# Extreme Points of Riccati Inequalities

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*Abstract*—**Relations between solutions of the algebraic Riccati equation and the associated quadratic matrix inequalities are discussed and explained.**

The purpose of this note is to clarify some of the relations that exist between solutions of the algebraic Riccati equation and the associated quadratic matrix inequalities. In particular, the main result of the note is to establish that there are *extreme points* in the solutions set of quadratic matrix inequalities that are *not* solutions of the algebraic Riccati equation. The history of this result is typical of many results involving Riccati equations in the engineering literature—total confusion. It has been part of the folklore for many years that the solutions of the algebraic Riccati equation are extreme points of the above-mentioned quadratic matrix inequalities. In Badawi's thesis [1] a very elegant proof is given, however, in review it was discovered that there had in fact appeared a proof in the literature. In Faurre *et al.* [3] there is indeed a proof and a footnote to the effect that there are extreme points other than the solutions of the algebraic Riccati equation. However, it has evolved as part of the folklore that the two sets coincide, even though it seems to be known that there are extreme points that are not solutions of the algebraic Riccati equation. However, we have been unable to find a proof. In this note we present a class of examples that establishes that there are other extreme points.

The example is based on the simple analysis presented in [2]. Following the notation of [1] we let $F$, $G$ and $H$ and $R$ be matrices such that $F$ is $2\times 2$, $G$ is $2\times 1$, and $H$ is $1\times 2$. We define then the function $W(P)=FP+PF'+(G-PH')R^{-1}(G-PH')'$. $W(P)$ is the Riccati operator. The matrix Riccati inequality referred to above is of course the inequality $W(P)\leqslant 0$ in the sense of positive definite matrices. The Riccati equation is the equation $W(P)=0$. Now we choose $F$, $G$, and $H$ such that the Hamiltonian associated with the Riccati equation has complex eigenvalues. The Hamiltonian is constructed by transforming the above equation to the more standard form (for the purposes of geometric analysis)

$$W(P)=(F-GR^{-1}H)P+P(F-GR^{-1}H)'+GR^{-1}G'+PH'R^{-1}HP$$

and writing the Hamiltonian $\mathscr{H}$

$$\mathscr{H}=\begin{pmatrix} A & D \\ -Q & -A' \end{pmatrix}$$

where we let $A=F-GR^{-1}H$, $D=-H'R^{-1}H$, and $Q=GR^{-1}G'$. This matrix has four complex eigenvalues which can be denoted by $r, r, -r, -r$. (A standard result about infinitesimal symplectic matrices.) It is trivial to establish that such matrices exist. For example, let $A$ have complex eigenvalues and let $Q=sI$. Since $\mathscr{H}$ has complex eigenvalues with $s=0$, it follows (from continuity) that when $s$ is sufficiently small, the eigenvalues of the Hamiltonian are also. Choose the matrices $A$, $B$, $Q$, and $D$ such that $(A, H)$ is controllable and $Q$ and $D$ are positive semidefinite. Then using the results of [2] there exist exactly two real solutions of the associated algebraic Riccati equation. Thus, if the set of extreme points of the quadratic matrix inequalities consists only of these two solutions then the solution set is a linear segment. This is not the case and hence there must exist other extreme points.

Manuscript received January 9, 1984. This work was supported in part by the National Science Foundation under Grant ECS-7903731, in part by the Air Force Office of Scientific Research under Grant AFOSR78-3519, and in part by NASA under Grant 2384 and DOE Contract DE-AC01-80RA-5256.

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# The Optimal Projection Equations for Fixed-Order Dynamic Compensation

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*Abstract*—**First-order necessary conditions for quadratically optimal, steady-state, fixed-order dynamic compensation of a linear, time-invariant plant in the presence of disturbance and observation noise are derived in a new and highly simplified form. In contrast to the pair of matrix Riccati equations for the full-order LQG case, the optimal steady-state fixed-order dynamic compensator is characterized by *four* matrix equations (two modified Riccati equations and two modified Lyapunov equations) coupled by a projection whose rank is precisely equal to the order of the compensator and which determines the optimal compensator gains. The coupling represents a graphic portrayal of the demise of the classical separation principle for the reduced-order controller case.**

## I. Introduction

Because of constraints imposed by on-line computations, dynamic controllers for high-order systems such as flexible spacecraft must be of relatively modest order. Hence, this paper is concerned with the design of quadratically optimal, fixed-order (i.e., reduced-order) dynamic compensation for a plant subject to stochastic disturbances and nonsingular measurement noise. Since white noise in all measurement channels precludes direct output feedback (see Section II), only purely dynamic controllers are considered. The requirements for resolution of this optimization problem include the following.

1) Conditions for the existence of an optimal, stabilizing compensator of the prescribed order. (In the full-order case these are the usual stabilizability and detectability conditions of LQG theory.)

2) Stationary conditions, i.e., first-order necessary conditions, rendered in a tractable form to facilitate developments in items 3) and 4) below. (In the full-order case these conditions are precisely the LQG gain relations together with the regulator and observer Riccati equations.)

3) Sufficiency conditions, i.e., additional restrictions on solutions of the first-order necessary conditions which characterize local minima and single out the *global* minimum. (In the full-order case the global minimum is distinguished by the unique nonnegative-definite solutions to the LQG Riccati equations.)

4) Convergent numerical algorithms for simultaneous satisfaction of the necessary and sufficient conditions. (In the full-order case numerical algorithms have been devised which take full advantage of the highly structured form of the Riccati equations.)

Manuscript received August 15, 1983; revised February 14, 1983. This paper is based on a prior submission of March 10, 1983. Paper recommended by P. R. Kumar, Past Chairman of the Stochastic Control Committee. This work was supported by the Department of the Air Force and was performed at Lincoln Laboratory, M.I.T.

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