_t$ (Lemma 2.11),

$$V_t^-(X_1)|_{X_2 \cap H^-} = V_t^-(X_2) \tag{3.2}$$

for $t \geq 0$. However, for any $X \in \mathcal{X}$,

$$V_t^-(X) = V_t(X)|_{X \cap H^-}$$

for all $t \in \mathbb{R}$, and hence (3.2) may be written

$$V_t(X_1)|_{X_2 \cap H^-} = V_t(X_2)|_{X_2 \cap H^-} \quad \text{for} \quad t \geq 0$$

which is a statement about groups and consequently holds for all $t \in \mathbb{R}$.

(iii) The proof follows from a symmetric argument to that used to prove (ii), first proving the the statement for $t \leq 0$ and then invoking the group property. ∎

Corollary 3.2. *Let $X \in \mathcal{X}$. Then*

$$V_t^-(X) = V_t^-(X_-)|_{X \cap H^-} = V_t(X_-)|_{X \cap H^-} \tag{3.3}$$

*and*

$$V_t^+(X) = V_t^+(X_+)|_{X \cap H^+} = V_t(X_+)|_{X \cap H^+} \tag{3.4}$$

*Proof.* To prove (3.3) just take $X_1 = X_-$ and $X_2 = X$ in Lemma 3.1, and then observe that $V_t^-(X_-) = V_t(X_-)$. A symmetric argument yields (3.4). ∎

We see from this lemma that, if $W$, $W_-$ and $W_+$ are the spectral factors of $X$, $X_-$ and $X_+$ respectively, then the stable zeros $W$ are also zeros of $W_-$ and the antistable zeros of $W$ are zeros of $W_+$. We also see that $W_-$ is the minimum phase spectral factor, all its zeros being stable, and $W_+$ is the maximum phase spectral factor with only antistable zeros.

Lemma 3.1 with Corollary 3.2 has a number of other important consequences which will be discussed below. Before turning to this, we shall however complete the analysis of the relation between subspace inclusion of the type exhibited in statement (i) of Lemma 3.1.

Lemma 3.3. *Let $X_1, X_2 \in \mathcal{X}_0$. Then, for each $X \in \mathcal{X}$,*

(i) $X_1 \leq X \iff X_1 \cap H^+ \subset X \cap H^+$

(ii) $X \leq X_2 \iff X_2 \cap H^- \subset X \cap H^-$

*Moreover, $X_1 = X_{0-}$ if and only if $X_1 \cap H^+ = X \cap H^+$ and $X_2 = X_{0+}$ if and only if $X_2 \cap H^- = X \cap H^-$.*

*Proof.* We begin by proving (i). In view of Lemma 3.1, it remains to prove that $X_1 \cap H^+ \subset X \cap H^+$ implies that $X_1 \leq X$, which, by Theorem 6.8(ii) in [16], is equivalent to $S_1 \subset S$. This in turn is certainly implied by $S_1 \subset S \cap H_0$.

Now, for any splitting subspace $X \sim (S, \bar{S})$, $S$ is itself a splitting subspace, namely $S \sim (S, H)$, and consequently Lemma 2.9 implies that

$$S \cap H_0 = H^- + X \cap H^+, \tag{3.5}$$

because, by (2.61) and (2.62), $S \cap H^- = H^-$ and $S \cap H^+ = S \cap \bar{S} \cap H^+ = X \cap H^+$.

Then, by (3.5), $X_1 \cap H^+ \subset X \cap H^+$ implies that $S_1 = S_1 \cap H_0 \subset S \cap H_0$, proving (i). A completely symmetric argument yields (ii). By Theorem 6.11 in [16], $X_1 = X_{0-}$ is equivalent to $S_1 = S \cap H_0$. This implies that $S_1 \cap H^+ = S \cap H^+$, i.e.

$$X_1 \cap H^+ = X \cap H^+ \tag{3.6}$$

On the other hand there is only one $X_1 \in \mathcal{X}_0$ satisfying (3.6), because (3.6) and

$$S_1 = H^- + X_1 \cap H^+$$

determine $S_1$ uniquely and for minimal Markovian splitting subspaces there is a one-one correspondence between $X$ and $S$ as can be seen from (2.65). Hence we have shown that (3.6) is equivalent to $X_1 = X_{0-}$. In the same way we show that

$$X_2 \cap H^- = X \cap H^-$$

is equivalent to $X_2 = X_{0+}$. □

THEOREM 3.4. *Let $X_1, X_2 \in \mathcal{X}_0$, and suppose $X_1 \leq X_2$. Then:*

(i) *For each $X \in \mathcal{X}$,*

$$X_1 \leq X \leq X_2 \iff X_1 \cap X_2 \subset X$$

*Moreover, $X_1 = X_{0-}$ if and only if $X_1 \cap X_2 = X \cap X_2$ and $X_2 = X_{0+}$ if and only if $X_1 \cap X_2 = X \cap X_1$.*

(ii) *If $X_1 \cap X_2 \subset X$, then $X_1 \cap X_2$ is a $V_t(X)$-invariant subspace for each $t \in \mathbb{R}$, i.e.*

$$G\,[X_1 \cap X_2] \subset X_1 \cap X_2. \tag{3.7}$$

*Conversely, any $G$-invariant subspace $Z \subset X \cap H_0$ takes the form $Z = X_1 \cap X_2$ for some unique $X_1, X_2 \in \mathcal{X}_0$ such that $X_1 \leq X \leq X_2$.*

The proof of this theorem is rather long and technical. For this reason we shall first give some interpretations of the results stated so far, and postpone the proof of Theorem 3.4 to the end of the section.

COROLLARY 3.5. *Let at least one of $X_1, X_2 \in \mathcal{X}$ be internal, and suppose that $X_1 \leq X_2$. Then*

$$V_t(X_1)|_{X_1 \cap X_2} = V_t(X_2)|_{X_1 \cap X_2} \tag{3.8}$$

*for all $t \in \mathbb{R}$.*

*Proof.* We want to prove that, for any $\lambda \in X_1 \cap X_2$,

$$V_t(X_1)\xi = V_t(X_2)\xi$$

for all $t \in \mathbb{R}$. To this end, first suppose that $t \geq 0$, and set $\xi_i := V_t(X_i)$, $I = 1, 2$. Then $\xi_i = \pi_t^{(i)} U_t \lambda$, where, for each $i = 1, 2$,

$$\pi_t^{(i)} : X_i \cap H_0 + H^+_{[0,t]} \to X_i \cap H_0$$

is the oblique projector onto $X_i \cap H_0$ parallel to $H^+_{[0,t]}$. Hence there are $\eta_1, \eta_2 \in H^+_{[0,t]}$ such that

$$U_t\lambda = \xi_1 + \eta_1 = \xi_2 + \eta_2.$$

Now, applying the invariance result of Theorem 3.4 twice, first taking $X = X_1$ and then $X = X_2$, wesee that both $\xi_1$ and $\xi_2$ must belong to $X_1 \cap X_2$. But

$$X_1 \cap X_2 + H^+_{[0,t]}$$

is a direct sum (Lemma 2.4), and hence we must have $\xi_1 = \xi_2$ (and $\eta_1 = \eta_2$) establishing (3.8) for $t \geq 0$. Because of the group property, (3.8) then actually holds for all $t \in \mathbb{R}$. □

Recalling the characterization of Proposition 2.8 of output-induced subspaces of $X \in \mathcal{X}$, we have immediately the following important corollary of Theorem 3.4.

COROLLARY 3.6. *The output-induced subspaces $Y \subset X \in \mathcal{X}$ are precisely the subspaces of the form $Y = X_1 \cap X_2$ where $X_1, X_2 \in \mathcal{X}_0$ are internal bounds of $X$, i.e. $X_1 \leq X \leq X_2$.*

As an illustration of Corollary 3.6 we shall give representations of the output-induced subspaces $X \cap H_0$, $X \cap H^-$ and $X \cap H^+$ as intersections of internal minimal Markovian splitting subspaces. As we have already seen, these output-induced subspaces are of special importance in the classification of the zero structure of minimal spectral factors.

PROPOSITION 3.7. *Let $X \in \mathcal{X}$ have tightest internal bounds $X_{0-}$ and $X_{0+}$. Then,*
(i) $X \cap H^- = X \cap X_- = X_{0+} \cap X_-$
(ii) $X \cap H^+ = X \cap X_+ = X_{0-} \cap X_+$
(iii) $X \cap H_0 = X \cap X_{0-} = X \cap X_{0+} = X_{0-} \cap X_{0+}$

*Proof.* In view of the last statement of Theorem 3.4(i), it only remains to prove that

$$X \cap H^- = X \cap X_-, \tag{3.9}$$

$$X \cap H^+ = X \cap X_+ \tag{3.10}$$

and

$$X \cap H_0 = X_{0-} \cap X_{0+}. \tag{3.11}$$

Taking $X_1 = X_-$ and $X_2 = X$ in Lemma 3.1(i) and recalling that $X_- \subset H^-$, we see that $X \cap H^- \subset X_- \cap H^- \subset X_-$, and hence $X \cap H^- \subset X \cap X_-$. Trivially, $X_- \subset H^-$ also implies that $X \cap X_- \subset X \cap H^-$, and hence (3.9) follows. Relation (3.10) follows by symmetry. To prove (3.11), let $X \sim \left(S, \bar{S}\right)$. Then, by Theorem 6.11 in [16], $S_{0-} = S \cap H_0$ and $\bar{S}_{0+} = \bar{S} \cap H_0$. Hence

$$X_{0-} \cap X_{0+} = S_{0-} \cap \bar{S}_{0+} = S \cap \bar{S} \cap H_0 = X \cap H_0$$

because $X_{0-} \leq X_{0+}$ and hence $S_{0-} \subset S_{0+}$ and $\bar{S}_{0+} \subset \bar{S}_{0-}$. □

Recall that the group $\{V_t(X)\}$ acting on the maximal output-induced subspace $X \cap H_0$ can be identified with the *zero dynamics* of the minimal spectral factor $W$ corresponding to $X$ because of the isomorphism of Theorem 2.10. Similarly the groups $\left\{V_t^-(X)\right\}$ and $\left\{V_t^+(X)\right\}$ on $X \cap H^-$ and $X \cap H^+$ respectively can be identified with the *stable* respectively the *antistable zero dynamics* of $W$. The partial ordering of minimal Markovian splitting subspaces induces a partial ordering of the stable and antistable zero dynamics of the corresponding spectral factors. We shall say that $\{V_t^-(X_1)\}$ acting on $X_1 \cap H^-$ is a *restriction* of $\{V_t^-(X_2)\}$ acting on $X_2 \cap H^-$ if $X_1 \cap H^- \subset X_2 \cap H^-$ and

$$V_t^-(X_1) = V_t^-(X_2)|_{X_1 \cap H^-}.$$

In the same way we can define restrictions of antistable zero dynamics. Clearly restriction is a partial-order relation.

THEOREM 3.8. *Let $X_1, X_2 \in \mathcal{X}$ with at least one of them be internal, and let $W_1$ and $W_2$ be the corresponding minimal spectral factors. Then, if $X_1 \leq X_2$,*

(i) *The stable zero dynamics of $W_2$ is a restriction of the stable zero dynamics $W_1$. In particular, all stable zeros of $W_2$ are zeros of $W_1$.*

(ii) *The antistable zero dynamics of $W_1$ is a restriction of the antistable zero dynamics $W_2$. In particular, all antistable zeros of $W_1$ are zeros of $W_2$.*

(iii) *The zero dynamics of $W_1$ and $W_2$ coincide on the intersection $X_1 \cap X_2$ (i.e. a relation such as (3.8) holds).*

*Proof.* Statements (i) and (ii) are just restatements of (ii) and (iii) of Lemma 3.1, while statement (iii) is a reformulation of Corollary 3.5. ◻

COROLLARY 3.9. *Let $X_{0-}$ and $X_{0+}$ be the tightest internal bounds of $X \in \mathcal{X}$, and let $W_{0-}$, $W_{0+}$ and $W$ be the corresponding minimal spectral factors. Then the zeros of $W$ are precisely the common zeros of $W_{0-}$ and $W_{0+}$.*

*Proof.* This follows immediately from Proposition 3.7(iii) and Theorem 3.8(iii). ◻

From Corollary 3.9 we see that, if $X_-$ and $X_+$ are the tightest internal bounds of $X$, which in fact is the "generic" situation, then the corresponding spectral factor has no zeros. In fact, $W_-$ and $W_+$ have no common zero. The other extreme is the situation when $X$ is internal so that $X_{0-} = X = X_{0+}$. Then $W$ has $n$ zeros.

The following corollary of Theorem 3.4 is a splitting-subspace version of an invariance result, due to Willems [27], formulated in the context of the algebraic Riccati equation. It will be used in Section 5.

COROLLARY 3.10. *Let $G_+$ be the zero generator of $X_+$. Then there is a one-one correspondence between $G_+$-invariant subspaces $Z \subset X_+$ and $X \in \mathcal{X}_0$ under which $Z = X \cap X_+$ and $X \sim (S, \bar{S})$ where*

$$S = H^- + Z \quad \text{and} \quad \bar{S} = H^+ \vee S^\perp$$

*Similarly, if $G_-$ is the zero generator of $X_-$, there is a one-one correspondence between $G_-$-invariant subspaces $Z \subset X_-$ and $X \in \mathcal{X}_0$ under which $Z = X \cap X_-$ and $X \sim (S, \bar{S})$ where*

$$\bar{S} = H^+ + Z \quad \text{and} \quad S = H^+ \vee \bar{S}^\perp$$

*Proof of Theorem 3.4(i).* ($\Rightarrow$): We first prove that if $X_1 \leq X_2$ and $X_1, X_2$ are internal, then

$$X_1 \cap X_2 = (X_1 \cap X_2 \cap H^-) + (X_1 \cap X_2 \cap H^+) \tag{3.12}$$

The inclusion $\supset$ is trivial and we use the procedure of the proof of Lemma 2.9 to prove the converse. To this end, take $\lambda \in X_1 \cap X_2$. Then, by Lemma 2.9,

$$\lambda = X_2 \cap \left[ \left( X_1 \cap H^- \right) + \left( X_1 \cap H^+ \right) \right]$$

Set $\lambda = \alpha + \beta$ where $\alpha \in X_1 \cap H^-$ and $\beta \in X_1 \cap H^+$. But, since $S_1 \subset S_2$ (see proof of Lemma 3.3),

$$X_1 \cap H^+ = S_1 \cap H^+ \subset S_2 \cap H^+ = X_2 \cap H^+ \subset X_2,$$

and therefore $\beta \in X_2$. Hence $\alpha = \lambda - \beta \in X_2$ so that $\alpha \in X_1 \cap X_2 \cap H^-$ and $\beta \in X_1 \cap X_2 \cap H^+$, as required. This proves (3.12). Now, if $X_1 \leq X \leq X_2$, then, by Lemma 3.3, $X_2 \cap H^- \subset X \cap H^-$, and therefore

$$X_1 \cap X_2 \cap H^- \subset X_1 \cap X \cap H^- = \bar{S}_1 \cap \bar{S} \cap H^-$$

where we also have used (2.61) and (2.62). But

$$\bar{S} \cap H^- \subset \bar{S} \cap H_0 = \bar{S}_{0+}$$

by Theorem 6.11 in [16], and, since $X_1 \leq X_{0+}$, $\bar{S}_{0+} \subset \bar{S}_1$. Hence

$$X_1 \cap X_2 \cap H^- \subset \bar{S} \cap H^- = X \cap H^-$$

In the same way we show that

$$X_1 \cap X_2 \cap H^+ \subset X \cap H^+$$

and therefore (3.12) and Lemma 2.9 imply that

$$X_1 \cap X_2 \subset X \cap H_0 \subset X$$

($\Leftarrow$): Next suppose that $X_1 \cap X_2 \subset X$. Then

$$X_1 \cap X_2 \cap H^+ \subset X \cap H^+$$

But $X_1 \cap X_2 \cap H^+ = S_1 \cap S_2 \cap H^+$, which in view of the fact that $X_1 \leq X_2$ and hence $S_1 \subset S_2$ (see above), is the same as $S_1 \cap H^+$. Since $S_1 \cap H^+ = X_1 \cap H^+$, we have

$$X_1 \cap H^+ \subset X \cap H^+$$

which, by Lemma 3.3, is equivalent to $X_1 \leq X$. In the same way we show that $X \leq X_2$.

We turn next to the second statement of the theorem, concerning tight internal bounds. Since $X_1 \leq X_2$ and $X_1$ and $X_2$ are internal, $S_1 \subset S_2$ and $\bar{S}_2 \subset \bar{S}_1$ (Theorem 6.8 in [16]). Hence, in view of (2.61),

$$X_1 \cap X_2 = S_1 \cap \bar{S}_2.$$

Now $S_1 = S \cap H_0$ if and only if $X_1 = X_{0-}$ (Theorem 6.11 in [16]), in which case

$$X_1 \cap X_2 = S \cap \bar{S}_2 = X \cap \bar{S}_2.$$

But, since $X_1 \cap X_2 \subset S_2$, this is the same as

$$X_1 \cap X_2 = X \cap \bar{S}_2 \cap S_2 = X \cap X_2.$$

The rest follows analogously. □

*Proof of Theorem 3.4(ii).* First suppose that $\xi \in X_1 \cap X_2 \subset X$, and let $t \geq 0$. Then, for $i = 1, 2$, $U_t\xi \in \bar{S}_i$, and therefore, since

$$(1 - \pi_t)U_t\xi \in H^+_{[0,t]} \subset \bar{S}_i$$

we have $\pi_t U_t \xi \in \bar{S}_i$, i.e.

$$V_t(X \cap H_0)\xi \in \bar{S}_i \quad \text{for} \quad i = 1, 2. \tag{3.13}$$

A symmetric argument yields

$$\bar{V}_t(X \cap H_0)\xi \in S_i \quad \text{for} \quad i = 1, 2. \tag{3.14}$$

Now, from (3.13) and (3.14) we have $G\xi \in \bar{S}_1 \cap \bar{S}_2$ and $\bar{G}\xi \in S_1 \cap S_2$. But the group property of Theorem 2.10 implies that $\bar{G} = -G$ so therefore

$$G\xi \in S_1 \cap S_2 \cap \bar{S}_1 \cap \bar{S}_2 = X_1 \cap X_2,$$

proving the invariance property (3.7).

Finally, we prove the converse statement on $G$-invariance. Thus, suppose that $Z \subset X \cap H_0$ is $G$-invariant. Then, in view of the decomposition (2.60) of Lemma 2.9 and the fact that both $X \cap H^-$ and $X \cap H^+$ are $G$-invariant, there is a decomposition

$$Z = Z_s + Z_u \tag{3.15}$$

such that $Z_s \subset X \cap H^-$ is $G_s$-invariant and $Z_u \subset X \cap H^+$ is $G_u$-invariant (Theorem 2.10).

We show first that there is a one-one correspondence between $G_u$-invariant subspaces $Z_u \subset X \cap H^+$ and splitting subspaces $X_u \in \mathcal{X}_0$ such that $X_u \leq X$, under which $Z_u = X_u \cap H^+$ and $S_u = H^- + Z_u$. To this end, take $t \geq 0$ and recall that $e^{G_u} Z_u = \pi_+ U_t^* Z_u$, and therefore, since $(1-\pi_+)U_t^* Z_u \subset H^-$, $G_u Z_u \subset Z_u$ is equivalent to

$$U_t^* \left(H^- + Z_u\right) \subset \left(H^- + Z_u\right),$$

because $U_t^* H^- \subset H^-$. Set $S_u := H^- + Z_u$ and $\bar{S}_u := H^+ \vee {Z_u}^\perp$. Then, $X_u \sim (S_u, \bar{S}_u)$ belongs to $\mathcal{X}_0$. (See the discussion in Section 2 and [15] or [16].) Since $S_u \sim (S_u, H_0)$ is itself a splitting subspace, Lemma 2.9 yields

$$S_u = H^- + \left(X_u \cap H^+\right), \tag{3.16}$$

for $S_u \cap H^- = H^-$ and $S_u \cap H^+ = X_u \cap H^+$. Hence we must have

$$Z_u = X_u \cap H^+,$$

and, since $Z_u \subset X$, we have $X_u \cap H^+ \subset X \cap H^+$, from which we see that $X_u \leq X$ (Lemma 3.3). Consequently we have established the required one-one correspondence between $G_u$-invariant $Z_u \subset X \cap H^+$ and $X_u \in \mathcal{X}_0$ such that $X_u \leq X$.

In the same way we prove the symmetric statement that there is a one-one correspondence between $G_s$-invariant subspaces $Z_s \subset X \cap H^-$ and $X_s \in \mathcal{X}_0$ such that $X_s \geq X$, under which $Z_s = X_s \cap H^-$ and $S_s = H^+ + Z_s$.

Now, returning to the decomposition (3.15), we have shown that there are splitting subspaces $X_1, X_2 \in \mathcal{X}_0$ such that $X_1 \leq X \leq X_2$ and

$$Z = \left(X_1 \cap H^-\right) + \left(X_2 \cap H^+\right).$$

Let $\tilde{X}$ be an abitrary element in $\mathcal{X}$ having $X_1$ and $X_2$ as tightest internal bounds. Then, by Lemma 3.3,

$$Z = \left(\tilde{X} \cap H^-\right) + \left(\tilde{X} \cap H^+\right),$$

i.e. $Z = \tilde{X} \cap H_0$ (Lemma 2.9). Proposition 3.7(iii) then yields $Z = X_1 \cap X_2$, proving the last statement of the theorem. □

COROLLARY 3.11. *Let $X \in \mathcal{X}$ and $X_0 \in \mathcal{X}_0$ be arbitrary, and let $G$ be the zero generator of $X$. Then*

$$G\left[X \cap X_0\right] \subset X \cap X_0.$$

*Conversely, any $G$-invariant subspace $Z$ can be written $Z = \tilde{X} \cap X_0$ where $\tilde{X} \in \mathcal{X}$, $X_0 \in \mathcal{X}_0$ and $X_0$ is either the tightest upper or tightest lower internal bound of $\tilde{X}$.*

*Proof.* Take $\xi \in X \cap X_0$ and $t \geq 0$. Then, by the same procedure as in the proof of Theorem 3.4, $V_t(X \cap H_0)\xi \in \bar{S}_0$ and $\bar{V}_t(X \cap H_0)\xi \in S_0$, i.e. $G\xi \in \bar{S}_0$ and $-G\xi \in S_0$, and consequently $G\xi \in S_0 \cap \bar{S}_0 = X_0$. But, by definition, $G\xi \in X \cap H_0 \subset X$, and therefore $G\xi \in X \cap X_0$. This proves the required invariance. The inverse statement follows from the proof of Theorem 3.4. In fact, $Z$ can be written $Z = X_1 \cap X_2$ where $X_1$ and $X_2$ are tight internal bounds of $\tilde{X} \in \mathcal{X}$. Then, from the last statement of Theorem 3.4(i), $Z = \tilde{X} \cap X_1 = \tilde{X} \cap X_2$. □

*Proof of Corollary 3.10.* Just noting that $G_s = G_-$ for $X = X_-$, $G_u = G_+$ for $X = X_+$, and $X_- \leq X \leq X_+$, the statements of the corollary are seen to be special cases of the corresponding results in the proof of Theorem 3.4. □

**4. Introducing coordinates.** In this section we shall, among other things, reformulate the geometric results of Section 3 in the dual deterministic setting of linear functionals of the state at time zero. This will lead to characterizations in terms of state covariances and will facilitate the application of some of these results to the algebraic Riccati inequality in Section 5.

To this end, we shall now equip each $X \in \mathcal{X}$ with a basis chosen uniformly over the family $\mathcal{X}$, in a way first suggested in [5]. Let $\{\xi_1, \xi_2, \ldots, \xi_n\}$ be an arbitrary basis in $X_+$. Such a basis corresponds to a model (1.1) with a state process $\{x_+(t); t \in \mathbb{R}\}$ such that

$$x_+(0) = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}$$

(See, e.g., [16] for the construction.) Now, for an arbitrary $X \in \mathcal{X}$, we define

$$x_k(0) = E^X \xi_k \quad k = 1, 2, \ldots, \tag{4.1}$$

This can be seen to be a basis in $X$, and $x(0)$ is the state vector at zero of a model (1.1) having the same $A$ and $C$ matrices as that of $x_+(0)$.

There are several reasons why this construction is the right one. First, if for each $X \in \mathcal{X}$ we define the state covariance

$$P = E\{x(0)x(0)'\}, \tag{4.2}$$

then it was shown in [16] that

$$X_1 \leq X_2 \Longleftrightarrow P_1 \leq P_2 \tag{4.3}$$

(where, as before, $P_1 \leq P_2$ means that $P_2 - P_1$ it positive semidefinite.) In particular, (3.1) corresponds to

$$P_- \leq P \leq P_+, \tag{4.4}$$

$\mathcal{X}$ to the solution set $\mathcal{P}$ of the algebraic Riccati inequality $\Lambda(P) \leq 0$, and $\mathcal{X}_0$ to the subfamily $\mathcal{P}_0$ of solutions of the algebraic Riccati equation $\Lambda(P) = 0$, thus connecting the geometric theory of stochastic realization with that of Anderson [1] and Faurre [9].

Secondly, the above family of bases is consistent in the sense that representations coincide on intersecting splitting subspaces as explained in the following lemma.

LEMMA 4.1. *Let $X_1, X_2 \in \mathcal{X}$. Then, if $\lambda \in X_1 \cap X_2$, then there is a unique $a \in \mathbb{R}^n$ such that*

$$\lambda = a'x_1(0) = a'x_2(0)$$

*where $x_1(0)$ and $x_2(0)$ are bases of $X_1$ and $X_2$ respectively constructed as in (4.1).*

*Proof.* Suppose that $\lambda = a_1'x_1(0) = a_2'x_2(0)$. Then, by Theorem 6.12 in [16],

$$E^{X_-}\lambda = a_1'x_-(0) = a_2'x_-(0)$$

and hence we must have $a_1 = a_2$, as claimed. □

Next we shall give a result which will be instrumental in establishing the correspondence between families of output-induced subspaces and covariance matrices $P$. To this end, given $X \in \mathcal{X}$ and the corresponding basis (4.1), define the linear map $T : \mathbb{R}^n \to X$ as

$$Ta = a'x(0) \tag{4.5}$$

This is a natural extension to $\mathbb{R}^n$ of the map $T$ defined in Section 2. Clearly $T$ is a bijection, and in view of Lemma 4.1,

$$T_1^{-1}|_{X_1 \cap X_2} = T_2^{-1}|_{X_1 \cap X_2} \tag{4.6}$$

if $T_1$ corresponds to $X_1$ and $T_2$ corresponds to $X_2$, and hence, with some care, we may simply write $T^{-1}$ whenever there is no risk for misunderstanding.

LEMMA 4.2. *Let $X_1, X_2 \in \mathcal{X}$ and $X_1 \leq X_2$, and let at least one of $X_1$ and $X_2$ be internal. Then*

$$T^{-1}(X_1 \cap X_2) = \ker(P_2 - P_1)$$

*where $P_1$ and $P_2$ are the covariances corresponding to $X_1$ and $X_2$ respectively.*

*Proof.* Let $\lambda \in X_1 \cap X_2$ and $T^{-1}(\lambda) = a$. Then $a'x_1(0) = a'x_2(0)$, and therefore

$$a'(P_2 - P_1)a = 0, \tag{4.7}$$

and therefore $a \in \ker(P_2 - P_1)$. Conversely, suppose that

$$a \in \ker(P_2 - P_1) \tag{4.8}$$

Since $X_1 \le X_2$ and at least one of $X_1$ and $X_2$ is internal

$$a'x_1(0) = E^{X_1} a'x_2(0)$$

(Proposition 6.12 in [16]), i.e. $[a'x_2(0) - a'x_1(0)] \perp a'x_1(0)$. Therefore, since

$$a'x_2(0) = [a'x_2(0) - a'x_1(0)] + a'x_1(0),$$

we have

$$E\,|a'x_2(0) - a'x_1(0)|^2 = a'(P_2 - P_1)\,a.$$

Consequently, by (4.8), $a'x_2(0) = a'x_1(0) \in X_1 \cap X_2$, i.e. $a \in T^{-1}(X_1 \cap X_2)$. ∎

We are now in a position to reformulate the first part of Theorem 3.4 in terms of covariances, thus obtaining an amplification of Theorem 9.1 and Lemma 9.3 in [16]. In the parameterization $\mathcal{P}$ of $\mathcal{X}$, the tightest internal bounds $X_{0-}$ and $X_{0+}$ of $X \in \mathcal{X}$, will be denoted $P_{0-}$ and $P_{0+}$ respectively. Recall that $(P_{0-}, P_{0+})$ denotes the *open tightest frame* of P, i.e. the set of all $P \in \mathcal{P}$ having $P_{0-}$ and $P_{0+}$ as their tightest upper and lower bounds in $\mathcal{P}_0$.

THEOREM 4.3. *Let $P_1, P_2 \in \mathcal{P}_0$ and $P \in \mathcal{P}$. Then,*

(i) $P_1 \le P \le P_2 \Longleftrightarrow \ker(P_2 - P_1) \subset \ker(P_2 - P)$
*with* $\ker(P_2 - P_1) = \ker(P_2 - P)$ *if and only if* $P_1 = P_{0-}$*; and*

(i) $P_1 \le P \le P_2 \Longleftrightarrow \ker(P_2 - P_1) \subset \ker(P - P_1)$
*with* $\ker(P_2 - P_1) = \ker(P - P_1)$ *if and only if* $P_2 = P_{0+}$.

*Proof.* Let $T : \mathbb{R}^n \to X$ be the bijection defined above, i.e. $T(a) = a'x(0)$. If $X_1 \le X \le X_2$, then $X_1 \cap X_2 \subset X$ by Theorem 3.4. Hence Lemma 4.2 can be applied with the same $T^{-1}$ so that $X_1 \cap X_2$, $X \cap X_2$ and $X \cap X_1$ correspond to $\ker(P_2 - P_1)$, $\ker(P_2 - P)$ and $\ker(P - P_1)$ respectively under the bijection. Therefore

$$\ker(P_2 - P_1) \subset \ker(P_2 - P) \cap \ker(P - P_1) \tag{4.9}$$

Also $X_2 = X_{0+}$ if and only if $X_1 \cap X_2 = X \cap X_2$, i.e. $\ker(P_2 - P_1) = \ker(P_2 - P)$. To prove the converse statement observe that any element $\xi \in X_1 \cap X_2$ can be written in the form $\xi = a'x_1(0) = a'x_2(0)$, where $a \in \ker(P_2 - P_1)$. So if $\ker(P_2 - P_1) \subset \ker(P_2 - P)$ then $a \in \ker(P_2 - P)$, i.e. $a'x_2(0) = a'x(0)$, and therefore $\xi \in X$ which implies that $X_1 \cap X_2 \subset X$ which is equivalent to $X_1 \le X \le X_2$ by Theorem 3.4. This proves (i). Statement (ii) is proved in the same way. ∎

We shall now provide an explicit representation of $\mathcal{V}^*$ and its $\Gamma'$-invariant subspaces $\mathcal{V}$ in terms of covariance matrices.

As pointed out in Section 1, the set $\mathcal{P}$ is a parametrization of the family $\mathcal{X}$ of minimal Markovian splitting subspaces. In fact, a uniform choice of bases produces a unique state process $x$ for each $X \in \mathcal{X}$ and hence a unique $P := E\{x(0)x(0)'\}$. Modulo orthogonal transformations in the input space, there is a unique minimal stochastic realization (1.9) corresponding to $x$ which may be written in standard form

$$(\Sigma)\quad \begin{cases} dx = Axdt + B_1 dw_1 + B_2 dw_2 \\ dy = Cxdt + R^{1/2} dw_1 \end{cases} \tag{4.10}$$

A uniform choice of bases also fixes the matrices $A$ and $C$ to be the same for all $X \in \mathcal{X}$. Conversely, for each $P \in \mathcal{P}$, we have a minimal spectral factor

$$W(s) = C(sI - A)^{-1}(B_1, B_2) + (R^{1/2}, 0)$$

where

$$B_1 = (\bar{C} - CP)' R^{-1/2} \tag{4.11}$$

and $B_2$ is a full-rank factor of $-\Lambda(P)$, i.e.

$$\Lambda(P) = -B_2 B_2', \tag{4.12}$$

and (in a suitable Hilbert space $\hat{H}$ as discussed in the beginning of Section 3) a unique stochastic realization (4.10), in turn defining a unique $X$.

Moreover, the uniform choice of bases associates to each $X \in \mathcal{X}$ a maximal output-nulling subspace $\mathcal{V}^* = \mathcal{V}^*(A', C', B', D')$ of the dual system (2.13) and a feedback matrix

$$\Gamma = A - B_1 R^{-1/2} C'. \tag{4.13}$$

We recall that $\mathcal{V}^* = \langle \Gamma \,|\, B_2 \rangle^{\perp}$. As explained in the proof of Lemma 2.4, eqn. (2.36), $\mathcal{V}^*$ can be decomposed into a direct sum

$$\mathcal{V}^* = \mathcal{V}^*_- + \mathcal{V}^*_+$$

of $\Gamma'$-invariant subspaces, $\mathcal{V}^*_-$ and $\mathcal{V}^*_+$, coresponding to the stable and the antistable modes of $\Gamma' |_{\mathcal{V}^*}$ respectively.

LEMMA 4.4. *Let $P \in \mathcal{P}$ and let $\mathcal{V}^*$ be the corresponding output-nulling subspace. Then*

(i) $\mathcal{V}^* = \ker(P - P_{0-}) = \ker(P_{0+} - P) = \ker(P_{0+} - P_{0-})$
(ii) $\mathcal{V}^*_- = \ker(P - P_-) = \ker(P_{0+} - P_-)$
(iii) $\mathcal{V}^*_+ = \ker(P_+ - P) = \ker(P_+ - P_{0-})$.

*Proof.* In view of Lemma 2.4 and (2.74), $\mathcal{V}^* = T^{-1}(X \cap H_0)$, $\mathcal{V}^*_- = T^{-1}(X \cap H^-)$ and $\mathcal{V}^*_+ = T^{-1}(X \cap H^+)$. Then applying Lemma 4.2 to Proposition 3.7 yields the desired result. □

Consider two covariance matrices $P_1$ and $P_2$ in $\mathcal{P}$ such that $P_1 \leq P_2$. We shall next establish the relation between the corresponding pairs of output-nulling subspaces $(\mathcal{V}^*_-)_1$, $(\mathcal{V}^*_+)_1$ and $(\mathcal{V}^*_-)_2$, $(\mathcal{V}^*_+)_2$ and the corresponding feedback matrices (4.13), $\Gamma_1$ and $\Gamma_2$. The following chain of results provides dual versions of Lemma 3.1, Corollary 3.2, and Lemma 3.3 in Section 3.

LEMMA 4.5. *Let at least one of $P_1, P_2 \in \mathcal{P}$ belong to $\mathcal{P}_0$, and suppose that $P_1 \leq P_2$. Then*

(i) $(\mathcal{V}^*_+)_1 \subset (\mathcal{V}^*_+)_2$ and $(\mathcal{V}^*_-)_2 \subset (\mathcal{V}^*_-)_1$
(ii) $\Gamma_1' |_{(\mathcal{V}^*_-)_2} = \Gamma_2' |_{(\mathcal{V}^*_-)_2}$
(iii) $\Gamma_1' |_{(\mathcal{V}^*_+)_1} = \Gamma_2' |_{(\mathcal{V}^*_+)_1}$

*Proof.* Follows directly by applying Proposition 3.7 and Lemma 4.2 to Lemma 3.1. □

The following corollary illustrates the role of $\mathcal{V}^*_-$ and $\mathcal{V}^*_+$ as the stable and unstable $\Gamma'$-invariant subspaces of $\mathcal{V}^*$.

COROLLARY 4.6. *Let $P \in \mathcal{P}$ and let $\Gamma$ be the corresponding feedback matrix (4.13). Then*

$$\Gamma' |_{\mathcal{V}_-^*} = \Gamma'_- |_{\mathcal{V}_-^*} \quad \text{and} \quad \Gamma' |_{\mathcal{V}_+^*} = \Gamma'_+ |_{\mathcal{V}_+^*},$$

*where $\Gamma_-$ and $\Gamma_+$ are the feedback matrices corresponding to $P_-$ and $P_+$ respectively.*

*Proof.* Take $P_1 = P_-$ and $P_2 = P$ in Lemma 4.5(ii) to prove (i). The second statement follows by setting $P_1 = P_+$ and $P_2 = P$ in Lemma 4.5(iii). □

LEMMA 4.7. *Let $P_1, P_2 \in \mathcal{P}_0$. Then for each $P \in \mathcal{P}$,*

(i) $P_1 \leq P \Longleftrightarrow \ker(P_+ - P_1) \subset \ker(P_+ - P)$
*with* $\ker(P_+ - P_1) = \ker(P_+ - P)$ *if and only if $P_1 = P_{0-}$; and*

(i) $P \leq P_2 \Longleftrightarrow \ker(P_2 - P_-) \subset \ker(P - P_-)$
*with* $\ker(P_2 - P_-) = \ker(P - P_-)$ *if and only if $P_2 = P_{0+}$.*

*In other words,*

(i) $P_1 \leq P \Longleftrightarrow (\mathcal{V}_+^*)_1 \subset \mathcal{V}_+^*$
*with $(\mathcal{V}_+^*)_1 = \mathcal{V}_+^*$ if and only if $P_1 = P_{0-}$; and*

(i) $P \leq P_2 \Longleftrightarrow (\mathcal{V}_-^*)_2 \subset \mathcal{V}_-^*$
*with $(\mathcal{V}_-^*)_2 = \mathcal{V}_-^*$ if and only if $X_2 = X_{0+}$.*

*Proof.* Follows immediately from Lemma 3.3. It is also a simple corollary of Theorem 4.3. □

The following theorem gives, for an arbitrary $P \in \mathcal{V}$, a complete characterization of all $\Gamma'$-invariant subspaces in $\mathcal{V}$, i.e. the output-nulling subspaces of the dual control system (2.13).

THEOREM 4.8. *Let $\Gamma$ be the feedback matrix (4.13) corresponding to $P \in \mathcal{P}$. Then, if $P_1, P_2 \in \mathcal{P}_0$ and $P_1 \leq P \leq P_2$, the subspace*

$$\ker(P_2 - P_1)$$

*is $\Gamma'$-invariant. Conversely, any $\Gamma'$-invariant subspace $\mathcal{V} \subset \mathcal{V}^*$ has a representation*

$$\mathcal{V} = \ker(P_2 - P_1)$$

*for some $P_1, P_2 \in \mathcal{P}_0$ such that $P_1 \leq P \leq P_2$.*

*Proof.* Follows by applying Lemma 4.2 to Theorem 3.4. □

Concerning Theorem 3.4, of which the above Theorem 4.8 is an isomorphic version, we may add that, thanks to Lemma 4.2, a simpler and more transparent proof of the invariances can be given. For example, to prove the $G$-invariance of $X_1 \cap X_2$ in Theorem 3.4, take $\xi \in X_1 \cap X_2$. Then, by Lemma 4.1, there is an $a \in \mathbb{R}^n$ such that

$$\xi = a'x_1(0) = a'x_2(0).$$

Since $X_1$ and $X_2$ are internal, the corresponding $B_2$-matrices are zero, i.e., for $t \geq 0$ and $i = 1, 2$,

$$U_t\xi = a'e^{\Gamma_i t}x_i(0) + \int_0^t a'e^{\Gamma_i(t-s)}(B_1)_i R^{-1/2} dy(s),$$

and therefore

$$e^{Gt}\xi = \pi_t U_t \xi = a'e^{\Gamma_1 t}x_1(0) = a'e^{\Gamma_2 t}x_2(0) \in X_1 \cap X_2.$$

Consequently $X_1 \cap X_2$ is $G$-invariant.

In the same way as above we obtain from Corollary 3.5 the following result characterizing intersecting zero dynamics.

LEMMA 4.9. *Let at least one of $P_1, P_2 \in \mathcal{P}$ belong to $\mathcal{P}_0$, and suppose that $P_1 \leq P_2$. Then*

$$\Gamma_1' \,|_{\ker(P_2-P_1)} = \Gamma_2' \,|_{\ker(P_2-P_1)}.$$

An important consequence of this lemma and the fact that $\mathcal{V}^*$ is constant over the open tightest frame $(P_{0-}, P_{0+})$ (Lemma 4.4) is that the zero dynamics is the same for all $P \in (P_{0-}, P_{0+})$. In fact, by Lemma 4.9,

$$\Gamma_{0-}' \,|_{\ker(P-P_{0-})} = \Gamma' \,|_{\ker(P-P_{0-})},$$

and, by Lemma 4.4, $\ker(P - P_{0-}) = \ker(P_{0+} - P_{0-})$.

The next proposition, which is due to Molinari [20] (also see [16]; Lemma 10.2), also belongs to the general area of invariance results described in this section and corresponds to Corollary 3.11.

PROPOSITION 4.10. *Let $P \in \mathcal{P}$ and $P_0 \in \mathcal{P}_0$ be arbitrary. Then all subspaces of the form*

$$\mathcal{V} = \ker(P - P_0)$$

*are $\Gamma'$-invariant subspaces of $\mathcal{V}^*$.*

**5. Invariant subspaces and the algebraic Riccati inequality.** In this section we shall generalize the well-known Potter-MacFarlane characterization of the (symmetric) solutions of the algebraic Riccati equation

$$\Lambda(P) = 0, \tag{5.1}$$

in terms of subspaces invariant under the Hamiltonian matrix, to the algebraic Riccati inequality

$$\Lambda(P) \leq 0. \tag{5.2}$$

Setting

$$F := A - \bar{C}' R^{-1} C, \tag{5.3}$$

we may write

$$\Lambda(P) = FP + PF' + PC'R^{-1}CP + \bar{C}'R^{-1}\bar{C}, \tag{5.4}$$

which corresponds to the *Hamiltonian* matrix

$$\mathcal{H} = \begin{bmatrix} F' & C'R^{-1}C \\ -\bar{C}'R^{-1}\bar{C} & -F \end{bmatrix}. \tag{5.5}$$

It is well-known ([17], [22], [18]) that the solution set $\mathcal{P}_0$ of the algebraic Riccati equation is in a one-one correspondence with the class of Lagrangian $\mathcal{H}$-invariant subspaces $\mathcal{L}$ of $\mathbb{R}^{2n}$. Recall that a subspace $\mathcal{L}$ is Lagrangian if it is *isotropic* in the sense that if $x, y \in \mathcal{L}$ then

$$x' \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} y = 0, \tag{5.6}$$

and it is of maximal dimension $n$. Under this correspondence $\mathcal{L} = \text{Im}\begin{bmatrix} I \\ P \end{bmatrix}$.The purpose of this section is to show that a similar correspondence holds for the solution set $\mathcal{P}$ of the algebraic Riccati inequality (5.2) and that this correspondence is related to the zero structure described above. In this respect a crucial observation is the following.

Proposition 5.1. *Let $P \in \mathcal{P}$ and let $\mathcal{V}^*$ be the maximal output-nulling subspace of the corresponding dual system (2.13). Then $\mathcal{V}^*$ is the largest $\Gamma'$-invariant subspace of $\mathbb{R}^n$ such that*

$$\Lambda(P)\,|_{\mathcal{V}^*} = 0 \tag{5.7}$$

*where $\Gamma$ is defined by (2.18) or, equivalently,*

$$\Gamma = F + PC^{'}R^{-1}C. \tag{5.8}$$

*Proof.* In view of (2.21), $\mathcal{V}^*$ is the largest $\Gamma^{'}$-invariant subspace orthogonal to the columns of $B_2$, and consequently, since $\Lambda(P) = -B_2B_2^{'}$, $\mathcal{V}^*$ is the largest $\Gamma^{'}$-invariant subspace for which (5.7) holds. □

Now, recall from Section 4 that to each $P \in \mathcal{P}$ there is a direct-sum decomposition

$$\ker(P - P_-) + \ker(P_+ - P) = \ker(P_{0+} - P_{0-}) \tag{5.9}$$

where $P_{0-}$, $P_{0+} \in \mathcal{P}_0$ are the tightest lower and upper internal bounds of P. In view of of Lemma 4.4, this is equivalent to

$$\mathcal{V}^* = \mathcal{V}^*_- + \mathcal{V}^*_+ \tag{5.10}$$

As we have seen in Section 4 $\mathcal{V}^*_-$ is $\Gamma^{'}_-$-invariant and $\mathcal{V}^*_+$ is $\Gamma^{'}_+$-invariant. Moreover, if $a \in \mathcal{V}^*_-$ and $b \in \mathcal{V}^*_+$, then $a^{'}(P_+ - P)b = a^{'}(P - P_-)b = 0$, and consequently $\mathcal{V}^*_-$ and $\mathcal{V}^*_+$ are $(P_+ - P_-)$-orthogonal, i. e.

$$a^{'}(P_+ - P_-)b = 0 \quad \text{for all } a \in \mathcal{V}^*_-, b \in \mathcal{V}^*_+ \tag{5.11}$$

In Section 4 (Lemma 4.4) we saw that $\mathcal{V}^*_- = \ker(P_{0+} - P_-)$ and $\mathcal{V}^*_+ = \ker(P_+ - P_{0-})$, so decomposition (5.9) may also be written

$$\ker(P_{0+} - P_-) + \ker(P_+ - P_{0-}) = \ker(P_{0+} - P_{0-}), \tag{5.12}$$

only involving covariance matrices belonging to $\mathcal{P}_0$.

If $P$ is a solution of the algebraic Riccati equation (5.1), i. e. $P \in \mathcal{P}_0$, then $P = P_{0-} = P_{0+}$, and both (5.9) and (5.12) reduce to the $(P_+ - P_-)$-orthogonal decompositon

$$\ker(P - P_-) + \ker(P_+ - P) = \mathbb{R}^n \tag{5.13}$$

of the whole $\mathbb{R}^n$. This corresponds to the situation studied by J. C. Willems [27]. To set up notations and make contact with the gemetric theory of splitting subspaces we shall here restate Willems' result.

To this end, let $X \in \mathcal{X}_0$ and consider the stochastic version of (5.13), namely

$$X = X \cap X_- + X \cap X_+, \tag{5.14}$$

obtained via Lemma 4.2 or directly from Lemma 2.9 and Proposition 3.7. Applying the projectors $\pi_-$ and $\pi_+$ of (2.75) to (5.14) shows that

$$\pi_- X = X \cap X_- \quad \text{and} \quad \pi_+ X = X \cap X_+,$$

which can be translated into $\mathbb{R}^n$ via the bijective map $T : \mathbb{R}^n \to X$ of (4.5) to yield

$$\operatorname{Im} \Pi_- = \ker(P - P_-) \quad \text{and} \quad \operatorname{Im} \Pi_+ = \ker(P_+ - P).$$

Here $\Pi_- : \mathbb{R}^n \to \mathbb{R}^n$ and $\Pi_+ : \mathbb{R}^n \to \mathbb{R}^n$ are complementary projection operators defined as $\Pi_- = T^{-1}\pi_-|_X T$ and $\Pi_+ = T^{-1}\pi_+|_X T$ respectively. Now take $a \in \mathbb{R}^n$ and form the projections $a_- := \Pi_- a$ and $a_+ := \Pi_+ a$. From (5.13) we see that $a = a_- + a_+$, $Pa_- = P_- a_-$, and $Pa_+ = P_+ a_+$ so that $Pa = P_- \Pi_- a + P_+ \Pi_+ a$ for all $a \in \mathbb{R}^n$. Consequently,

$$P = P_- \Pi_- + P_+ \Pi_+.$$

LEMMA 5.2 (J. C. WILLEMS). *Let $\Gamma_-$ and $\Gamma_+$ be the feedback matrices, given by (4.13) and (4.11), corresponding to $P_-$ and $P_+$ respectively. Then:*

(i) *There is a one-one correspondence between $\Gamma'_-$-invariant subspaces $\mathcal{V}_- \subset \mathbb{R}^n$ and $P \in \mathcal{P}_0$ under which*

$$\mathcal{V}_- = \ker(P - P_-) \tag{5.15}$$

*and*

$$P = P_- \Pi_- + P_+(I - \Pi_-), \tag{5.16}$$

*where $\Pi_-$ is the $(P_+ - P_-)$-orthogonal projector of $\mathbb{R}^n$ onto $\mathcal{V}_-$.*

(ii) *Dually, there is a one-one correspondence between $\Gamma'_+$-invariant subspaces $\mathcal{V}_+ \subset \mathbb{R}^n$ and $P \in \mathcal{P}_0$ under which*

$$\mathcal{V}_+ = \ker(P_+ - P) \tag{5.17}$$

*and*

$$P = P_-(I - \Pi_+) + P_+ \Pi_+, \tag{5.18}$$

*where $\Pi_+$ is the $(P_+ - P_-)$-orthogonal projector of $\mathbb{R}^n$ onto $\mathcal{V}_+$.*

*Proof.* By Lemma 4.2, $\mathcal{V}_-$ corresponds to $Z = X \cap X_-$ and $\mathcal{V}_+$ to $Z = X \cap X_+$ in Corollary 3.10. Moreover, $\Gamma_-$ and $\Gamma_+$ correspond to $G_-$ and $G_+$ respectively, and therefore the lemma follows. □

In summary, by Lemma 5.2, any $P \in \mathcal{P}_0$ corresponds to two subspaces, $\mathcal{V}_-^* = \ker(P - P_-)$, invariant for $\Gamma'_-$, and $\mathcal{V}_+^* = \ker(P_+ - P)$, invariant for $\Gamma'_+$, which by (5.13) are complementary, i.e. sum up to all of $\mathbb{R}^n$. If $P \in \mathcal{P}$ does not belong to $\mathcal{P}_0$, however, (5.13) is replaced by (5.9). Therefore, if we insist on representing the invariant subspaces $\mathcal{V}_-^*$ and $\mathcal{V}_+^*$ in terms of solutions of the algebraic Riccati equation, as stated in Lemma 5.2, then there will still be representations of the type $\mathcal{V}_-^* = \ker(P_0 - P_-)$ and $\mathcal{V}_+^* = \ker(P_+ - P_0)$, but now we can no longer use the same $P_0$. Formula (5.12) is precisely a manifestation of this fact.

The following notation will be used in the sequel. If $\mathcal{L}$ is a $k$-dimensional subspace of $\mathbb{R}^{2n}$ with basis matrix $L \in \mathbb{R}^{2n \times k}$, define $\tau(\mathcal{L})$ to be the subspace in $\mathbb{R}^n$ spanned by the truncated matrix obtained by removing the bottom $n$ rows of $L$.

We are now in a position to state the main result of this section.

THEOREM 5.3. *Let $\mathcal{P}$ be the solution set of the matrix Riccati inequality (5.2) and let $\mathcal{H}$ be the Hamiltonian matrix (5.5). Then there is a one-one correspondence between the isotropic $\mathcal{H}$-invariant subspaces $\mathcal{L} \subset \mathbb{R}^{\in\setminus}$ of dimension $k \leq n$, and the family of open tightest frames $(P_{0-}, P_{0+})$ of $\mathcal{P}$. Under this correspondence*

$$\mathcal{L} = \begin{bmatrix} I \\ P \end{bmatrix} \mathcal{V}^* \tag{5.19}$$

*for any $P \in (P_{0-}, P_{0+})$,where $\mathcal{V}^* \subset \mathbb{R}^n$ is the subspace of zero directions*

$$\mathcal{V}^* = \ker(P_{0+} - P_{0-}) \tag{5.20}$$

*and $k = \dim \mathcal{L}$ is the number of zeros of the spectral factor $W$ corresponding to $P$. Conversely, given any isotropic $\mathcal{H}$-invariant subspace $\mathcal{L} \subset \mathbb{R}^{2n}$ of dimension $k \leq n$, the matrices $P_{0-}$ and $P_{0+}$ are obtained from Lemma 5.2, formulas (5.16) and (5.18), as the elements in $\mathcal{P}_0$ corresponding to the invariant subspaces $\mathcal{V}_- = \tau(\mathcal{L}_-)$ and $\mathcal{V}_+ = \tau(\mathcal{L}_+)$, where $\mathcal{L}_-$ and $\mathcal{L}_+$ are the subspaces of $\mathcal{L}$ consisting of sums of stable and antistable eigenspaces of $\mathcal{H}$.*

*Proof.* First suppose that $P \in \mathcal{P}$ has the tightest local frame $(P_{0-}, P_{0+})$, and define $\mathcal{L}$ by (5.19) and (5.20). Clearly, (5.19) is independent of the choice of $P \in (P_{0-}, P_{0+})$. In fact, if $P_1, P_2 \in (P_{0-}, P_{0+})$, then, by Lemma 4.4, $\mathcal{V}^* = \ker(P_1 - P_{0-}) = \ker(P_2 - P_{0-})$, and hence it follows that $(P_2 - P_1)a = 0$ for all $a \in \mathcal{V}^*$. Now, a straightforward calculation, using (5.4) and the fact that $\Lambda(P)\mathcal{V}^* = 0$ (Proposition 5.1), shows that

$$\mathcal{H}\mathcal{L} = \begin{bmatrix} I \\ P \end{bmatrix} \Gamma' \mathcal{V}^*.$$

Since $\Gamma' \mathcal{V}^* \subset \mathcal{V}^*$, this yields $\mathcal{H}\mathcal{L} \subset \mathcal{L}$ as claimed. The fact that $P' = P$ insures that $\mathcal{L}$ is isotropic.

Conversely, suppose that $\mathcal{L} \subset \mathbb{R}^{2n}$ is any $\mathcal{H}$-invariant isotropic subspace of dimension $k \leq n$. Then $\mathcal{L}$ is a direct sum of generalized eigenspaces of $\mathcal{H}$, and, since these eigenspaces are contained in either $\operatorname{Im} \begin{bmatrix} I \\ P_- \end{bmatrix}$ or $\operatorname{Im} \begin{bmatrix} I \\ P_+ \end{bmatrix}$ (for $\mathbb{R}^{2n}$ is a direct sum of these subspaces), we have the direct sum decomposition

$$\mathcal{L} = \mathcal{L}_- + \mathcal{L}_+ \tag{5.21}$$

where $\mathcal{L}_- := \mathcal{L} \cap \operatorname{Im} \begin{bmatrix} I \\ P_- \end{bmatrix}$ and $\mathcal{L}_+ := \mathcal{L} \cap \operatorname{Im} \begin{bmatrix} I \\ P_+ \end{bmatrix}$ are both $\mathcal{H}$-invariant, because $\operatorname{Im} \begin{bmatrix} I \\ P_- \end{bmatrix}$ and $\operatorname{Im} \begin{bmatrix} I \\ P_+ \end{bmatrix}$ are. Therefore there are full-rank matrices $M_-$ and $M_+$ such that

$$\mathcal{L}_- = \operatorname{Im} \begin{bmatrix} I \\ P_- \end{bmatrix} M_- \quad \text{and} \quad \mathcal{L}_+ = \operatorname{Im} \begin{bmatrix} I \\ P_+ \end{bmatrix} M_+. \tag{5.22}$$

But $\operatorname{Im} \begin{bmatrix} I \\ P_- \end{bmatrix}$ is $\mathcal{H}$-invariant and

$$\mathcal{H} \begin{bmatrix} I \\ P_- \end{bmatrix} = \begin{bmatrix} I \\ P_- \end{bmatrix} \Gamma'_-$$

and consequently

$$\mathcal{H}\begin{bmatrix} I \\ P_- \end{bmatrix} M_- = \begin{bmatrix} I \\ P_- \end{bmatrix} \Gamma_-^{'} M_- .$$

Therefore, since $\mathcal{L}_-$, represented by (5.22), is $\mathcal{H}$-invariant, $\operatorname{Im} M_-$ must be $\Gamma_-^{'}$-invariant. In the same way we show that $\operatorname{Im} M_+$ is $\Gamma_+^{'}$- invariant. Consequently, it follows from Lemma 5.2 that there are unique $P_{0-}, P_{0+} \in \mathcal{P}_0$ so that

$$\mathcal{V}_- := \operatorname{Im} M_- = \ker(P_{0+} - P_-) \tag{5.23}$$

and

$$\mathcal{V}_+ := \operatorname{Im} M_+ = \ker(P_+ - P_{0-}). \tag{5.24}$$

It remains to show that $P_{0-} \le P_{0+}$ so that $(P_{0-}, P_{0+})$ may form a tightest local frame and we may identify $\mathcal{V}_-$ and $\mathcal{V}_+$ with $\mathcal{V}_-^*$ and $\mathcal{V}_+^*$ respectively. To this end, note that since

$$\mathcal{L} = \operatorname{Im} \begin{bmatrix} M_- & M_+ \\ P_- M_- & P_+ M_+ \end{bmatrix}$$

is isotropic,

$$\begin{bmatrix} M_- & M_+ \\ P_- M_- & P_+ M_+ \end{bmatrix}^{'} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} M_- & M_+ \\ P_- M_- & P_+ M_+ \end{bmatrix} = 0,$$

i. e.

$$M_-^{'}(P_+ - P_-)M_+ = 0.$$

Consequently $\mathcal{V}_-$ and $\mathcal{V}_+$ are $(P_+ - P_-)$-orthogonal. In other words,

$$\mathcal{V}_+ \subset (\mathcal{V}_-)^\circ \tag{5.25}$$

where $^\circ$ denotes the $(P_+ - P_-)$-orthogonal complement in $\mathbb{R}^n$. Now, in view of (5.23) and decomposition (5.13),

$$(\mathcal{V}_-)^\circ = \ker(P_+ - P_{0+}). \tag{5.26}$$

Therefore,

$$\ker(P_+ - P_{0-}) = \mathcal{V}_+ \subset (\mathcal{V}_-)^\circ = \ker(P_+ - P_{0+}),$$

so it follows from Lemma 4.7 that $P_{0-} \le P_{0+}$, as claimed.

Now, let $P \in \mathcal{P}$ be an arbitrary element in the open tightest frame $(P_{0-}, P_{0+})$. Then, by (5.12), (5.23) and (5.24),

$$\mathcal{V} := \mathcal{V}_- + \mathcal{V}_+ = \ker(P_{0+} - P_{0-}),$$

and hence, by Lemma 4.4, $\mathcal{V} = \mathcal{V}^*$, the space of zero directions corresponding to $P$. Moreover, $\mathcal{V}_-$ and $\mathcal{V}_+$ are actually $\mathcal{V}_-^*$ respectively $\mathcal{V}_+^*$. □

Theorem 5.3 is a generalization of the well-known result linking solutions in $\mathcal{P}_0$ to $\mathcal{H}$-invariant Lagrangian subspaces ([17], [22], [18]), in which special situation the equivalence classes of Theorem 5.3 are singletons, and the invariant subspaces are n-dimensional. The fewer zeros the spectral factor corresponding to $P$ has, the larger is the equivalence class (the tightest local frame) and then smaller is the dimension of the invariant subspace $\mathcal{L}$.

**Appendix.** In this appendix we shall give the proofs deferred from Section 2.

*Proof of Proposition 2.2.* Suppose that

$$U_t Y \subset Y \vee H^+_{[0,t]} \tag{A.1}$$

Let $\xi \in Y$. Then

$$U_t \xi = \lambda_t + \eta_t \tag{A.2}$$

where $\lambda_t \in Y$ and $\eta_t \in H^+_{[0,t]}$. Since $Y \subset X \cap H_0$, we must have $\lambda_t = e^{Gt}\xi$, and therefore, applying the orthogonal projector $E^X$ to (A.2), we obtain

$$e^{Ft}\xi = e^{Gt}\xi + E^X \eta_t. \tag{A.3}$$

Hence, for $t > 0$,

$$\frac{1}{t}(e^{Ft} - I)\xi = \frac{1}{t}(e^{Gt} - I)\xi + \frac{1}{t}E^X \eta_t, \tag{A.4}$$

and consequently

$$\lim_{t \downarrow 0} \frac{1}{t} E^X \eta_t \tag{A.5}$$

exists and, by the definition (2.25) of the operator $N$, must belong to $\operatorname{Im} N$. Therefore, since $e^{Gt}\xi \in Y$ (Proposition 2.8), we have $\xi \in Y \vee \operatorname{Im} N$, i.e.

$$FY \subset Y \vee \operatorname{Im} N \tag{A.6}$$

as claimed. □

*Remark.* Let $\nu \in \operatorname{Im} N$ be the limit (A.5). Then, from (A.5), we see that

$$\nu = (F - G)\xi$$

is a linear function of $\xi$, and therefore there is a map $L : Y \to \mathbb{R}^m$ such that $\nu = NL\xi$, and consequently

$$G = F - NL.$$

*Proof of Lemma 2.5.* If $\xi \in X \cap H_0$. Then $\xi = a'x(0)$ with $a \in \mathcal{V}^*$. Therefore it follows immediately from (2.43) and the fact that $\mathcal{V}^*$ is $\Gamma'$-invariant that, for $t \geq 0$,

$$U_t \xi \in X \cap H_0 \vee H^+_{[0,t]}$$

and

$$U_t^* \xi \in X \cap H_0 \vee H^-_{[-t,0]},$$

so it only remains to show that these vector sums are direct, i.e. that $X \cap H^+_{[0,t]} = 0$ and $X \cap H^-_{[-t,0]} = 0$.

By stationarity $X \cap H^-_{[-t,0]} = 0$ if and only if $(U_t X) \cap H^+_{[0,t]} = 0$. To prove the latter, suppose $\eta \in (U_t X) \cap H^+_{[0,t]}$. We want to prove that $\eta$ must be zero. To this end, note that

$$\hat{\eta} := E^{H^+_{[0,t]}} \eta = \eta \tag{A.7}$$

Now, there is an $a \in \mathbb{R}^n$ such that $\eta = a'x(t)$ and hence

$$\hat{\eta} = a'\hat{x}(t),$$

where $\hat{x}(t)$ is the Kalman-filter estimate. It is well-known that $\Pi(t) := E\{\hat{x}(t)\hat{x}(t)'\}$ satisfies the Riccati differential equation

$$\dot{\Pi} = \Lambda(\Pi), \quad \Pi(0) = 0$$

which has the limit $P_-$ as $t \to \infty$; see e.g. [9] or [16]. It is now easy to see that $Q := P_- - \Pi$ satisfies the homogeneous Riccati equation

$$\dot{Q} = \Gamma_- Q + Q\Gamma_-' - QC'R^{-1}CQ, \quad Q(0) = P_- > 0.$$

Since $Q(0) > 0$, $M(t) = Q(t)^{-1}$ exists on some finite interval $[0, t_1]$ and it is readily seen that it satisfies the Lyapunov differential equation

$$\dot{M} = -M\Gamma_- - \Gamma_-' M + C'R^{-1}C, \quad M(0) = P_-^{-1} > 0$$

there. Integrating we obtain

$$M(t) = e^{-\Gamma_-' t} M(0) e^{-\Gamma_- t} + \int_0^t e^{-\Gamma_-'(t-s)} C'R^{-1}C e^{-\Gamma_-(t-s)} ds$$

where the first term is positive definite and the second nonnegative definite. Consequently, $M(t) > 0$ for all finite t and hence $Q(t) > 0$ for all finite $t$.

Now, from (A.7) we have that

$$a' \left[ P - \Pi(t) \right] a = 0.$$

But $P - \Pi(t) \geq P_- - \Pi(t) = Q > 0$. Hence $a = 0$, and therefore $\eta = 0$. The proof that $X \cap H^+_{[0,t]} = 0$ follows from a symmetric argument. □

REFERENCES

[1] B. D. O. Anderson, *The inverse problem of stationary covariance generation*, J. Statistical Physics, 1 (1969), pp. 133–147.
[2] B. D. O. Anderson, *Algebraic properties of minimal degree spectral factors*, Automatica, 9 (1973), pp. 491–500.
[3] G. Basile and G. Marro, *Controlled and conditioned invariant subspaces in linear system theory*, J. Optimization Theory and Appl., 3 (1973), pp. 306–316.
[4] C. I. Byrnes and A. Isidori, *A frequency domain philosophy for nonlinear systems, with applications to stabilization and to adaptive control*, Proc. 23rd CDC, Las Vegas, Nevada, 1984, pp. 1569–1573.
[5] P. E. Caines and D. Delchamps, *Splitting subspaces, spectral factorization and the positive real equation: structural features of the stochastic realization problem*, Proc. 1980 IEEE Conf. Decision and Control, 1980, pp. 358–362.
[6] P. E. Caines, *Linear Stochastic Systems*, Wiley, 1988.
[7] C. A. Desoer and J. D. Schulman, *Zeros and poles of matrix transfer functions and their dynamical interpretations*, IEEE Trans. Circuits and Systems CAS-21, (1974), pp. 3–8.
[8] H. Dym and H. P. McKean, *Gaussian Processes, Function Theory, and the Inverse Spectral Problem*, Academic Press, 1976.
[9] P. Faurre, M. Clerget and F. Germain, *Opérateurs Rationnels Positifs*, Dunod, 1979.
[10] L. E. Faibusovich, *Matrix Riccati inequality: Existence of solutions*, Systems & Control Letters, 9 (1987), pp. 59–64.
[11] M. Green, *Balanced stochastic realizations*, Linear Algebra Appl., 98 (1983), pp. 369–402.

[12] M. L. J. Hautus and L. M. Silverman, *System structure and and singular control*, Linear Algebra Appl., 50 (1974), pp. 3–8.

[13] V. Kučera, *Algebraic Riccati equation: Hermitian and definite solutions*, in The Riccati Equation, S. Bittanti, A. J. Laub and J. C. Willems, eds., Springer Verlag, 1991, pp. 53–88.

[14] A. Lindquist and G. Picci, *Forward and backward semimartingale representations for stationary increment processes*, Stochastics, 15 (1985), pp. 1–50.

[15] ———*Realization theory for multivariate stationary Gaussian processes*, SIAM J. Control and Optimization, 23 (1985), pp. 809–857.

[16] ———*A geometric approach to modelling and estimation of linear stochastic systems*, Journal of Mathematical Systems, Estimation and Control, 1 (1990), pp. 241–333.

[17] A. G. J. MacFarlane, *An eigenvector solution of the optimal linear regulator problem*, J. Electron. Contr., 14 (1963), pp. 496–501.

[18] K. Mårtensson, *On the matrix Riccati equation*, Inform. Sci., 3 (1971), pp. 17–49.

[19] Gy. Michaletzky, *Zeros of (non-square) spectral factors and canonical correlations*, Proc. 11th IFAC World Congress, Tallinn, Estonia, 1991, pp. 167–172; also in preprint collection: 1990, pp. 221–226.

[20] B. P. Molinari, *The time-invariant linear-quadratic optimal-control problem*, Automatica, 13 (1977), pp. 347–357.

[21] A. Isidori and C. Moog, *On the nonlinear equivalent of the notion tranmsmission zeros*, in Modelling and Adaptive Control (C. I. Byrnes and A. Kurzhansky, eds.), Springer-Verlag, 1986.

[22] J. E. Potter, *Matrix quadratic solutions*, SIAM J. Appl. Math., 14 (1966), pp. 496–501.

[23] E. A. Robinson, *Properties of the Wold decomposition of stationary Gaussian stochastic processes*, Th. Prob. Appl., 8 (1963), pp. 187–195.

[24] Yu. A. Rosanov, *Stationary Random Processes*, Holden-Day, 1963.

[25] C. Scherer, *The solution set of the algebraic Riccati equation and the algebraic Riccati inequality*, Linear Algebra Appl., 153 (1991), pp. 99–122.

[26] M. Shayman, *Geometry of the algebraic Riccati equation*, SIAM J. Control and Optim., 21 (1983), pp. 375–394.

[27] J. C. Willems, *Least squares stationary optimal control and the algebraic Riccati equation*, IEEE Trans. Autom. Control AC-16, (1971), pp. 621–634.

[28] W. M. Wonham, *Linear Multivariable Control: a Geometric Approach*, Springer-Verlag, 1977.