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A geometric approach to modeling and estimation of linear stochastic systems *

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Abstract

A comprehensive theory for linear state-space modeling of stationary-increments random processes is presented. The theory of [33], which deals with stationary processes and internal models, is completed and extended to describe general non-internal representations and several new geometric concepts are introduced. Applications to a prototype noncausal linear estimation problem are discussed, and in this context new results on the invariant-sets geometry of the Riccati equation and on the zero structure of (generally nonsquare) spectral factors are presented. The emphasis is on coordinate-free representations and on geometric methods based on elementary Hilbert space concepts.

Contents

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1. Introduction

The theory of modeling, estimation and signal processing in the setting of linear systems and second-order random processes is often presented as a disparate collection of topics and methods and it has been felt that a more conceptual and unified framework is needed, both for economy of basic principles and also in view of the applicability to more general situations.

In this paper we present a comprehensive theory for linear state-space modeling of random processes and discuss applications to estimation. The emphasis is on coordinate-free representations and on geometric methods based on elementary Hilbert space concepts.

The theory presented here should be regarded as a natural and logically consistent way of building up linear stochastic systems theory. Traditionally there has been little attention paid even to the most elementary structural concepts in linear stochastic systems, like, for example, minimality. This has lead to derivations of filtering algorithms by formula manipulations without deeper understanding of why the estimates satisfy recursive equations and whether the algorithms obtained are of minimal complexity, etc. It is a fact that many structural properties important in dynamic estimation, such as, for example, the existence of recursive (i.e. differential-equation type) solutions, the minimality of filtering algorithms, and processing of specific observed signals, possibly with a noncausal information pattern, are best formulated and understood in a coordinate-free form, using the geometric language of Hilbert space theory. The use of coordinates may sometimes only obscure the basic issues.

All this motivates us to study the geometric structure of stochastic models and to investigate the natural geometric formulations of some of the system-theoretic properties mentioned above. This is basically the scope of the approach initiated in [1, 44, 47] and developed in [26-34, 48-51] into a geometric theory of stochastic realization, leading to a extensive literature in the past fifteen years; see, e.g., [5, 6, 11, 12, 23, 24].

The introduction of coordinate-free geometric descriptions is based on factoring out equivalent models with respect to a natural equivalence relation existing among models. In this respect, the basic viewpoint taken in this paper is to regard a stochastic model, say a state-space model of the form

$$\begin{cases} dx = Axdt + Bdw \\ dy = Cxdt + Ddw \end{cases} \tag{1.1}$$

(which will be discussed in much greater detail in Section 3 below), merely as a mechanism for generating trajectories of the output process y, which is considered as the only dynamical variable given and fixed in advance. Other variables in the model, like the *state process* x and the *generating noise* w, even if physically motivated, are regarded as *auxilary variables* which may be modified or even eliminated provided the model generates the same process y. This viewpoint, which underlies *stochastic realization theory* and has been implicit in much of our previous work, provides the natural equivalences on which the geometric theory is founded. Note the obvious difference to classical input-state-output models in the deterministic setting, where the input function is an external variable (like y) which is assigned from the outside and cannot be substituted by other variables.

The classification of dynamic variables described above is part of a general view of stochastic modeling according to which different models of a random process y are just different mathematical representations of y corresponding to different choices of auxilary variables. The auxilary variables are introduced in order to convey in explicit form certain additional statistical information regarding the process, useful in particular types of applications.

The auxilary dynamic variables which enter in a *stochastic state-space model* are of two types. The *state*, which is defined by the *Markovian splitting property* of rendering the past and future evolutions of the joint output-state process conditionally independent at each time t, given the current state at time t, and the *generating noise*, i.e. a white (Wiener) process which generates y when filtered by a suitable deterministic input-output map. A Markovian splitting variable for, say, a stationary mean-square continuous process y, produces a representation of y as a memoryless function of a Markov process. Generating noises lead instead to representations of y as a functional of a white noise process. The latter concept is classical in probability theory and is encountered already in the Wold representation (also called Wold's decomposition) of discrete-time stationary processes, whereby y is expressed as the output of a particular, causal and causally invertible, linear system driven by white noise.

Under the equivalence mentioned above the state variables correspond to certain fundamental geometric quantities which are called *Markovian splitting subspaces* and the generating noises correspond to *scattering pairs* of incoming and outgoing subspaces for the unitary shift group attached to the stationary (or stationary-increment) process y. Most of the important properties of state-space systems which make them useful both as models of random signals and as filtering algorithms are intrinsic defining properties of these subspaces. For example, the Markovian property of a subspace X is just the coordinate-free version of *recursiveness*, i.e. of the property that any basis in X propagates in time as the solution of a stochastic differential equation. Likewise, to say a that a Markovian splitting subspace X is *internal* (i.e. X is contained in the subspace generated by the output process) is to say that the input noise w of any state-space model corresponding to X is constructible from the process y by means of a suitable whitening filter and hence the state-space model itself can be viewed as a recursive algorithm processing y.

The geometric approach leads to a very clear notion of minimality and to geometric conditions for observability, constructibility, minimality of spectral factors, etc., which provide economy of representation and which play important roles in many questions of stochastic systems theory. There is a fundamental representation of Markovian splitting subspaces in terms of *scattering pairs* which clarifies the role of *causality* in the representations.

In the linear-stationary setting, it is common to consider only causal state-space models (1.1) where A is assumed to be a stability matrix (i.e. with eigenvalues strictly inside the left half plane) and therefore with x, and hence y a function of the *past noise* only. This is both natural and useful in the context of classical estimation problems with a causal information pattern, but less so in more general situations. The geometric representation of Markovian splitting subspaces by scattering pairs introduces a more symmetric treatment of past and future and leads *naturally*, after a choice of basis x in X, to the simultaneous consideration of *pairs* of state-space models, called the *forward* and *backward* realizations, in which the same state process x is expressed both as a causal and as a anticausal function of the generating noises. In fact, this simultaneous consideration of the two models turns out to be quite useful, for example, in testing minimality of a model.

The geometric theory of stochastic realization, besides providing general and natural tools for studying linear stochastic systems and estimation problems, also provides a better understanding of the fine structure of the solution set of the Riccati equation. It is probably not so widely known that, together with linear-quadratic control and Kalman filtering, state space modeling of random processes is an area in which the Riccati equation in various forms, both differential and algebraic, but especially in the form of a quadratic matrix inequality, plays a very fundamental role. The Riccati equation enters into stochastic modeling because of the quadratic nature of the problem, which is essentially based on spectral factorization. Very roughly speaking, state space models of a random process are based on realizations in the deterministic sense of pairs of spectral factors (here assumed to be rational) of the process to be represented.

The study and classification of state space models, in particular the characterization of minimality etc., must then involve the study and classification of spectral factors and hence of the entire family of solutions of a corresponding quadratic inequality of Riccati-type. This need to consider the whole solution set of an algebraic Riccati inequality is a peculiar feature of the stochastic modeling problem. The classification of particular subclasses of models, for example the so-called internal realizations, leads to the study of certain subsets of the solution set of the algebraic Riccati inequality. For example, internal models correspond to extreme points at which the algebraic Riccati inequality becomes an *algebraic Riccati equation*. In this context the reader should note that the results on the local structure of the solution set $\mathcal{P}$ of the algebraic Riccati inequality, the geometry of the invariant sets for the corresponding Riccati differential equation, and the relation to zeros of spectral factors, discussed in Sections 10 and 11, are based on a geometric notion of partial *ordering* and on the notion of *tightness* of the ordering of minimal Markovian splitting subspaces. These concepts are introduced in a geometric framework and have a very natural interpretation in terms of the underlying noncausal estimation problem. It seems much less natural (and much harder) to develop these concepts in a purely matrix-theoretic context.

Since this is a rather long paper, we shall provide the reader with a "navigation chart" through the various sections. The first part of the paper, consisting of Section 3, 4 and 5, deals with stochastic realization theory. *Section 3* motivates the geometric approach and introduces the notions of Markovian splitting subspace and Markovian representation in the context of stationary increments processes and not necessarily internal realizations. This is a wider class of models than covered in [33] , which deals with stationary processes and internal models only. *Section 4* extends the basic geometric theory of stochastic realization based on scattering pairs, as presented in [33], to the noninternal, stationary-increments setting. The concept of minimality is introduced and geometric characterizations of minimality are given. Ruckebusch [48-50] has studied the stochastic realization problem from a somewhat different, but conceptually similar, angle, and the early development of the theory has profited from important cross-fertilization.

Section 5 ties up the geometric theory to spectral factorization and the computation of the generating processes (i.e. input noises) of the resulting state space models, thereby translating the geometry of Section 4 to an isomorphic coordinate-free description in the frequency domain. The *inner triplet* of a Markovian splitting subspace is introduced and minimality is characterized in terms of various coprimeness conditions. Forward-backward pairs of realizations are discussed and related to the corresponding pairs of spectral factors.

Next, in *Section 6*, a partial ordering of minimal Markovian splitting subspaces is introduced and a fundamental approximation theorem bounding a minimal Markovian splitting subspace from above and from below by *internal* minimal Markovian splitting subspaces is presented (Theorem 6.11). Based on this ordering one can equip the family $\mathcal{X}$ of minimal Markovian splitting subspaces with a natural *uniform choice of bases*. This leads to the parametrization of minimal stochastic realizations by $n \times n$ state covariance matrices P.

In *Section 7* this parametrization is first analyzed in the framework of the classical Anderson-Faurre theory of "stationary covariance generation"[3, 10]. The parametrization of $\mathcal{X}$ by the solution set $\mathcal{P}$ of an *algebraic Riccati inequality* is discussed. This set consists of the state covariance matrices of minimal models in a given uniform choice of bases and the ordering of $\mathcal{X}$ becomes the positive semidefinite ordering of symmetric matrices.

In *Section 8* , after all necessary tools have been introduced, we study a fundamental noncausal estimation problem which serves as a motivation and provides stochastic interpretations for the results described in the rest of the paper. The solution of this problem, presented in geometric terms, is given by the tightest pair of internal minimal Markovian splitting subspaces, bounding a given X (Theorem 6.11), and their vector sum is the *local frame space* of X. The local frame space serves as a minimal state space for the noncausal estimator.

In *Section 9* the geometric conditions for tightness of subspaces are reformulated in terms of state covariance matrices $P \in \mathcal{P}$. The results give necessary and sufficient conditions for a pair of solutions (P_1, P_2) of the algebraic Riccati equation to be the tightest bound for a given $P \in \mathcal{P}$. In fact, this *tightest frame* of P can be computed as the limits as $t \to \pm\infty$ of the solution of the corresponding Riccati differential equation initialized at $P(0) = P$. This interesting result, which emanates from the invariant-sets decomposition of $\mathcal{P}$ mentioned above, is presented in *Section 10.* It also provides a computational tool for constructing the noncausal filter, thus generalizing the role of the Riccati equation in Kalman filtering and giving a very natural filtering interpretation to *all* the solutions of the algebraic Riccati equation.

Section 11, the last section, gives a different characterization of the tightest frame about $P \in \mathcal{P}$ in terms of the *zeros* of the corresponding minimal spectral factor $W(s)$. The relation between zeros and the local frame spaces is given in Theorem 11.4 . As a byproduct of the analysis we get a simple geometric description of the local frame space of any minimal realization (Theorem 11.5) and an explicit computation of its dimension.

The results presented in Sections 9-11 were first announced in an IMA plenary lecture in Glasgow in September, 1988 and have appeared in condensed form in two conference proceedings [35, 36]. Independently, Michaletzky [39] recently presented results on zeros of spectral factors some of which are similar to ours. Finally, we would like to thank Paul Fuhrmann for some advice helpful in proving Lemma 6.7 and Christopher Byrnes for alerting us to the fact that our results on zeros of spectral factors are connected to geometric control theory.

2. Hilbert spaces of random variables

The geometric theory of linear stochastic systems is formulated in terms of subspaces of certain Hilbert spaces H of zero-mean second-order random variables, having the inner product

$$\langle \xi \eta \rangle = E\{\xi \eta\} \tag{2.1}$$

where E denotes mathematical expectations. Such Hilbert spaces may be constructed from any underlying finite or infinite set M of second order random variables by taking the closure in the Hilbert space topology (2.1) of the space of all finite linear combinations of elements in M. For example, to set notations, if $\{z(t) \; ; \; t \in \mathrm{R}\}$ is a stationary m-dimansional vector process, $M := \{z_k(t) \; ; \; t \in \mathrm{R}, \; k = 1, 2, \ldots, m\}$ defines the Hilbert space $H(z)$, and if it is a m-dimensional vector process with stationary increments then $M := \{z_k(t) - z_k(s) \; ; \; t, s \in \mathrm{R}, \; k = 1, 2, \ldots, m\}$ generates the Hilbert space $H(dz)$.

Given any subspace X of H we shall denote by $E^X \eta$ the orthogonal projection of $\eta \in H$ onto X. In terms of Hilbert spaces of Gaussian random variables this may be interpreted as the *conditional expectation* given the random variables generating X. If z is a random vector, $E^X z$ will denote the random vector with components $E^X z_i$. Moreover, we shall write $A \perp B$ to denote that two subspaces A and B are orthogonal and $A \perp B|X$ to denote that they are *conditionally orthogonal* given X, i.e. that

$$\langle \alpha - E^X \alpha, \beta - E^X \beta \rangle = 0 \qquad \text{for all} \qquad \alpha \in A, \beta \in B \tag{2.2}$$

Finally, $A \vee B$ is the closure of the set $\{\alpha + \beta | \alpha \in A, \beta \in B\}$, $A \oplus B$ is orthogonal direct sum, and $C := A \ominus B$ is the subspace such that $B \oplus C = A$. Sometimes we write $H \ominus A$ as $A^\perp$.

The following proposition can be found e.g. in [33].

Proposition 2.1. *The following statements are equivalent*

(i) $A \perp B|X$

(ii) $B \perp A|X$

(iii) $(A \vee X) \perp B|X$

(iv) $E^{A \vee X}\beta = E^X \beta \qquad$ *for all* $\qquad \beta \in B$

(v) $(A \vee X) \ominus X \perp B$

(vi) $E^A \beta = E^A E^X \beta \qquad$ *for all* $\qquad \beta \in B.$

Hilbert spaces generated by random processes, such as $H(z)$ and $H(dz)$, come naturally equipped with a time structure. We define the *past space* $H^-(z)$ of $H(z)$ as the subspace generated by $\{z(t)\ ;\ t \leq 0\}$ and the *future space* as the subspace generated by $\{z(t)\ ;\ t \geq 0\}$. The past space $H^-(dz)$ and the future space $H^+(dz)$ are defined analogously. We shall only consider processes which are continuous in mean-square. Then there is a strongly continuous group $\{U_t\ ;\ t \in \mathsf{R}\}$ of unitary operators on $H(z)$ and $H(dz)$ called the *shift* induced by z or dz, respectively, defined by extending the operators U_t

$$U_t z_k(s) = z_k(s+t) \tag{2.3}$$

and

$$U_t[z_k(s) - z_k(\tau)] = z_k(s+t) - z_k(\tau + t) \tag{2.4}$$

to $H(z)$ and $H(dz)$ respectively in the standard way [46]. The shift U_t has the adjoint $U_t^* = U_{-t}$. In terms of the shift we have the invariance properties

$$U_t^* H^-(z) \subset H^-(z) \ \text{ and } \ U_t H^+(z) \subset H^+(z) \tag{2.5}$$

for $t \geq 0$, respectively,

$$U_t^* H^-(dz) \subset H^-(dz) \ \text{ and } \ U_t H^+(dz) \subset H^+(dz) \tag{2.6}$$

Invariances of this type will play an important part in this paper. We refer the reader to Appendix A for details.

3. Stochastic models and Markovian representations

A basic object of our study are linear stochastic systems of the type

$$(\Sigma) \quad \begin{cases} dx = Axdt + Bdw \\ dy = Cxdt + Ddw \end{cases} \tag{3.1}$$

defined for all $t \in \mathsf{R}$, where w is a p-dimensional vector Wiener process, and A, B, C, D are constant matrices with A being a *stability matrix*, i.e. having all its eigenvalues in the open left half-plane. The system is in statistical steady state so that the n-dimensional *state* process x and the increments of the m-dimensional *output* process y are jointly stationary. We shall think of Σ as a representation of the (increments of the) process y; such a representation will be called a (finite-dimensional) *stochastic realization* of dy. The number of state variables n will be called the *dimension* of Σ, denoted $\dim \Sigma$.

Systems of this type have been used in the engineering literature since the early 1960's as models for random signals. An alternative but, as we shall see below, not entirely equivalent way of representing the signal dy is obtained by eliminating the state x from (3.1). In this way we obtain a scheme which generates dy by passing white noise dw through a shaping filter with rational transfer function

$$W(s) = C(sI - A)^{-1}B + D \tag{3.2}$$

as explained in Appendix B. This produces a stationary increment process dy with the spectral representation

$$y(t) - y(s) = \int_{-\infty}^{\infty} \frac{e^{i\omega t} - e^{i\omega s}}{i\omega} W(i\omega) d\hat{w} \tag{3.3}$$

and hence with the rational spectral density

$$W(s)W(-s)' = \Phi(s) \tag{3.4}$$

where prime ($'$) denotes transpose. In other words, W is a *spectral factor* of Φ, which, in view of the fact that A is a stability matrix, is *analytic*, i.e. has all its poles in the open left halfplane.

However, the model Σ is more than just a representation of a stochastic process in terms of white noise. Much more important in applications is that the model (3.1) contains a state process x which serves as a dynamical memory for dy. A formalization of this idea will be the starting point for the geometric theory developed in this paper. Before getting into this, however, we shall present some preliminary observations about stochastic models.

3.1. Minimality and nonminimality of models

We shall say that Σ is *minimal* if dy has no other stochastic realization of smaller dimension. Occasionally, as for example in noncausal estimations, we shall also need to consider nonminimal Σ. Therefore, it is important to understand the relation between $\deg W$, the McMillan degree of W, and $\dim \Sigma$.

Before turning to this point, we need to recall a few well-known facts about the state process x. Since A is a stability matrix, we have

$$x(t) = \int_{-\infty}^{t} e^{A(t-\tau)} B dw(\tau) \tag{3.5}$$

from which it is seen that the state process is a stationary wide-sense Markov process with a constant covariance matrix

$$P := E\{x(t)x(t)^{'}\} = \int_{0}^{\infty} e^{A\tau} BB^{'} e^{A^{'}\tau} d\tau \tag{3.6}$$

which clearly satisfies the Lyapunov equation

$$AP + PA^{'} + BB^{'} = 0 \tag{3.7}$$

From (3.6) it is seen that P is the reachability Grammian for the pair (A, B), and therefore the system Σ is reachable if and only if P is positive definite ($P > 0$), i.e. if and only if $\{x_1(0), x_2(0), \dots, x_n(0)\}$ is a basis in the space

$$X = \text{span}\{x_1(0), x_2(0), \dots, x_n(0)\} \tag{3.8}$$

consisting of all linear combinations of the components of $x(0)$. The space X will play a fundamental role in what follows, being the abstract representation of Σ in the geometric theory. We should, however, immediately alert the reader to the fact that X and Σ cannot be equivalent representations, as trivially there may be redundancy in Σ due to nonreachability which cannot be seen in X. The following proposition makes this point more precise and gives a preview of some facts concerning X and W to be studied in detail in Section 5.

PROPOSITION 3.1. *Let dy be a stationary-increment process with a rational spectral density Φ having a finite-dimensional stochastic realization Σ of type (3.1) with spectral factor W given by (3.2), and let X be the state space (3.8). Then*

$$\frac{1}{2} deg\Phi \leq degW \leq \dim X \leq \dim \Sigma \tag{3.9}$$

Moreover, $\deg W = \dim X$ *if and only if (C, A) is observable, and* $\dim X = \dim \Sigma$ *if and only if (A, B) is reachable.*

The statements concerning the last of inequalities (3.9) follows immediately from the preceding discussion while those concerning the second inequality are a consequence of Theorem 5.13 in Section 5.5 below. The first inequality in the chain is proved in [2].

From Proposition 3.1 we may learn several things about stochastic realizations. First, for Σ to be minimal it is *not* sufficient that Σ is both observable and reachable. For this we must also have

$$\deg W = \frac{1}{2} \deg \Phi \tag{3.10}$$

A W satisfying this condition will be called a *minimal* spectral factor [2, 3]. Secondly, reachability plays no role in the geometric theory since the basic object of it is X and not Σ.

3.2. The idea of state space and Markovian representations

There is a trivial equivalence relation between realizations of dy corresponding to a change of coordinates in the state space and constant orthogonal transformations of the input Wiener process dw, which we would like to factor out before undertaking the study of the family of (minimal *and* nonminimal) stochastic realizations. The equivalence classes are defined by

$$(A, B, C, D, dw) \sim (T_1 A T_1^{-1}, T_1 B T_2^{-1}, C T_1^{-1}, D T_2^{-1}, T_2 dw) \tag{3.11}$$

where T_1, is an $n \times n$ nonsingular matrix and T_2 is a $p \times p$ orthogonal matrix. Clearly, the state space X , defined by (3.8), is an invariant of this equivalence, and we shall look for conditions under which this invariant is complete in the sense that there is bijective correspondence between equivalence classes $\Sigma]$ and spaces X. Since realizations Σ and $\tilde{\Sigma}$ such that $\begin{bmatrix} B \\ D \end{bmatrix} dw = \begin{bmatrix} \tilde{B} \\ \tilde{D} \end{bmatrix} d\tilde{w}$ give rise to the same X, an obvious necessary condition is that

$$\operatorname{rank} \begin{bmatrix} B \\ D \end{bmatrix} = p. \tag{3.12}$$

Moreover, as pointed out in Section 3.1, it is necessary to consider only models Σ for which

$$(A, B) \text{ reachable.} \tag{3.13}$$

We shall prove that under these two conditions the above one-one correpondence holds.

We proceed to characterize these X spaces. Given a realization Σ, first denote by H and H_0 the spaces of random variables

$$H := H(dw) \qquad H_0 := H(dy) \tag{3.14}$$

and let $\{U_t; t \in \mathrm{R}\}$ be the *shift* induced by dw, i.e. the strongly continuous group of unitary operators on H such that

$$U_t[w(\tau) - w(\sigma)] = w(\tau + t) - w(\sigma + t). \tag{3.15}$$

Obviously X and H_0 are subspaces of H, H_0 being doubly invariant for the shift, so that $U_t x(\tau) = x(\tau + t)$ and

$$U_t[y(\tau) - y(\sigma)] = y(\tau + t) - y(\sigma + t) \tag{3.16}$$

Next define

$$X^- := H^-(x), \;\; X^+ := H^+(x), \;\; H^- := H^-(dy) \text{ and } H^+ := H^+(dy) \tag{3.17}$$

Now solving (3.1) we have

$$x(t) \;\; = \;\; e^{At}x(0) + \int_0^t e^{A(t-\tau)} B dw(\tau) \tag{3.18a}$$

$$y(t) - y(0) \;\; = \;\; \int_0^t C e^{A\tau} d\tau x(0) + \int_0^t [\int_0^t C e^{A(\tau-\sigma)} B d\sigma + D] dw(\tau) \tag{3.18b}$$

Therefore, since $H^+(dw) \perp H^-(dw) \supset H^- \vee X^-$,

$$E^{H^- \vee X^-}\lambda = E^X \lambda \quad \text{for all } \lambda \in H^+ \vee X^+ \tag{3.19}$$

which is, as pointed out in Section 2, the conditional orthogonality

$$H^- \vee X^- \perp H^+ \vee X^+ | X. \tag{3.20}$$

This is the *state space* property of the subspace X which will play a central role in what follows. In general, given a Hilbert space H of random variables containing H_0 with a shift $\{U_t\}$ satisfying (3.16), a subspace X of H is said to be a *Markovian splitting subspace* if it satisfies the conditional orthogonality relation (3.20) with X^- and X^+ defined as

$$X^- := \vee_{t \leq 0} U_t X \quad \text{and} \quad X^+ := \vee_{t \geq 0} U_t X \tag{3.21}$$

Note that (3.20) implies that

$$X^- \perp X^+ | X \tag{3.22}$$

and

$$H^- \perp H^+ | X \tag{3.23}$$

A subspace X is said to be *Markovian* if it satisfy (3.22) with X^- and X^+ given by (3.21) and *splitting* if it satisfies (3.23). Note that, in general, (3.22) and (3.23) do not imply the joint conditional orthogonality relation (3.20), and therefore being a Markovian splitting subspace is a more stringent condition than being both a Markovian space and a splitting subspace.

In view of (3.5), $H^- \vee X^- \subset H^-(dw)$ which is purely nondeterministic (p.n.d.); see Appendix A. Hence the subspace $H^- \vee X^-$ is also p.n.d. In the finite-dimensional case it can be shown (as will be done below) that $H^- \vee X^-$ is p.n.d. if and only if $H^+ \vee X^+$ is. In general we say that the Markovian splitting subspace is *proper* if both these conditions hold.

We shall now give a precise statement describing the parametrization of equivalent classes $[\Sigma]$ of realizations in terms of Markovian splitting subspaces. To this end, we need the following definition.

DEFINITION 3.2. *A Markovian representation* of dy is a triplet $(H, \{U_t\}, X)$ where X is a Markovian splitting subspace in the Hilbert space

$$H = H_0 \vee \overline{\text{span}}\{U_t X; t \in R\}, \tag{3.24}$$

called the *ambient space* of the representation, $\{U_t\}$ is a shift on H such that (3.16) holds, and $\overline{\text{span}}$ denotes closed span. A Markovian representation is said to be *internal* if $X \subset H_0$, in which case $H = H_0$, and *proper* if X is proper. The *dimension* of a Markovian representation is the dimension of X. When there is no reason for misunderstanding, we shall write (H, U, X) for short.

THEOREM 3.3 *There is a one-one correspondence between equivalence classes* $[\Sigma]$ *of stochastic realizations of* dy *satisfying conditions* (3.12) *and* (3.13) *and proper finite-dimensional Markovian representations* $(H, \{U_t\}, X)$ *of* dy *under which* $H(dw) = H$ *and the state* $x(0) = \{x_1(0), x_2(0), \dots, x_n(0)\}$ *of each* $\Sigma \in [\Sigma]$ *is a basis of* X.

Proof. We showed above that A being a stability matrix implies that $H^- \vee X^-$ is p.n.d. In [25] it was shown that to each realization Σ there corresponds a backward realization with $\bar{A}$ similar to $-A'$. (See (3.30) below.) Hence applying the same argument in reversed time, we also have $H^+ \vee X^+$ p.n.d., i.e. X is proper. Consequently, it follows from the construction above that to each equivalence class $[\Sigma]$ of realizations there corresponds a unique proper Markovian representation, of the same dimension as Σ, having the stated properties. It remains to show that to each finite dimensional, proper Markovian representation $(H, \{U_t\}, X)$ of a process dy with stationary increments there corresponds a realization Σ satisfying (3.12) and (3.13) and such that $\{x_1(0), x_2(0), \dots, x_n(0)\}$ is a basis of X and $H(dw) = H$. To this end, set $S := H^- \vee X^-$. Then, by assumption S is p.n.d. Moreover, S is full-range because of (3.24). Therefore, there is a Wiener process dw uniquely defined modulo multiplication by a constant orthogonal matrix T_2, such that $S = H^-(dw)$ and $H = H(dw)$; (Theorem A.2). Let $\{\xi_1, \xi_2, \dots, \xi_n\}$ be a basis in X. Then, since X is Markovian,

$$x(t) = \begin{bmatrix} U_t\xi_1 \\ U_t\xi_2 \\ \vdots \\ U_t\xi_n \end{bmatrix} \qquad -\infty < t < \infty \tag{3.25}$$

is a stationary, p.n.d., vector Markov process. From (3.19) we see that there is a matrix function $\Phi(t)$ such that

$$E^S x(t) = E^X x(t) = \Phi(t)x(0) \qquad \text{for} \quad t \geq 0 \tag{3.26}$$

Moreover, for $t, s \geq 0$,

$$E^S x(t+s) = E^S E^{U_t S} x(t+s) = E^S \Phi(s) x(t) = \Phi(s)\Phi(t)x(0), \tag{3.27}$$

that is, $\Phi(t)$ is a continuous semigroup on R^n, and hence of the form e^{At} with A being an asymptotically stable $n \times n$ matrix. Stability follows from the fact that

$$\| E^S x(t) \| = \| E^{U_{-t}S} x(0) \| \to 0 \ \text{ as } \ t \to \infty \tag{3.28}$$

because S is p.n.d. Then, by the same argument as in [34, Theorem 3.1], it is seen that there is a constant matrix B such that

$$x(t) - x(0) - \int_0^t Ax(s)ds = \int_0^t B dw(s) \tag{3.29}$$

i.e. $x(t)$ satisfies a stochastic differential equation of type (3.1). Since $x(0)$ is a basis in X, (A, B) must be reachable. Next, we note that, because of finite dimensionality of x the process y is conditionally Lipschitz with respect to S [34] and therefore it has a semimartingale representation as in (3.1). Clearly, condition (3.12) holds, for otherwise $H^- \vee X^-$ would be a proper subspaces of $H(dw)$ contrary to the construction. (A similar construction can be found in [37].) □

Consequently, we have reduced stochastic realizations of dy to geometric objects in Hilbert space. We shall commence our study of this in Section 4 where a geometric characterization of all Markovian representations will be given.

3.3. Anticausal stochastic realizations

As will be quite clear from the geometric theory to follow, there is complete symmetry in stochastic realization theory under reversal of time. In fact, in [25] it was shown that there is a natural one-one correspondence between (forward) stochastic realizations Σ and backward stochastic realizations

$$(\bar{\Sigma}) \qquad \begin{cases} d\bar{x} = \bar{A}\bar{x}dt + \bar{B}d\bar{w} \\ dy = \bar{C}\bar{x}dt + \bar{D}d\bar{w} \end{cases} \tag{3.30}$$

with $\bar{A}$ is *antistable*, i.e. having all its eigenvalues in the right open halfplane, in the sense that $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, d\bar{w})$ is uniquely determined by (A, B, C, D, dw) and vice versa and that Σ and $\bar{\Sigma}$ have the same state space, i.e.

$$\text{span}\,\{\bar{x}_1(0), \bar{x}_2(0), \dots, \bar{x}_n(0)\} = X \tag{3.31}$$

Such a stochastic system $\bar{\Sigma}$ evolves backward in time and its spectral factor

$$\bar{W}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} \tag{3.32}$$

is *coanalytic*, i.e. it has all its poles in the open right half plane.

Naturally, there is a backward version of Proposition 3.1 so that

$$\frac{1}{2}\deg\Phi \le \deg\bar{W} \le \dim X \le \dim\bar{\Sigma} \tag{3.33}$$

where $\bar{W}$ is called a *minimal* coanalytic spectral factor if the first inequality is satisfied with equality. Also, $\deg\bar{W} = \dim X$ if and only if $(\bar{C}, \bar{A})$ is observable and $\dim X = \dim\bar{\Sigma}$ if and only if $(\bar{A}, \bar{B})$ is reachable. In accordance with Kalman's definitions [19], we shall say that the *backward* system $\bar{\Sigma}$ is *constructible* if $(\bar{C}, \bar{A})$ is observable and *controllable* if $(\bar{A}, \bar{B})$ is reachable.

This duality between forward and backward, between causal and anticausal, will emerge very naturally in the geometric theory.

4. Geometric theory of Markovian representations

Although the primary concern in stochastic systems theory is the study of finite-dimensional systems Σ of type (3.1) the geometric theory and many of the results and concepts based on it hold under more general conditions. Observability and constructability, as well as minimality of W and $\bar{W}$ will be given simple geometric characterizations which make sense also in the infinite dimensional case. In fact, the concept of dimension plays a secondary role in the geometric theory. Therefore, unless otherwise stated, no assumption of finite-dimensionality will be made, and when so is done it is for technical reasons.

4.1. The fundamental representation theorem

The following theorem, which is a generalization of the corresponding results first presented in [28, 29], provides basic geometric description of the class of Markovian representations.

THEOREM 4.1. *Let $H \supset H_0$ be a Hilbert space of random variables with a shift $\{U_t\}$ satisfying (3.9), and let X be a subspace of H such that*

$$H = H_0 \vee \overline{\mathit{span}}\,\{U_t X\ ;\ t \in R\} \tag{4.1}$$

Then (H, U, X) is a Markovian representation if and only if

$$X = S \cap \bar{S} \tag{4.2}$$

for some pair $(S, \bar{S})$ of subspaces of H such that

$$\text{(i)} \begin{cases} H^- \subset S \\ H^+ \subset \bar{S} \end{cases} \qquad \text{(ii)} \begin{cases} U_t^* S \subset S & \text{for } t \geq 0 \\ U_t \bar{S} \subset \bar{S} & \text{for } t \geq 0 \end{cases} \tag{4.3}$$

and

$$\text{(iii)}\ \ H = \bar{S}^\perp \oplus (S \cap \bar{S}) \oplus S^\perp \tag{4.4}$$

where $\perp$ denotes the orthogonal complement in H. Moreover, the correspondence $X \leftrightarrow (S, \bar{S})$ is one-one. In fact,

$$\begin{cases} S = H^- \vee X^- \\ \bar{S} = H^+ \vee X^+ \end{cases} \tag{4.5}$$

Finally, X is proper if and only if both $S^\perp$ and $\bar{S}^\perp$ are full range, or, equivalently, both S and $\bar{S}$ are p.n.d.

Proof. (if) By Proposition 2.4 and Theorem 3.2 in [33], (iii) is equivalent to the conditions

$$S \ \perp\ \bar{S} \mid X \tag{4.6}$$

$$S \ \vee\ \bar{S} = H \tag{4.7}$$

where X is given by (4.2). Now, together with (4.2), (i) and (ii) imply that

$$H^- \vee X^- \subset S \qquad \text{and} \qquad H^+ \vee X^+ \subset \bar{S} \tag{4.8}$$

and therefore (3.20) follows from (4.6). (only if): Define S and $\bar{S}$ by (4.5). Then (i) and (ii) hold. Moreover, (3.20) is the same as (4.6), and the definition (4.1) of H insures that (4.7) holds. It remains to show that X is given by (4.2). However, this follows from Theorem 3.1 in [33]. (one-one): By Theorem 3.1 in [33] and (3.20), there is only one pair $(S, \bar{S})$ satisfying (4.6) and (4.8), namely that defined by (4.5). The last statement of the theorem, finally, follows from the fact that S $[\bar{S}]$ is p.n.d. if and only if $S^\perp$ $[\bar{S}^\perp]$ is full range. □

In the sequel, we shall write $X \sim (S, \bar{S})$ to exhibit the one-one correspondence of Theorem 4.1, and we shall call $(S, \bar{S})$ the *scattering pair representation* of X. This terminology comes from the fact that S and $\bar{S}$ are incoming and outgoing subspaces for the unitary group $\{U_t\}$ in the sense of Lax-Phillips [21]. Note that Theorem 4.1 provides a *different* scattering framework for each Markovian splitting subspace X. Let us define the *multiplicity* of a proper Markovian representation (H, U, X) with $X \sim (S, \bar{S})$ to be the common multiplicity of $S, \bar{S}$ and H. (See Theorem A.1.).

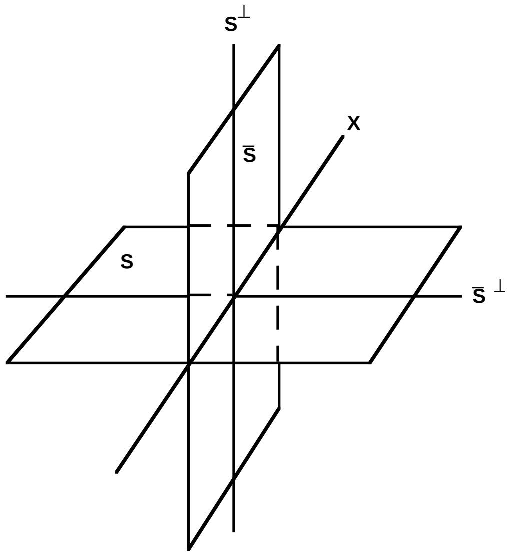

Figure 4.1: The splitting geometry

The geometric interpretation of Condition(iii) of Theorem 4.1 is that S and $\bar{S}$ *intersect perpendicularly* as depicted in Figure 4.1. It is clear from this condition that (4.2) can be replaced by

$$X = E^S \bar{S} = E^{\bar{S}} S \tag{4.9}$$

Alternative geometric characterizations of perpendicular intersection can be found in [33]. In particular, S and $\bar{S}$ intersect perpendicularly if and only if $\bar{S}^\perp \subset S$, the orthogonal complement $S \ominus \bar{S}^\perp$ being precisely equal to X.

Given a Markovian representation (H, U, X), for each $t \geq 0$, let $U_t(X) : X \to X$ be the compressed shift

$$U_t(X) = E^X U_{t|X} \tag{4.10}$$

Note that, since $\bar{S}^\perp$ is a U_t^*-invariant subspace of S, its orthogonal complement in S, which is precisely X, is invariant for the adjoint of the restricted backward shift $U^*_{|S}$. This is the same as saying that $E^S U_{t|X} = U_t(X)$, and so $\{U_t(X);\ t \in \mathrm{R}\}$ is a strongly continuous semigroup, i.e.

$$U_t(X) U_s(X) = U_{t+s}(X) \tag{4.11}$$

and, if X is proper, $U_t(X)$ tends strongly to zero as $t \to \infty$; see, e.g., [33; Thm 6.2]. In particular, if (H, U, X) is finite-dimensional and corresponds to the stochastic realization (3.1), $E^X a' x(t) = a' e^{At} x(0)$ for any $a \in \mathrm{R}$, i.e.

$$U_t(X) a' x(0) = a' e^{At} x(0) \tag{4.12}$$

and consequently $U_t(X)$ plays the role of e^{At} in the geometric theory. A dual argument exchanging S and $\bar{S}$ yields the backward semigroup $U_t(X)^* = E^X U^*_{t|X}$, which corresponds to the matrix representation $e^{-A' t}$. These are the semigroups which govern the dynamics of the forward and backward models corresponding to X, mentioned in Section 3, and to be reintroduced in Section 5.5.

4.2. Geometric characterizations of minimality

We say that a Markovian splitting subspace is *minimal* if it contains no other Markovian splitting subspace as a proper subspace. A *minimal Markovian representation* is a Markovian representation for which X is minimal. Now the inclusions $\bar{S}^\perp \subset S$ and $S^\perp \subset \bar{S}$ together with condition (i) of Theorem 4.1 imply that the scattering pair $(S, \bar{S})$ of a Markovian splitting subspace must satisfy the constraints

$$S \supset H^- \vee \bar{S}^\perp \qquad \bar{S} \supset H^+ \vee S^\perp \tag{4.13}$$

Moreover, it is obvious from the representations (4.2) and (4.5) that we have the inclusion $X_1 \subset X_2$ of Markovian splitting subspaces if and only if there is a subspace inclusion of the corresponding $(S, \bar{S})$ pairs, i.e. $S_1 \subset S_2$, $\bar{S}_1 \subset \bar{S}_2$. Therefore, in order to achieve minimality, in view of (4.2), we should reduce S and $\bar{S}$ as much as possible but without violating the constraints (4.13). The following theorem, which is a generalization to the not necessarily internal case of a result in [29] appearing as Theorem 3.3 in [33] , provides a procedure for this reduction.

THEOREM 4.2. *Let (H, U, X) be a Markovian representation and let $X \sim (S, \bar{S})$. Set*

$$\bar{S}_1 := H^+ \vee S^\perp \tag{4.14a}$$
$$S_1 := H^- \vee \bar{S}_1^\perp \tag{4.14b}$$

where $\perp$ denotes the orthogonal complement in H. Then $X_1 \sim (S_1, \bar{S}_1)$ is a minimal Markovian splitting subspace such that $X_1 \subset X$, and (H_1, U, X_1) is a minimal Markovian representation with $H_1 = S_1 \vee \bar{S}_1$. If (H, U, X) is proper, so is (H_1, U, X_1) and $H_1 = H$. In particular, multiplicity is preserved.

Proof. The proof that $X_1 \sim (S_1, \bar{S}_1)$ is a minimal Markovian splitting subspace such that $X_1 \subset X$ is the same as that of Theorem 3.3 in [33]. From this and Theorem 4.1, it follows that (H_1, U, X_1) with $H_1 = S_1 \vee \bar{S}_1 \subset H$ is a minimal Markovian representation. It remains to prove the last statement of the theorem. To this end, suppose (H, U, X) is proper. As a corollary of the proof in [33] we have that $X_0 \sim (S, \bar{S}_1)$ is a Markovian splitting subspace and that $\bar{S}_1 \subset \bar{S}$. From Theorem 4.1 it follows that $S = H^- \vee X_0^-$ and that $\overline{span}\{U_t S;\ t \in \mathrm{R}\} = H$, and therefore H is also the ambient space of X_0. Moreover, $\bar{S}_1^\perp \supset \bar{S}^\perp$ so $\bar{S}_1^\perp$ is full range. Therefore (H, U, X_0) is a proper Markovian splitting subspace, which, using the same argument again for the next step of reduction, in turn implies that (H, U, X_1) is a proper Markovian representation. □

EXAMPLE 4.3. It is immediately seen that (H_0, U, H^-) is a Markovian representation and that $H^- \sim (H^-, H_0)$. Applying Theorem 4.2 we obtain $S_1 = H^-$ and $\bar{S}_1 = H^+ \vee (H^-)^\perp$, and in view of (4.9), the corresponding Markovian splitting subspace is

$$X_1 = E^{H^-}[H^+ \vee (H^-)^\perp] = E^{H^-} H^+$$

Moreover, $S_1 \vee \bar{S}_1 = H_0$. Therefore, the *predictor space*

$$H^{+/-} := E^{H^-} H^+ \tag{4.15}$$

is an internal Markovian splitting subspace and $H^{+/-} \sim (H^-, (N^-)^\perp)$, where

$$N^- := H^- \cap (H^+)^\perp \tag{4.16}$$

This subspace will play an important role below. □

EXAMPLE 4.4 In the same way, applying Theorem 4.2 to the Markovian representation (H_0, U, H^+), we see that the backward predictor space

$$H^{-/+} := E^{H^+} H^- \tag{4.17}$$

is an internal minimal Markovian splitting subspace, and with

$$N^+ := H^+ \cap (H^-)^\perp \tag{4.18}$$

we have $H^{-/+} \sim ((N^+)^\perp, H^+)$. □

COROLLARY 4.5. *A Markovian representation* (H, U, X) *with* $X \sim (S, \bar{S})$ *is minimal if and only if*

$$\bar{S} = H^+ \vee S^\perp \tag{4.19a}$$

$$S = H^- \vee \bar{S}^\perp \tag{4.19b}$$

From this corollary and Conditions (i) and (iii) of Theorem 4.1 we see that any minimal X must be orthogonal to the two subspaces N^- and N^+, defined by (4.16) and (4.18) respectively. This implies that

$$E^{H_0} X \subset H^\square \tag{4.20}$$

where the *frame space* $H^\square$ is defined by the orthogonal decomposition

$$H_0 = N^- \oplus H^\square \oplus N^+ \tag{4.21}$$

By Theorem 4.1, $H^\square$ is an internal Markovian splitting subspace with representation $H^\square \sim ((N^+)^\perp, (N^-)^\perp)$. In general, $H^\square$ is nonminimal. In fact, it is easy to check (see (4.25) below), that

$$H^\square = H^{+/-} \vee H^{-/+} \tag{4.22}$$

and that $H^\square$ is the closed linear hull of all internal minimal Markovian splitting subspaces [33]. Decomposition (4.21) partitions the output space H_0 into three parts. The subspace N^- is the part of the past H^- which is orthogonal to the future H^+, and N^+ is the part of the future which is orthogonal to the past. Consequently, the inclusion (4.20) reflects the fact that the spaces N^- and N^+ play no role in the interaction between past and future and hence in minimal state space construction. The following result, which has interesting interpretations in Kalman filtering (see Section 7.4 below), provides further support to this interpretation.

LEMMA 4.6. *Let* X *be a Markovian splitting subspace. Then*

$$E^{H^-} X = H^{+/-} \tag{4.23}$$

if and only if $X \perp N^-$*, and*

$$E^{H^+}X = H^{-/+} \tag{4.24}$$

if and only if $X \perp N^+$*.*

A simple proof of this lemma follow the lines of [33; Theorem 3.5].

Conditions (4.19) of Corollary 4.5 can be interpreted as minimality conditions on S and $\bar{S}$ respectively. The systems-theoretic interpretations of these conditions is that of *observability* and *constructibility* respectively. In fact, applying the decomposition

$$A = E^A B \oplus (A \cap B^\perp) \tag{4.25}$$

(which is easily seen to hold for any pair of subspaces A and B with $B^\perp$ being the orthogonal complement of B in any space H containing both A and B) to X and H^+ or H^- we obtain

$$E^X H^+ \oplus [X \cap (H^+)^\perp] = X = E^X H^- \oplus [X \cap (H^-)^\perp] \tag{4.26}$$

In these two decompositions of $X, X \cap (H^+)^\perp$ is the *unobservable space* of X, i.e. the subspace consisting of all elements in X which are orthogonal to the future H^+, and hence unobservable in a sense which is the natural generalization of that of deterministic systems theory [19]. Symmetrically, $X \cap (H^-)^\perp$ is called the *unconstructible subspace* of X. Consequently, X is said to be *observable* if $X \cap (H^+)^\perp = 0$ and *constructible* if $X \cap (H^-)^\perp = 0$. In accordance with this terminology, $E^X H^+$ and $E^X H^-$ are *the observable* and *constructible subspaces* of X respectively [48].

THEOREM 4.7. *Let* $X \sim (S, \bar{S})$ *be a Markovian splitting subspace. Then* X *is observable if and only if (4.19a) holds and constructible if and only if (4.19b) holds.*

The proof follows precisely the lines of that of Theorem 4.1 in [33]. As a corollary to this theorem we have a theorem first presented by Ruckebusch [48] with a different proof.

THEOREM 4.8. *A Markovian splitting subspace is minimal if and only if it is both observable and constructible.*

COROLLARY 4.9. *Let* X *be a Markovian splitting subspace, and let* N^- *and* N^+ *be defined by (4.15) and (4.17). Then, if* X *is observable,* $X \perp N^-$*, and, if* X *is constructible,* $X \perp N^+$*.*

Proof. Condition (4.18b) and $\bar{S} \supset H^+$ implies that $N^+ = H^+ \cap (H^-)^\perp \subset \bar{S} \cap (H^-)^\perp = S^\perp$, which is orthogonal to X by Condition (iii) of Theorem 4.1. Hence constructibility of X implies $X \perp N^+$. The rest follows by symmetry. □

The following theorem, which has important systems-theoretical consequences to be discussed in Section 5.4, states that the minimality conditions of Theorem 4.8 can be relaxed in that either the observability or the constructibility condition can be replaced by the corresponding weaker condition of Corollary 4.9. In a sense, this theorem is a geometric version (and generalization) of Proposition 3.1. In fact, as we shall see in Section 5.4, the condition $X \perp N^+$ is equivalent to the spectral factor W being minimal.

THEOREM 4.10. *Let X be a proper Markovian splitting subspace. Then the following condition are equivalent.*

(i) X minimal

(ii) X observable and $X \perp N^+$

(iii) X constructible and $X \perp N^-$

We shall here give a proof along the lines of [33; p.823] for the case that $H^{+/-}$ and $H^{-/+}$ are finite-dimensional. The proof for the general case requires some concepts to be introduced in Section 5 and will therefore be postponed and proved as a corollary to Theorem 5.4.

Proof (finite-dimensional case). It follows trivially from Theorem 4.8 and Corollary 4.9 that (i) implies (ii) and (iii). Conversely, if (ii) or (iii) hold, X is orthogonal to both N^- and N^+ (Corollary 4.9), and therefore, since

$$H^- = H^{+/-} \oplus N^- \text{ and } H^+ = H^{-/+} \oplus N^+ \tag{4.27}$$

(decomposition (4.25)), the splitting condition $H^- \perp H^+|X$ can be replaced by the reduced condition

$$H^{+/-} \perp H^{-/+}|X \tag{4.28}$$

where N^- and N^+ have been removed from the past and the future. Now, introduce the corresponding (reduced) observability operator $\mathcal{O} : X \to H^{-/+}$ and constructibility operator $\mathcal{C} : X \to H^{+/-}$ defined by

$$\mathcal{O} := E^{H^{-/+}}{}_{|X} \quad \text{and} \quad \mathcal{C} := E^{H^{+/-}}{}_{|X} \tag{4.29}$$

respectively. Then, in view of (4.26),

$$\ker\mathcal{O} = X \cap [(H^+)^\perp \oplus N^+] = X \cap (H^+)^\perp$$

since $X \perp N^+$, and hence $\mathcal{O}$ is injective if and only if X is observable. Likewise, $\mathcal{C}$ is injective if and only if X is constructible. Moreover, in view of (4.26) and the assumptions $X \perp N^-$ and $X \perp N^+$, Lemma 4.6 implies that $Im\mathcal{O} = H^{-/+}$ and $Im\mathcal{C} = H^{+/-}$ (if they are finite dimensional), and consequently $\mathcal{O}$ and $\mathcal{C}$ are always surjective. Now, suppose (ii) holds. Then $\mathcal{O}$ is bijective, and hence invertible. Now, it is easy to see that the reduced splitting condition (4.28) is equivalent to the factorization

$$\mathcal{O}\mathcal{C}^* = \mathcal{O}_- \tag{4.30}$$

where $\mathcal{O}_-$ is the reduced observability operator of $H^{+/-}$, i.e. $\mathcal{O}_- := E^{H^{-/+}}{}_{|H^{+/-}}$ (Proposition 2.1(vi)), which of course is also bijective, since $H^{+/-}$ is minimal and hence observable. Therefore, it follows from (4.29) that $\mathcal{C}^* = \mathcal{O}^{-1}\mathcal{O}_-$ is bijective, and consequently X is also constructible. Hence (i) holds (Theorem 4.8). A symmetric argument shows that (iii) implies (i). □

The following corollaries will be needed later.

Corollary 4.11. *Let* $\dim H^{+/-} =: n < \infty$. *Then all minimal Markovian splitting subspaces have dimension* n.

Proof. Since $\mathcal{C} : X \to H^{+/-}$ is a bijection, X must have the same dimension as $H^{+/-}$.□

Corollary 4.12. *A Markovian splitting subspace* X *is observable [constructible] if and only if* $\mathcal{O}$ $[\mathcal{C}]$, *defined by (4.29), is injective with dense range (or bijective in the finite-dimensional case).*

Corollary 4.13. *Let* X *be a Markovian splitting subspace such that* $X \perp N^+$. *Then*

$$U_t(X)\mathcal{O}^* = \mathcal{O}^* U_t(H^{-/+}) \tag{4.31}$$

Proof. Let $\xi \in H^{-/+}$. Then, since $\xi \in \bar{S}$, $X^\perp = S^\perp \oplus \bar{S}^\perp$ and $U_t S^\perp \subset S^\perp$ for $t \geq 0$ (Theorem 4.1), $U_t(X)\mathcal{O}^* = E^X U_t E^X \xi = E^X U_t \xi$. But, since $X \perp N^+$, this equals $E^X E^{(N^+)^\perp} U_t \xi = \mathcal{O}^* U_t(H^{-/+})\xi$ because $U_t \xi \perp (H^+)^\perp$. □

Corollaries 4.12 and 4.13 imply that minimal X have similar $U_t(X)$ in the finite-dimensional case and, as we shall see in the next section, quasi-similar $U_t(X)$ in the general case [41].

5. Construction of Markovian representations

In this section we tie up the geometric notion of Markovian representation with analytic and coanalytic solutions of the spectral factorization problem

$$W(s)W(-s)' = \Phi(s) \tag{5.1}$$

where Φ is the $m \times m$ incremental spectral density of dy. To begin with, we shall only assume that the m-dimensional stationary increment process dy is mean-square continuous and p.n.d. and that the spectral density Φ is full rank almost everywhere on the imaginary axis I. Later, in Section 5.5, we shall consider the case when Φ is rational. (See Appendices B and C).

5.1. From Markovian representations to spectral factors

Given a proper Markovian representation (H, U, X) of multiplicity $p \geq m$ with $X \sim (S, \bar{S})$, there is a pair $(dw, d\bar{w})$ of p-dimensional Wiener processes such that $H(dw) = H(d\bar{w}) = H$ and

$$\begin{cases} S = H^-(dw) \\ \bar{S} = H^+(d\bar{w}) \end{cases} \tag{5.2}$$

(Theorem A.1 in the Appendix.) These processes are called the *generating processes* of the Markovian representation, and they are uniquely determined modulo multiplication by a constant $p \times p$ orthogonal matrix.

By (5.2), every random variable in S [in $\bar{S}$] can be represented by a stochastic integral (B.1) of a *causal* function $f \in L^2_p(\mathrm{R})$ [an *anticausal* function $\bar{f} \in L^2_p(\mathrm{R})$] with respect to dw $[d\bar{w}]$. In particular, this naturally leads to representations of dy by means of a causal and an anticausal input-output map driven by the white noise processes dw and $d\bar{w}$, respectively. The most efficient way to study these representations in the current stationary framework, is by spectral-domain techniques. To this end, recall that there are two unitary maps I_w and $I_{\bar{w}}$ from $L^2_p(\mathrm{I})$ to H establishing unitary isomorphisms between S and $\bar{S}$ and the Hardy spaces H^2_p and $\bar{H}^2_p$ respectively. (See Appendix C.) In fact, under each of these isomorphisms the shift U_t becomes multiplication by $e^{i\omega t}$, as can be seen from (B.12), and, recalling (C.1),

$$I_w H^2_p = H^-(dw) = S \quad \text{and} \quad I_{\bar{w}} \bar{H}^2_p = H^+(d\bar{w}) = \bar{S} \tag{5.3}$$

Moreover, since $I_w^{-1} I_{\bar{w}}$ is a unitary operator which commutes with the shift on $L^2_p(\mathrm{I})$, it can be represented by a multiplication operator

$$I_w^{-1} I_{\bar{w}} = M_K \tag{5.4}$$

where $M_K f = fK$ and K is a unitary $p \times p$ matrix function on I [13, 15, 52]. An isometry which sends analytic functions to analytic functions is called *inner.* A $p \times q$ matrix function V on I such that $H^2_p V$ is dense in H^2_q is called *outer* [52]. Functions with the corresponding properties with respect to the conjugate Hardy space $\bar{H}^2_p$ will be called *conjugate inner* and *conjugate outer* respectively.

In the Appendix C we introduce the modified Hardy spaces $\mathcal{W}^2_p$ and $\bar{\mathcal{W}}^2_p$ consisting of the $p-$dimensional row vector functions g and $\bar{g}$ respectively such that $\bar{\chi}_h g \in H^2_p$ and $\chi_h \bar{g} \in \bar{H}^2_p$, where $\chi_h(i\omega) = (e^{i\omega t} - 1)/i\omega$ and $\bar{\chi}_h(i\omega) = \chi_h(-i\omega)$. For reasons explained in Appendix C, a spectral factor W with rows in $\mathcal{W}^2_p$ will be called *analytic* and a spectral factor $\bar{W}$ with rows in $\bar{W}^2$ *coanalytic.*

LEMMA 5.1. *Let (H, U, X) be a proper Markovian representation with generating processes $dw, d\bar{w}$. Then there is a unique pair $(W, \bar{W})$ of spectral factors, the first being analytic and the second coanalytic, such that*

$$d\hat{y} = W d\hat{w} = \bar{W} d\hat{\bar{w}} \tag{5.5}$$

Moreover the matrix function K defined by (5.4) is inner, and satisfies

$$W = \bar{W} K \tag{5.6}$$

In particular,

$$d\hat{\bar{w}} = K d\hat{w} \tag{5.7}$$

Proof. For a fixed $h > 0$, define two $m \times p$ matrix-valued functions W and $\bar{W}$ on I with rows

$$W_k = \bar{\chi}_h^{-1} I_w^{-1} [y_k(-h) - y_k(0)] \tag{5.8a}$$

$$\bar{W}_k = \chi_h^{-1} I_{\bar{w}}^{-1} [y_k(h) - y_k(0)] \tag{5.8b}$$

for $k = 1, 2, \dots, m$. Then

$$y(-h) - y(0) = \int_{-\infty}^{\infty} \frac{e^{-i\omega h} - 1}{i\omega} W(i\omega) d\hat{w}(i\omega) \tag{5.9a}$$

$$y(h) - y(0) = \int_{-\infty}^{\infty} \frac{e^{i\omega h} - 1}{i\omega} \bar{W}(i\omega) d\hat{\bar{w}}(i\omega) \tag{5.9b}$$

from which we see that W and $\bar{W}$ are spectral factors, independent of the choice of h in the definitions (5.8), and that (5.5) holds. (See Appendix C). In fact from the spectral representation (B.19) and from (B.21) we obtain

$$E\{[y(h_1) - y(0)][y(h_2) - y(0)]^{'}\} = \int_{-\infty}^{+\infty} \chi_{h_1}(i\omega)\Phi(i\omega)\bar{\chi}_{h_2}(i\omega)\frac{d\omega}{2\pi}$$

and by making the same computation starting from, say, (5.9b) we obtain instead

$$\int_{-\infty}^{+\infty} \chi_{h_1}(i\omega)\bar{W}(i\omega)\bar{W}(i\omega)^{*}\bar{\chi}_{h_2}(i\omega)d\omega/2\pi$$

Since the functions $\{\chi_h;\ h \in \mathrm{R}\}$ are dense in $L^1(\mathrm{I})$, by comparing the two expressions we indeed get $\Phi(i\omega) = \bar{W}(i\omega)\bar{W}(i\omega)^{*}$ a.e. Conversely, if W and $\bar{W}$ satisfy (5.5) and hence (5.9), they satisfy (5.8), proving uniqueness. Since the components of $y(-h) - y(0)$ belong to H^-, for $h > 0$, and hence to S, it follows from (5.9a) and (5.3) that the rows of $\bar{\chi}_h W$ belong to H^2_p. In the same way, we see that $H^+ \subset \bar{S}$ implies that the rows of $\chi_h \bar{W}$ belong to $\bar{H}^2_p$. That K is inner follows from perpendicular intersection. In fact, in view of (5.2), $\bar{S}^{\perp} \subset S$ may be written $H^-(d\bar{w}) \subset H^-(dw)$, for $H^-(d\bar{w}) \oplus H^+(d\bar{w}) = H$. Therefore, it follows from (5.3) and (5.4) that $H^2_p K \subset H^2_p$, showing that K is inner. Moreover, for any $f \in L^2_p(\mathrm{I})$, $I_w^{-1} I_{\bar{w}} f = fK$, i.e.

$$\int f d\hat{\bar{w}} = \int f K d\hat{w}$$

proving (5.7). Then (5.6) follows from (5.5) and (5.7). □

It follows from the analysis above that the spectral factors W and $\bar{W}$ are uniquely determined by the subspaces S and $\bar{S}$, once a specific choice of generating process $dw, d\bar{w}$ has been made. According to Theorem A.1, this amounts to say that W and $\bar{W}$ are determined by S and $\bar{S}$ modulo right multiplication by a constant $p \times p$ orthogonal matrix. The equivalence class of $m \times p$ spectral factors

$$[W] = \{WT;\ T \text{ orthogonal}\, p \times p \text{ matrix}\} \tag{5.10}$$

will sometimes be denoted by the symbol $W mod O(p)$, where $O(p)$ is the p-dimensional orthogonal group, or merely $W mod O$ if the dimension of W need not be mentioned.

Hence, given a proper Markovian representation (H, U, X) with $X \sim (S, \bar{S})$, we determine a unique ($mod O$) pair $(W, \bar{W})$ of $m \times p$ spectral factors, one being analytic and corresponding to S, and the other coanalytic and corresponding to $\bar{S}$. In terms of the splitting geometry the analyticity of W reflects the condition $S \supset H^-$, the coanalyticity of $\bar{W}$ the condition $\bar{S} \supset H^+$, and K being inner the perpendicular intersection between S and $\bar{S}$. We shall call a triplet $(W, \bar{W}, K)$ where W and $\bar{W}$ are $m \times p$ spectral factors for some $p \geq m$ and K is a $p \times p$ matrix function satisfying the equation $W = \bar{W}K$ a *Markovian triplet* if W is analytic, $\bar{W}$ coanalytic and K inner.

In view of (5.4), K is uniquely determined by the Markovian representation (H, U, X) modulo right and left multiplication by orthogonal constant matrices, and we shall call it the *structural function* of (H, U, X). It follows that the Markovian triplets corresponding to a Markovian representation are all related by the equivalence

$$(W, \bar{W}, K) \sim (WT_1, \bar{W}T_2, T_2^{-1}KT_1) \; ; \qquad T_1, T_2 \in O(p) \tag{5.11}$$

We shall denote the corresponding equivalence class of Markovian triplets $(W, \bar{W}, K)$ $modO$ or $[W, \bar{W}, K]$.

Note that if the proper Markovian representation (H, U, X) is internal, then its multiplicity p equals m so that W and $\bar{W}$ are square and hence, since Φ is full rank, invertible. There are two square spectral factors of particular importance, namely the outer spectral factor W_- and the conjugate outer spectral factor $\bar{W}_+$. As explained in Appendix C, the outer property implies that the corresponding Wiener process, in view of (5.5) uniquely defined as $d\hat{u}_- := W_-^{-1}d\hat{y}$ and $d\hat{\bar{u}} := \bar{W}_+^{-1}d\hat{y}$, satisfy $H^-(du_-) = H^-$ and $H^+(d\bar{u}_+) = H^+$, and consequently, du_- is the *(forward) innovation process* of dy and $d\bar{u}_+$ the *backward* one.

We shall see in Section 5.4 that there are proper Markovian representations only if the frame space $H^\square \sim ((N^+)^\perp, (N^-)^\perp)$, defined in Section 4.2, is proper, which is equivalent to $(N^+)^\perp$ and $(N^-)^\perp$ being p.n.d. (Theorem 4.1). In this case, there are two m-dimensional Wiener processes $d\bar{u}_-$ and du_+ such that $H^-(du_+) = (N^+)^\perp$ and $H^+(d\bar{u}_-) = (N^-)^\perp$ (Theorem A.1) and a corresponding analytic spectral factor W_+ such that $d\hat{y} = W_+ d\hat{u}_+$ and a coanalytic one $\bar{W}_-$ such that $d\hat{y} = \bar{W}_- d\hat{\bar{u}}_-$ (Lemma 5.1). Then the two predictor spaces $H^{+/-}$ and $H^{-/+}$, defined in Section 4.2, have Markovian triplets $(W_-, \bar{W}_-, K_-)$ and $(W_+, \bar{W}_+, K_+)$ respectively, where $K_- := \bar{W}_-^{-1}W_-$ and $K_+ := \bar{W}_+^{-1}W_+$. The condition that $(N^-)^\perp$ and $(N^+)^\perp$ are p.n.d. is equivalent to the *strict noncyclicity* introduced in Section 5.4.

5.2. From spectral factors to Markovian representations

Conversely, we shall now proceed to show that all Markovian representations can be constructed starting from Markovian pairs of spectral factors. To this end, we first have to give a procedure for constructing the generating processes of $X \sim (S, \bar{S})$ starting from $(W, \bar{W}, K)$. In the internal case this is a simple matter since W and $\bar{W}$ can be inverted in (5.5) yielding unique dw and $d\bar{w}$. In general, the systems (5.5) are underdetermined, introducing nonuniquiness in the corresponding generating processes.

LEMMA 5.2. *All p-dimensional Wiener processes dw satisfying*

$$d\hat{y} = W d\hat{w} \tag{5.12}$$

are given by

$$d\hat{w} = W^\sharp d\hat{y} + d\hat{z} \tag{5.13}$$

where $W^\sharp$ is the right inverse

$$W^\sharp = W^*\Phi^{-1} \tag{5.14}$$

of W (asterisk denoting conjugation and transposition) and dz is any p-dimensional stationary increment process with incremental spectral density

$$\Pi := I - W^\sharp W \tag{5.15}$$

and such that $H(dz) \perp H_0$*. The processes* dz *and* dw *are related by* $d\hat{z} = \Pi d\hat{w}$*. Moreover,* $\Pi(i\omega)$ *is a* $p \times p$ *orthogonal projection matrix for almost all* $\omega \in \mathsf{R}$.

Proof. First note that since $\Pi(i\omega)^2 = \Pi(i\omega)$ and $\Pi(i\omega)^* = \Pi(i\omega)$, $\Pi(i\omega)$ is an orthogonal projection matrix. Let $d\hat{w}$ be a solution to (5.12). Then $W^\sharp d\hat{y} = (I - \Pi)d\hat{w}$ and we obtain formula (5.13) where

$$d\hat{z} = \Pi d\hat{w} \tag{5.16}$$

Now $E\{d\hat{z}d\hat{z}^*\} = \Pi^2 d\omega = \Pi d\omega$, and hence Π is the incremental spectral density of dz. Moreover, $E\{d\hat{y}d\hat{z}^*\} = W\Pi d\omega = 0$ implying the orthogonality $H(dz) \perp H_0$. Conversely, given a process dz with a spectral density (5.15) and with $H(dz) \perp H_0$, define dw by formula (5.13). Then dw is a Wiener process and $Wd\hat{w} = d\hat{y}$. ☐

Consequently, given a Markovian triplet $(W, \bar{W}, K)$, by Lemma 5.2 we can construct pairs of generating processes

$$d\hat{w} = W^\sharp d\hat{y} + d\hat{z} \tag{5.17a}$$

$$d\hat{\bar{w}} = \bar{W}^\sharp d\hat{y} + d\hat{\bar{z}} \tag{5.17b}$$

where the spectrum of dz is given by (5.15) and that of $d\bar{z}$ is

$$\bar{\Pi} := I - \bar{W}^\sharp \bar{W} \tag{5.18}$$

We now build the space H corresponding to the Markovian representation so that $H = H(dw) = H(d\bar{w})$. Of course, in order to do this, we must choose dz and $d\bar{z}$ in such a way that

$$H(d\bar{z}) = H(dz) \tag{5.19}$$

Note that in this case the multiplication operators M_Π and $M_{\bar{\Pi}}$ both represent the projection $E^{H_0^\perp}$ from H onto the doubly invariant subspace $H_0^\perp = H(d\bar{z}) = H(dz)$. More specifically, $I_w M_\Pi I_w^{-1} = I_{\bar{w}} M_{\bar{\Pi}} I_{\bar{w}}^{-1}$, i.e. $M_\Pi I_w^{-1} I_{\bar{w}} = I_w^{-1} I_{\bar{w}} M_{\bar{\Pi}}$. Therefore, in view of (5.4),

$$K\Pi = \bar{\Pi}K \tag{5.20}$$

from which we see that $\bar{\Pi}d\hat{\bar{w}} = K\Pi d\hat{w}$, i.e.

$$d\hat{\bar{z}} = Kd\hat{z}. \tag{5.21}$$

The following theorem describes the relation between Markovian representations and Markovian triplets $(W, \bar{W}, K)$.

Theorem 5.3. *There is a one-one correspondence between proper Markovian representations* (H, U, X) *and pairs* $([W, \bar{W}, K], dz)$ *where* $[W, \bar{W}, K]$ *is an equivalence class of Markovian triplets and* dz *is a vector stationary-increment process (defined modO) with spectral density* $\Pi := I - W^\sharp W$ *such that* $H(dz) \perp H_0$*. Under this correspondence*

$$H = H_0 \oplus H(dz) \tag{5.22}$$

and

$$X = H^-(dw) \cap H^+(d\bar{w}) \tag{5.23}$$

where $(dw, d\bar{w})$ *are the generating processes given by (5.17).*

Proof. Given a Markovian representation (H, U, X), we have shown above that there is a unique equivalence class $[W, \bar{W}, K]$ of Markovian triplets and a corresponding pair of generating processes $(dw, d\bar{w})$, defined *modO* and consequently a unique $d\hat{z} = \Pi d\hat{w}$ having the required properties. Conversely, given a triplet $(W, \bar{W}, K)$ and a process dz with the stated properties, we define $(d\hat{w}, d\hat{\bar{w}})$ by (5.17) and set $S := H^-(dw)$ and $\bar{S} := H^+(d\bar{w})$. Then since $(W, \bar{W}, K)$ is a Markovian triplet, W is analytic implying that $S \supset H^-$, $\bar{W}$ is coanalytic implying that $S \supset H^+$, and K is inner which is equivalent to perpendicular intersection. Hence, by Theorem 4.1, $X = S \cap \bar{S}$ is a Markovian splitting subspace with ambient space $H = H_0 \oplus H(dz)$, for the invariance condition (ii) is trivially satisfied. The shift is induced by dy and dz. □

At this point we have designed a spectral-domain framework, isomorphic to the geometric framework of Markovian representations, in which all random variables have concrete representations as functions in certain subspaces of H^2_p or $\bar{H}^2_p$. We shall next introduce a general functional model for Markovian splitting subspaces which is of the type studied in [21] and [13] in connection with deterministic scattering theory and linear systems in Hilbert space. Using this representation the characterization of various structural conditions of Markovian splitting subspaces (observability, constructibility and minimality) takes a very elegant form which is actually independent of any finite dimensionality assumption. These questions will be studied in Section 5.4.

Theorem 5.4. *Let* X *be a proper Markovian splitting subspace with structural function* K *and generating processes* $(dw, d\bar{w})$*. Then,*

$$X = \int H(K) d\hat{w} = \int \bar{H}(K^*) d\hat{\bar{w}} \tag{5.24}$$

where $H(K) := H^2_p \ominus H^2_p K$ *and* $\bar{H}(K^*) := \bar{H}^2_p \ominus \bar{H}^2_p K^*$*. Moreover,* X *is finite dimensional if and only if* K *is rational, in which case* $\dim X$ *equals the McMillans degree of* K*.*

Proof. By Theorem 4.1 and (5.4),

$$X = S \ominus \bar{S}^\perp = H^-(dw) \ominus H^-(d\bar{w}) \tag{5.25}$$

Therefore the first of equations (5.24) follows from (5.3) and (5.4). A symmetric argument yields the second equation. For a proof of the last statement, see [33]. □

5.3. The structure of Markovian triplets

The Markovian triplets $(W, \bar{W}, K)$ contain all the system-theoretic information needed for the construction of explicit stochastic-differential equation representations for dy. In particular, the structural function K determines the state space and hence the state equations, while W and $\bar{W}$ serve as the transfer functions of two stochastic realizations having the same state space determined by K, namely a causal one driven by the forward generating process dw and an anticausal driven by the backward generating process $d\bar{w}$. We shall now investigate the relation between $W, \bar{W}$ and K.

A Markovian triplet is called *tight* if K is uniquely determined by W and $\bar{W}$. This is always the case for internal Markovian representations, for then $K = \bar{W}^{-1}W$. In the noninternal case nontightness is due to modes of the state process which evolve independently of dy, and there are geometric conditions to exclude this modeling anomality, which, however, will not be discussed in this paper. In fact, tightness will be implied by either observability or constructibility, which conditions, as we shall see below, are equivalent to the coprimeness of the factorizations $W = \bar{W}K$ and $\bar{W} = WK^*$ respectively. Such coprime factorizations are known to be unique $(modO)$ [13]. Consequently, $\bar{W}$ and K are uniquely determined by W in the observable case, and W and K are uniquely determined by $\bar{W}$ in the constructible case.

Constructing K in the noninternal case starting from W and $\bar{W}$ can be regarded as a dilation problem. Given an arbitrary $m \times p$ analytic spectral factor W, define $Q := W_-^{-1}W$. Then $Q^*Q = W^\sharp W$ is the multiplicative operator corresponding to the orthogonal projection E^{H_0}, and

$$d\hat{u}_- = Qd\hat{w}. \tag{5.26}$$

In the internal case $H = H_0, W$ is square and Q is inner and $m \times m$ so that $d\hat{w}$ can be determined directly by inverting (5.26) to yield $d\hat{w} = Q^*d\hat{u}_-$. However, in the noninternal case when $H \neq H_0$, Q is an analytic $m \times p$ partial isometry with $Q^*Q = W^\sharp W$ and solving (5.26) for dw is then equivalent to finding an analytic $(p-m)\times p$ matrix function P such that $\begin{bmatrix} Q \\ P \end{bmatrix}$ is a $p \times p$ inner dilation of Q. For, by unitarity, we get

$$\begin{bmatrix} QQ^* & QP^* \\ PQ^* & PP^* \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad Q^*Q + P^*P = I \tag{5.27}$$

from which we see that P should be chosen a full rank analytic solution of the spectral factorization problem

$$P^*P = \Pi \tag{5.28}$$

Similarly, we construct a $p \times p$ inner dilation of $\bar{Q} := \bar{W}_+^{-1}\bar{W}$ where $\bar{W}_+$ is the conjugate outer spectral factor. This is achieved precisely by choosing a full-rank coanalytic solution $\bar{P}$ of the spectral factorization problem

$$\bar{P}^*\bar{P} = \bar{\Pi} \tag{5.29}$$

Then defining the $(p-m)$-dimensional Wiener processes

$$d\hat{\eta} = Pd\hat{w} \quad \text{and} \quad d\hat{\bar{\eta}} = \bar{P}d\hat{\bar{w}} \tag{5.30}$$

we obtain the representations

$$d\hat{w} = Q^*d\hat{u}_- + P^*d\hat{\eta} \tag{5.31a}$$

$$d\hat{\bar{w}} = \bar{Q}^*d\hat{\bar{u}}_+ + \bar{P}^*d\hat{\bar{\eta}} \tag{5.31b}$$

which are equivalent to (5.17). From this it follows that in order for (5.19) to hold we must have $H(d\eta) = H(d\bar{\eta})$. In other words, there must be a $(p-m) \times (p-m)$ unitary matrix functions Θ such that

$$d\hat{\bar{\eta}} = \Theta d\hat{\eta} \tag{5.32}$$

Therefore, we can base our construction of the generating processes $(dw, d\bar{w})$ on the choice of a single $(p-m)$-dimensional Wiener process $d\eta$, once P and $\bar{P}$ have been fixed.

Now comparing (5.7), (5.13) and (5.32) we obtain the following formula for the structural function:

$$K = \bar{Q}^* T_0 Q + \bar{P}^* \Theta P \tag{5.33}$$

where T_0 is the unitary function

$$T_0 = \bar{W}_+^{-1} W_- \tag{5.34}$$

representing the interface between the past and future, which is uniquely defined by the process dy. Formula (5.33) represents K as the sum of an internal part, $\hat{K} := \bar{Q}^* T_0 Q$, and the external part, $\tilde{K} := \bar{P}^* \Theta P$. While $\hat{K}$ is always uniquely determined by W and $\bar{W}$, $\tilde{K}$ is unique only when $(W, \bar{W}, K)$ is tight.

5.4. Spectral conditions for minimality

Next we turn to characterizations of minimality and to the family of minimal Markovian representations. As we wish to remain for a while in the general, possibly infinite-dimensional setting, we need a condition to insure that the process dy admits proper X. To this end, we quote the following result from [33].

Proposition 5.5. *Set $T_0 := \bar{W}_+^{-1} W_-$, and let N^- and N^+ be given by (4.16) and (4.18). Then the following statements are equivalent.*

(i) All minimal Markovian splitting subspaces are proper

(ii) Both N^- and N^+ are full range

(iii) There are square inner functions J_1, J_2, J_3 and J_4 such that

$$T_0 = J_1 J_2^* = J_3^* J_4$$

If Condition (iii) holds, we say that T_0 is *strictly noncyclic* [13; p.254]. In particular, this condition always holds when Φ is rational [33]. From (5.6) or (5.33) it is immediately seen that T_0 has a factorization

$$T_0 = \bar{Q} K Q^* \tag{5.35}$$

a different one for each proper Markovian splitting subspace X. We shall refer to $(K, Q, \bar{Q})$ as the *inner triplet* of X.

Theorem 5.5. *Let X be a proper Markovian splitting subspace with inner triplet $(K, Q, \bar{Q})$. Then X is constructible if and only if K and Q are right coprime, i.e. they have no nontrivial common right inner factor, and X is observable if and only if K^* and $\bar{Q}$ are right coprime, i.e. they have no nontrivial common right conjugate inner factor.*

Proof. By Theorem 4.7, X is constructible if and only if $S = H^- \vee \bar{S}^\perp$, i.e. $H^-(dw) = H^-(du_-) \vee H^-(d\bar{w})$, which under the isomorphism I_w takes the form

$$H_p^2 = (H_m^2 Q) \vee (H_p^2 K) \tag{5.36}$$

For (5.36) to hold, Q and K must clearly be right coprime. Conversely, suppose that Q and K are right coprime, and consider the right member of (5.36). Clearly it is a full-range invariant subspace of H^2_p, because H^2_pK is, and therefore, by the Beurling-Lax Theorem [14], it has the form H^2_pJ where J is inner. But then J must be a common right inner factor of Q and K, and hence $J = I$, concluding the proof of the constructibility criterion. The proof of the observability part is by symmetry. □

Theorem 5.5, which was first presented in the internal setting in [29], allows us to interpret minimality in terms of the factorization (5.36) of T_0. In fact, by Theorem 4.8, X is minimal if and only if this factorization is reduced as far as possible in the sense that no further cancellations are possible. The reduction procedure of Theorem 4.2 could be interpreted in terms of such cancellations.

COROLLARY 5.6. *Let X be an observable proper Markovian splitting subspace with analytic spectral factor W. Then its Markovian triplet $(W, \bar{W}, K)$ is tight and $\bar{W}$ and K are the unique (modO) coprime factors of*

$$W = \bar{W}K \tag{5.37}$$

such that $\bar{W}$ is $m \times p$ coanalytic and K is $p \times p$ inner. Similarly, if X is constructible with coanalytic spectral factor $\bar{W}$, its Markovian triplet $(W, \bar{W}, K)$ is tight, and W and K^* are the unique (modO) coprime factors of

$$\bar{W} = WK^* \tag{5.38}$$

The proof follows from the uniqueness of the coprime factorizations (5.37) and (5.38); see [13; p.254].

The structural functions of two minimal proper Markovian splitting subspaces may be quite different (in the multivariate case). In fact, they may not even take values in the same space, being matrices of different sizes. If they are finite dimensional, they have the same degree (Theorem 5.4 and Corollary 4.11). In the general case, there are still some important invariants, namely the nontrivial invariant factors. Recall that the invariant factors of a $p \times p$ inner function K are p scalar inner functions $k_1, k_2, \dots k_p$ defined in the following way. Set $\gamma_0 = 1$, and, for $i = 1, 2, \dots, p$ define γ_i to be the greatest common inner divisor of all $i \times i$ minors of K. Then set $k_i := \gamma_i/\gamma_{i-1}$ for $i = 1, 2, \dots, p$. Clearly, these functions are inner, for γ_{i-1} divides γ_i. The following theorem is a generalization of [31]; also see [51, 11] for related results.

THEOREM 5.7. *Let T_0 be strictly noncyclic. Then all internal minimal Markovian splitting subspaces have the same invariant factors; let us denote them*

$$k_1, k_2, k_3, \dots, k_m \tag{5.39}$$

Moreover, a Markovian splitting subspace of multiplicity p is minimal if and only if m invariant factors are given by (5.39) and the remaining $p - m$ are identically one.

Proof. Let X be an arbitrary minimal Markovian splitting subspace with structural function K and multiplicity p. Let K_+ denote the structural function of $H^{-/+}$, which of course has multiplicity m, being internal. Corollaries 4.12 and 4.13 (together with

Corollaries 4.8 and 4.9) imply that $U_t(H^{-/+})$ is a *quasi-affine transformation* of $U_t(X)$, i.e. that there is an injective operator T such that $TU_t(H^{-/+}) = U_t(X)T$. Now, $U_t(X)I_w = I_w S_t(K)$, where $S_t(K)$ is the shift $e^{i\omega t}$ in H^2_p compressed to $H(K)$, and therefore $U_t(X)$ is similar to $S_t(K)$. Similarly, $U_t(H^{-/+})$ is similar to $S_t(K_+)$, but it is a simple calculation to see that it is also similar to

$$\hat{K}_+ = \begin{bmatrix} K_+ & 0 \\ 0 & I_{p-m} \end{bmatrix} \tag{5.40}$$

where I_k is the $k \times k$ identity. The inner functions $\hat{K}_+$ and K are the same size, $p \times p$, and $S_t(\hat{K}_+)$ is a quasi-affine transformation of $S_t(K)$. Therefore, we can apply Theorem 4 in [41] to see that $\hat{K}_+$ and K are quasi-equivalent, which is equivalent to having the same invariant factors [13]. Conversely, we want to show that any $X \sim (S, \bar{S})$ whose structural function is quasi-equivalent to $\hat{K}_+$ is minimal. To this end, apply the two-step reduction algorithm of Theorem 4.2 to X. First consider the Markovian splitting subspace $X_0 \sim (S, \bar{S}_1)$ obtained after the first step. Then $X_0 \subset X$, and hence, since they have the same S-space, $H(K_0) \subset H(K)$, where K_0 is the structural function of X_0 (Theorem 5.4). Therefore $H^2_p K \subset H^2_p K_0$ so there must be an inner function J such that $K = JK_0$ (see, e.g., [13, 52]). Next, consider $X_1 \sim (S_1, \bar{S}_1)$ with structural function K_1, obtained in the second step. Then X_1 is minimal and $X_1 \subset X_0$, and therefore $\bar{H}(K_1^*) \subset \bar{H}(K_0^*)$, for X_0 and X_1 have the same $\bar{S}$-space. Consequently, $\bar{H}^2_p K_0^* \subset \bar{H}^2_p K_1^*$, and hence there is a conjugate inner function $\bar{J}$ such that $K_0^* = \bar{J} K_1^*$, i.e. $K_0 = K_1 \bar{J}^*$. Combining the two factorizations we obtain

$$K = JK_1\bar{J}^* \tag{5.41}$$

where both J and $\bar{J}^*$ are inner. In particular,

$$detK = detJ \cdot detK_1 \cdot det\bar{J}^*$$

i.e. a product of scalar inner functions. However, X_1 is minimal and hence, by the first part of the proof, K_1 has the same invariant factors as $\hat{K}_+$, and, by assumption, as K. Therefore, $detK = detK_1$, and consequently, $detJ = det\bar{J}^* = 1$, which implies that $J = \bar{J}^* = I$. This implies that $X_1 = X_0 = X$, proving that X is minimal. □

COROLLARY 5.8. *Let X_1 and X_2 be two minimal Markovian splitting subspaces. Then $U_t(X_1)$ and $U_t(X_2)$ are quasi-similar [41], or, in the finite-dimensional case, similar.*

As another corollary to Theorem 5.6 we have that Theorem 4.10 holds without the finite-dimensionality assumption.

Proof of Theorem 4.10 (general case). It remains to show that (ii) or (iii) implies (i). Suppose that (ii) holds. Then $\mathcal{O}^*$ is injective with dense range (Corollary 4.12), and therefore $U_t(H^{-/+})$ is a quasi-affine transformation of $U_t(X)$ (Corollary 4.13). Then it follows from the proof of Theorem 5.7 that K and $\hat{K}_+$ have the same invariant factors and hence that X is minimal. A symmetric argument shows that (iii) implies (i) also.□

In view of Theorem 4.10, we shall say that an analytic spectral factor W is *minimal* if we have $S \perp N^+$ for a corresponding S-space. This is a consistent definition, for if S_1

and S_2 both correspond to W they differ only by the choice of dz, which is orthogonal to H_0 and hence to N^+. Moreover, it can be seen that minimality thus defined reduces to minimality of degree, as in Section 3, whenever W is rational [33]. Likewise, we say that a coanalytic spectral factor $\bar{W}$ is *minimal* if any corresponding $\bar{S}$-space is orthogonal to N^-. We can now state the following corollary of Theorem 5.3, which of course has a symmetric "backward" counterpart. This type of minimality was also discussed in [49].

COROLLARY 5.9. *Let T_0 be strictly noncyclic. Then there is a one-one correspondence (mod O) between minimal Markovian representations (H, U, X) and pairs (W, dz) where W is a minimal spectral factor and dz is a stationary increment process with the properties prescribed in Theorem 5.3.*

Proof. By Theorem 4.10, X is minimal if and only if X is observable and $S \perp N^+$, i.e. W is minimal. From the observability condition $\bar{S} = H^+ \vee S^\perp$ (Theorem 4.7) we see that $\bar{W}$ is determined once W has been chosen (Lemma 5.1). $\square$

5.5. Forward and backward realizations

Given a Markovian representation (H, U, X) determined by its Markovian triplet $(W, \bar{W}, K)$ and its generating processes $(dw, d\bar{w})$, in this section we shall derive two stochastic realizations having the same state space $X \sim (S, \bar{S})$, namely a forward realization Σ corresponding to S with transfer function W and generating noise dw, and a backward one $\bar{\Sigma}$ corresponding to $\bar{S}$ with transfer function $\bar{W}$ and generating noise $d\bar{w}$. There are several reasons why it is natural and useful to study such pairs $(\Sigma, \bar{\Sigma})$ of stochastic realizations. There is an intrinsic symmetry between past and future in the geometric theory which naturally carries over to the state-space representation Σ and $\bar{\Sigma}$. Recall, for example, that minimality is characterized by the two conditions of observability and constructability which are symmetric with respect to direction of time. As we shall see, observability is a property of Σ and constructibility a property of $\bar{\Sigma}$. In applications to noncausal estimation it is natural to consider, not only backward models, but also nonminimal representations which are best understood in terms of pairs $(\Sigma, \bar{\Sigma})$.

To avoid entering into technical questions, we shall consider only finite-dimensional Markovian representations, referring the reader to [32, 33] for a procedure to tackle the general case. The Markovian triplets will therefore consist of rational functions.

Consequently, let the structural function K be a rational $p \times p$ inner function of degree n, and let

$$K(s) = I - \bar{B}'(sI - A)^{-1}B \tag{5.42}$$

be a minimal realization, i.e. (A, B) and $(A', \bar{B})$ are reachable. Since $(W, \bar{W}, K)$ is defined modulo orthogonal transformations, we can always choose a version of K such that $K(\infty) = I$. Since K is analytic, the eigenvalues of A lie in the open left complex halfplane.

THEOREM 5.10. *Let (H, U, X) be an n-dimensional Markovian representation with generating processes $(dw, d\bar{w})$ and structural function K given by (5.42), and consider the vector Markov processes x and $\bar{x}$ defined by*

$$x(t) = \int_{-\infty}^{t} e^{A(t-\tau)} B dw(\tau) \tag{5.43a}$$

$$\bar{x}(t) = -\int_t^{\infty} e^{A'(\tau - t)}\bar{B}d\bar{w}(\tau) \tag{5.43b}$$

Then $x(0)$ and $\bar{x}(0)$ are two bases in X. The processes x and $\bar{x}$ are related by the linear transformation

$$\bar{x}(t) = P^{-1}x(t) \tag{5.44}$$

where $P := E\{x(t)x(t)'\}$ is the unique solution of the Lyapunov equation

$$AP + PA' + BB' = 0 \tag{5.45}$$

Moreover

$$\bar{B} = P^{-1}B \tag{5.46}$$

and

$$d\bar{w} = dw - \bar{B}'xdt \tag{5.47}$$

We need the following lemma, the proof of which can be found in Section 8 of [33].

LEMMA 5.11. *Let K be a rational inner function with minimal realization (5.42), and let $H(K)$ and $\bar{H}(K)$ be the subspaces defined in Theorem 5.4. Then, the rows of $(i\omega - A)^{-1}B$ form a basis in $H(K)$ and the rows of $(i\omega + A')^{-1}B$ form a basis in $\bar{H}(K^*)$.*

Proof of Theorem 5.10. Since A is a stability matrix, the integrals (5.43) are well-defined. In view of Proposition 5.4, Lemma 5.11 implies that $x(0)$, as defined by (B.18), is a basis in X. But, in view of (B.9), (B.18) defines the same process as (5.43a). The proof that $\bar{x}(0)$ is a basis is analogous. Hence $P > 0$, and $\bar{P} := E\{\bar{x}(0)\bar{x}(0)'\} > 0$. It follows from (5.43) that P and $\bar{P}$ are the unique positive definite solutions of the Lyapunov equations (5.45) respectively

$$A'\bar{P} + \bar{P}A + \bar{B}\bar{B}' = 0 \tag{5.48}$$

because (A, B) and (A', B) are reachable. Next, proceeding along the lines of [12; Lemma 5.1] we note that

$$K(s)^{-1} = I + \bar{B}'(sI - A - B\bar{B}')^{-1}B$$

and that

$$K(-s)' = I + B'(sI + A')^{-1}\bar{B}$$

But K is inner so we must have $K(s)^{-1} = K(-s)'$, and consequently there is a regular $n \times n$ matrix T such that $(A + B\bar{B}',\ B,\ \bar{B}') = (-TA'T^{-1},\ T\bar{B},\ B'T^{-1})$. In particular, this implies that T satisfies the Lyapunov equation (5.45), and hence we must have $T = P$. Also $B = T\bar{B}$ so (5.46) holds. Next, multiplying (5.45) from left and right by P^{-1} and comparing with (5.48) we see that $\bar{P} = P^{-1}$. Hence, we must have $\bar{x}(0) = P^{-1}x(0)$ from which (5.44) follows. Finally, (5.47) is obtained from (5.7), (B.13) and (B.18). □

THEOREM 5.12. *Let dy be a stationary-increments process with rational spectral density and let (H, U, X) in Theorem 5.10 be one of its finite-dimensional Markovian representations. Then, for a fixed choice of bases in X as described in Theorem 5.10, there are unique matrices $C, \bar{C}$ and D such that*

$$dy = Cxdt + Ddw \tag{5.49a}$$

$$dy = \bar{C}\bar{x}dt + Dd\bar{w} \tag{5.49b}$$

Moreover, $D = W(\infty) = \bar{W}(\infty)$ *and* C *and* $\bar{C}$ *are* $m \times n$ *matrices such that*

$$\bar{C} = CP + DB^{'} \tag{5.50}$$

and, the spectral factors W *and* $\bar{W}$ *have realizations*

$$W(s) = C(sI - A)^{-1}B + D \tag{5.51a}$$
$$\bar{W}(s) = \bar{C}(sI + A^{'})^{-1}\bar{B} + \bar{D} \tag{5.51b}$$

Proof. Since $X \sim (S, \bar{S})$ is finite dimensional, dy is conditionally Lipschitz with respect to S, i.e. the conditional derivative

$$z(t) = \lim_{h \downarrow 0} \frac{1}{h} E^{U_t S}[y(t+h) - y(t)]$$

exists [34]. Moreover, the components of $z(0)$ belong to $E^S H^+$ which, by Theorem 4.1 and Proposition 2.1(iv), is contained in X. Consequently, since $x(0)$ is a basis in X, there is a unique $m \times n$ matrix C such that $z(0) = Cx(0)$, that is $z(t) = Cx(t)$ for all $t \in \mathsf{R}$. Moreover, in view of the fact that $S = H^-(dw)$, there is a D such that dy has the semimartingale representation

$$dy = zdt + Ddw \tag{5.52}$$

[34], which is the same as (5.49a). But x is given by (5.43a) and therefore S must correspond to the spectral factor (5.51a). In particular, $W(\infty) = D$. Next, inserting $dw = d\bar{w} + B^{'}\bar{x}dt$, obtained from (5.47), (5.44) and (5.46), into (5.49a), we obtain

$$dy = (CP + DB^{'})\bar{x}dt + Dd\bar{w}$$

where (5.44) has been used. This is the corresponding backward semimartingale representation with respect to $\bar{S} = H^+(d\bar{w})$, and hence (5.49b) and (5.50) as well as (5.51b) have been established. □

Combining the representations of Theorems 5.10 and 5.12, we have now constructed a forward stochastic realization

$$(\Sigma) \quad \begin{cases} dx = Axdt + Bdw \\ dy = Cxdt + Ddw \end{cases} \tag{5.53}$$

corresponding to the analytic spectral factor W and the forward generating process dw and a companion backward realization

$$(\bar{\Sigma}) \quad \begin{cases} d\bar{x} = -A^{'}\bar{x}dt + \bar{B}d\bar{w} \\ dy = \bar{C}\bar{x}dt + Dd\bar{w} \end{cases} \tag{5.54}$$

corresponding to the coanalytic spectral factor $\bar{W}$ and the backward generating process $d\bar{w}$. At this point it should be emphasized that the forward and backward character respectively of the (5.53) and (5.54) is a consequence of the splitting property

$$H = H^-(d\bar{w}) \oplus X \oplus H^+(dw) \tag{5.55}$$

In fact, the future input noise in (5.53) is orthogonal to present state X and past output $H^- \subset H^-(dw)$ making the system *forward*, and the past input noise of (5.54) is orthogonal to present state and future output H^+ making (5.54) a *backward* system.

Instead of starting from a state space realization (5.42) of K, we might have K given in a matrix fraction description

$$K(s) = \bar{M}(s)M(s)^{-1} \tag{5.56}$$

where $M, \bar{M}$ are $p \times p$-matrix polynomials with $detM(s)$ having all its zeros in the open left complex half plane and $det\bar{M}(s)$ having all its zeros in the open right halfplane. Since K is inner, $K^{-1} = K^*$ and hence M and $\bar{M}$ must satisfy

$$M(-s)'M(s) = \bar{M}(-s)'\bar{M}(s) \tag{5.57}$$

From (5.6) we see that $WM = \bar{W}\bar{M}$ which function we shall name N. Since WM is analytic in the right half plane and $\bar{W}\bar{M}$ in the left, $N(s)$ must be an $m \times p$ matrix polynomial. Therefore

$$W(s) = N(s)M(s)^{-1} \tag{5.58a}$$

$$\bar{W}(s) = N(s)\bar{M}(s)^{-1} \tag{5.58b}$$

which matrix fractions representations may not be coprime. In conclusion, in the rational case, a Markovian triplet corresponds uniquely to three matrix polynomials $(M, \bar{M}, N)$ of which M and $\bar{M}$ are related by the spectral factorization relation (5.57). The proof of the following theorem follows the lines of the analogous result in [33].

THEOREM 5.13. *Let (H, U, X) be a Markovian representation with forward realization Σ and backward realization $\bar{\Sigma}$. Then the following statements are equivalent*

(i) X observable

(ii) (C, A) observable

(iii) $\bar{W} = N\bar{M}^{-1}$ is coprime

Symmetrically the following statements are equivalent

(i) X constructible

(ii) $(\bar{C}, A')$ observable

(iii) $\bar{W} = N\bar{M}^{-1}$ is coprime

COROLLARY 5.14. *A stochastic realization Σ is minimal if and only if (i) (C, A) is observable, (ii) (A, B) is reachable, and (iii) $(A, PC' + BD')$ is reachable.*

Note that minimality of a stochastic realization is a condition that involves both the forward and the backward realization. Moreover, the minimal realizations are characterized by the numerator polynomial matrix N, W and $\bar{W}$ having the same zeros. We shall return to this in Section 11.

Theorems 5.13 and 4.10 suggest a procedure for determining a coprime factorizaton of $W = \bar{W}K$ for any analytic rational spectral factor.

Corollary 5.15. *Let W be an analytic rational spectral factor, let $W = NM^{-1}$ be a coprime matrix fraction representation, and let $\bar{M}$ be the solution of the matrix polynomial factorization problem (5.57) with all its zeros in the right half plane. Then the coprime factorization problem $W = \bar{W}K$ has the solution $K = \bar{M}M^{-1}$ and $\bar{W} = N\bar{M}^{-1}$, where the latter representation is coprime if and only if W is a minimal spectral factor.*

Proof. Since $W = NM^{-1}$ is coprime, the corresponding X is observable (Theorem 5.13). Then K^* and $\bar{Q}$ are right coprime (Theorem 5.5), i.e. the factorization $W = \bar{W}K$ is coprime. Then $\bar{W} = NM^{-1}$ is coprime if and only if X is minimal (Theorem 5.13), which in turn holds if and only if W is minimal (Theorem 4.10). $\square$

6. Partial ordering of minimal Markovian representations

The purpose of this section is to study the structure of the family of minimal Markovian representations. To this end, first we introduce a partial ordering on the set of minimal Markovian splitting subspaces.

Definition 6.1. *Given two minimal Markovian splitting subspaces, X_1 and X_2, let $X_1 < X_2$ denote the ordering*

$$\| E^{X_1}\lambda \| \leq \| E^{X_2}\lambda \| \qquad \textit{for all } \lambda \in H^+ \tag{6.1}$$

where the norms are those of the respective ambient spaces H_1 and H_2.

This partial ordering has the following interpretation. If $X_1 < X_2$, then X_2 is closer to the future H^+ than X_1 (or, loosely speaking, contains more information about the future than X_1) in the sense that for every subspace A of H^+ we have

$$\alpha(X_1, A) \leq \alpha(X_2, A) \tag{6.2}$$

where $\alpha(X, A)$ is the *angle* between the subspaces X and A [13, p.228]. This partial ordering, which turns out to be the natural one, is much ”finer” than that proposed in [50].

6.1. The partially ordered set $\mathcal{X}$

The partial ordering (6.1) has actually a symmetric interpretation with respect to the past.

Lemma 6.2. *The relation $X_1 < X_2$ holds if and only if*

$$\| E^{X_2}\lambda \| \leq \| E^{X_1}\lambda \| \qquad \text{for all } \lambda \in H^- \tag{6.3}$$

Proof. Since X_1 and X_2 are minimal, they are orthogonal to N^- and to N^+ (Theorem 4.11), and therefore, in view of (4.27), the condition (6.1) is equivalent to

$$\| E^{X_1}\lambda \| \leq \| E^{X_2}\lambda \| \qquad \text{for all } \lambda \in H^{-/+} \tag{6.4}$$

and the condition (6.3) to

$$\| E^{X_2}\lambda \| \leq \| E^{X_1}\lambda \| \qquad \text{for all } \lambda \in H^{+/-} \tag{6.5}$$

Now, for $i = 1, 2$, let $\mathcal{O}_i$ and $\mathcal{C}_i$ be the restricted observability and constructibility operator respectively of X_i, as defined by (4.29), and let $\mathcal{O}_i^*$ and $\mathcal{C}_i^*$ be their adjoints. By Corollary 4.12, these operators are injective with dense range. Clearly, (6.4) holds if and only if $\| \mathcal{O}_1^*(\mathcal{O}_2^*)^{-1}\lambda \| \leq \| \lambda \|$ on the dense domain of the bounded operator $\mathcal{O}_1^*(\mathcal{O}_2^*)^{-1}$, i.e. if and only if $\| \mathcal{O}_2^{-1}\mathcal{O}_1 \| = \| \mathcal{O}_1^*(\mathcal{O}_2^*)^{-1} \| \leq 1$. But, in view of the factorization result (4.30), $\mathcal{O}_1\mathcal{C}_1^* = \mathcal{O}_2\mathcal{C}_2^*$, i.e. $\mathcal{O}_2^{-1}\mathcal{O}_1 = \mathcal{C}_2^*(\mathcal{C}_1^*)^{-1}$, and therefore, by continuity, (6.4) is also equivalent to $\| \mathcal{C}_2^*(\mathcal{C}_1^*)^{-1}\lambda \| \leq \| \lambda \|$ on the dense domain of this operator, and hence to $\| \mathcal{C}_2^*\xi \| \leq \| \mathcal{C}_1^*\xi \|$ for all $\xi \in H^{+/-}$, i.e. (6.5). □

THEOREM 6.3. *The family of minimal Markovian splitting subspaces has a unique minimal element X_- and a unique maximal element X_+, i.e.*

$$X_- < X < X_+ \tag{6.6}$$

for all minimal X, and these are precisely the predictor spaces

$$X_- := H^{+/-} = E^{H^-}H^+ \tag{6.7}$$

$$X_+ := H^{-/+} = E^{H^+}H^- \tag{6.8}$$

defined in Section 4.

Proof. Since $E^X{}_{|X_+}$ is a projector,

$$\| E^X\lambda \| \leq \| \lambda \| \qquad \text{for all } \lambda \in X_+ \tag{6.9}$$

But, $\| E^{X_+}\lambda \| = \| \lambda \|$ for all $\lambda \in X_+$ and consequently, in view of (6.4), $X < X_+$. Moreover, for each $X \neq X_+$, there is a λ in X_+ for which strict inequality holds in (6.9), which proves uniqueness. A symmetric argument using Lemma 6.2 gives the rest. □.

Whenever both $X_1 < X_2$ and $X_2 < X_1$ hold, we say that X_1 and X_2 are *equivalent*, writing $X_1 \sim X_2$. In Section 6.3 we shall see that, if at least one of X_1 and X_2 is internal, $X_1 \sim X_2$ implies $X_1 = X_2$. In the noninternal case, however, the equivalence classes cannot be singletons. Indeed, noninternal Markovian splitting subspaces with the same Markovian triplets are equivalent but may differ trivially by the choice of external process dz. Hence, this equivalence factors out the uninteresting arbitrariness inherent in the choice of probability space for dz.

Let us define $\mathfrak{X}$ to be the family of all equivalence classes of minimal Markovian splitting subspaces, and let $\mathfrak{X}_0$ be the subset of those X which are internal ($X \subset H_0$). Then the order relation (6.1) makes $\mathfrak{X}$ into a partially ordered set with a maximal and minimal element. Note that each equivalence class in $\mathcal{X}_0$ is a singleton, and consequently $\mathfrak{X}_0$ is just a family of minimal X (Corollary 6.9).

6.2. Ordering in terms of covariance matrices

In this section we shall illustrate the meaning of the partial ordering defined above in terms of covariance matrices. This will require that the discussion in this subsection be limited to the finite-dimensional case, i.e. to the case of rational spectral density Φ. We shall parametrize $\mathcal{X}$ by a certain family of positive definite matrices. To this end, following [6], we introduce a uniform choice of bases on $\mathcal{X}$. Let $x_+(0)$ be an arbitrary basis in X_+ (see Theorem 5.10 for notations) and define

$$x(0) = E^X x_+(0) \tag{6.10}$$

for every minimal Markovian splitting subspace X.

Lemma 6.4. *The n-dimensional random vector $x(0)$ is a basis in X.*

Proof. Since $\mathcal{O}^* := E^X{}_{|X_+}$ is a bijection (Corollary 4.12)), it sends a basis into a basis. □

Now, to each basis vector $x(0)$ we associate the covariance matrix

$$P = E\{x(0)x(0)'\} \tag{6.11}$$

which is symmetric and positive definite. For a fixed choice of $x_+(0)$, let $\mathcal{P}$ be the family of all covariance matrices obtained as X varies over all minimal Markovian splitting subspaces, and let $\mathcal{P}_0$ be the subfamily generated by the internal X. Note that $\mathcal{P}$ is equipped with the natural ordering: $P_1 \leq P_2$ if and only if $P_2 - P_1$ is nonnegative definite.

Theorem 6.5. *There is a one-one correspondence between $\mathcal{X}$ and $\mathcal{P}$ which is order-preserving in the sense that $P_1 \leq P_2$ if and only if $X_1 < X_2$.*

Proof. To each $\lambda \in X_+$, there corresponds a unique $a \in \mathrm{R}^n$ such that $\lambda = a' x_+(0)$. By (6.10), $E^X\lambda = a' x(0)$, and hence

$$\| E^X\lambda \|^2 = a' P a \tag{6.12}$$

Therefore, in view of the ordering condition (6.4), $X_1 < X_2$ if and only if $P_1 \leq P_2$. Moreover, from (6.12) we see that two X have the same P if and only if they are equivalent, establishing the one-one correspondence between $\mathcal{X}$ and $\mathcal{P}$. □

We shall return to a more thorough analysis of the set $\mathcal{P}$ in the context Anderson-Faurre theory, in Section 7.

The symmetry between the future and the past allows us to introduce a uniform choice of bases also by first choosing a basis $\bar{x}_-(0)$ in X_- and then observing that

$$\bar{x}(0) = E^X \bar{x}_-(0) \tag{6.13}$$

is a basis in X for each minimal Markovian splitting subspace X. The following lemma, to be used in Section 7, shows that the uniform choices of bases (6.10) and (6.13) can be made consistently to reflect the forward-backward structure of Theorem 5.10.

Lemma 6.6. *Let $x_-(0)$ be the basis in X_- corresponding to the uniform choice (6.10), let P_- be the corresponding covariance matrix, and set $\bar{x}_-(0) := P_-^{-1}x_-(0)$. Then the bases (6.10) and (6.13) are related by*

$$\bar{x}(0) = P^{-1}x(0) \tag{6.14}$$

where P is the covariance matrix of $x(0)$.

Proof. First note that (6.14) is the unique basis in X for which $E\{x(0)\bar{x}(0)'\} = I$, i.e. for which

$$\langle a' x(0), b' \bar{x}(0)\rangle = a' b \quad \text{for all} \quad a, b \in \mathrm{R}^n \tag{6.15}$$

But, inserting (6.10) in (6.15), the left member becomes

$$\langle E^X a' x_+(0),\ b' \bar{x}(0)\rangle = \langle a' x_+(0),\ b' \bar{x}(0)\rangle = \langle a' x_+(0),\ E^{X_+} b' \bar{x}(0)\rangle$$

which together with (6.15) implies that

$$\bar{x}_+(0) = E^{X_+}\bar{x}(0) \tag{6.16}$$

In particular, we have

$$\bar{x}_+(0) = E^{X_+}\bar{x}_-(0) \tag{6.17}$$

Now, since $X_- \perp X_+ \mid X$ (c.f.(4.28) and Theorem 6.3), the right member of (6.17) equals $E^{X_+}E^X\bar{x}_-(0)$ (Proposition 2.1(vi)) which together with (6.16) yields

$$E^{X_+}\bar{x}(0) = E^{X_+}E^X\bar{x}_-(0)$$

But then, since $\mathcal{O} := E^{X_+}\,_{|X}$ is a bijection (Corollary 4.12), (6.13) follows with $\bar{x}_-(0)$ defined as in the lemma. □

6.3. Ordering and scattering pairs

One advantage with the geometric theory of Markovian representation is that it does not require any finite-dimensionality assumptions. Of course, our definition (6.1) of ordering is completely general, and therefore the results presented below in this section will be independent of any rationality assumption on Φ.

In subsequent sections our analysis requires that the ordering between minimal X be expressed in terms of geometric conditions of subspace inclusions. To this end, we need the following lemma.

Lemma 6.7. *Let $X_1 \sim (S_1, \bar{S}_1)$ and $X_2 \sim (S_2, \bar{S}_2)$ be two minimal Markovian splitting subspaces. Then $X_1 < X_2$ if and only if*

$$\| E^{S_1}\lambda \| \leq \| E^{S_2}\lambda \| \qquad \textit{for all}\ \lambda \in H_0 \tag{6.18a}$$

or equivalently

$$\| E^{\bar{S}_2}\lambda \| \leq \| E^{\bar{S}_1}\lambda \| \qquad \text{for all}\ \lambda \in H_0 \tag{6.18b}$$

Proof. To show that condition (6.18a) is equivalent to $X_1 < X_2$, we need to prove that (6.1) implies (6.18a). To this end, first note that, in view of the splitting property (3.12), (6.1) is equivalent to

$$\|E^{S_1}\lambda\| \;\leq\; \|E^{S_2}\lambda\| \quad \text{for all } \lambda \in H^+ \tag{6.19}$$

Now, for $i = 1, 2$, let Z_i be the orthogonal complement of H^- in S_i, i.e. $S_i = H^- \oplus Z_i$. Then

$$\|E^{S_i}\lambda\|^2 \;=\; \|E^{H^-}\lambda\|^2 + \|E^{Z_i}\lambda\|^2$$

so it only remains to prove that, if

$$\|E^{Z_1}\lambda\| \;\leq\; \|E^{Z_2}\lambda\| \tag{6.20}$$

holds for all $\lambda \in H^+$, then (6.20) holds for all $\lambda \in H_0$. Therefore, suppose (6.20) holds for all $\lambda \in H^+$. Since $Z_i \subset (H^-)^\perp := H_0 \ominus H^-$ for $i = 1, 2$, it follows that

$$\|E^{Z_1}E^{(H^-)^\perp}\lambda\| \;\leq\; \|E^{Z_2}E^{(H^-)^\perp}\lambda\| \quad \text{for all } \lambda \in H^+$$

But, from the decomposition rule (4.25) and the fact that $(H^- \vee H^+)^\perp = 0$ we have

$$E^{(H^-)^\perp}H^+ = (H^-)^\perp$$

and consequently (6.20) holds for all $\lambda \in (H^-)^\perp$. The extension from $(H^-)^\perp$ to all of H_0 is then trivial. In fact, let $\eta \in H_0$. Then there is a unique representation $\eta = \lambda + \mu$, where $\lambda \in (H^-)^\perp$ and $\mu \in H^-$. Moreover, $E^{Z_i}\eta = E^{Z_i}\lambda$ for $i = 1, 2$ so if (6.20) holds for all $\lambda \in (H^-)^\perp$ then it also holds for all $\eta \in H_0$. This concludes the proof that (6.18a) is equivalent to (6.1). A symmetric argument shows that (6.18b) is equivalent to (6.3). Then the rest follows from Lemma 6.2. □

THEOREM 6.8 Let $X_1 \sim (S_1, \bar{S}_1)$ *and* $X_2 \sim (S_2, \bar{S}_2)$ *be minimal Markovian splitting subspaces. Then:*

(i) *if* $X_1, X_2 \in \mathcal{X}_0$, *then*

$$X_1 < X_2 \iff S_1 \subset S_2 \iff \bar{S}_2 \subset \bar{S}_1$$

(ii) *if* $X_1 \in \mathcal{X}_0$, *then*

$$X_1 < X_2 \iff S_1 \subset S_2 \iff E^{H_0}\bar{S}_2 \subset \bar{S}_1$$

(iii) *if* $X_2 \in \mathcal{X}_0$, *then*

$$X_1 < X_2 \iff E^{H_0}S_1 \subset S_2 \iff \bar{S}_2 \subset \bar{S}_1$$

Proof. First, prove that

$$\text{if } X_1 \in \mathcal{X}_0, \text{ then } X_1 < X_2 \iff S_1 \subset S_2 \tag{6.21a}$$

using (6.18a). It is trivial that $S_1 \subset S_2$ implies $X_1 < X_2$, and to prove the converse, we take $\lambda \in S_1 \subset H_0$ in (6.18a), thereby obtaining $\| \lambda \| \leq \| E^{S_2}\lambda \|$ which implies that $\lambda \in S_2$, and therefore $S_1 \subset S_2$. Obviously, by symmetry and (6.18b), (6.21a) has a backward version, namely

$$\text{if } X_2 \in \mathcal{X}_0, \text{ then } X_1 < X_2 \iff \bar{S}_2 \subset \bar{S}_1 \tag{6.21b}$$

Secondly, prove that

$$\text{if } X_2 \in \mathcal{X}_0, \text{ then } X_1 < X_2 \iff E^{H_0}S_1 \subset S_2 \tag{6.22}$$

To see this, use (6.21b), noting that $\bar{S}_2 \subset \bar{S}_1$ if and only if $\bar{S}_1^{\perp} \subset \bar{S}_2^{\perp} \oplus (H_1 \ominus H_0)$, where H_1 is the ambient space of X_1. By the constructibility condition (4.18b), this is equivalent to

$$S_1 \subset S_2 \oplus (H_1 \ominus H_0) \tag{6.23}$$

from which follows that

$$E^{H_0}S_1 \subset S_2 \tag{6.24}$$

Conversely, if (6.24) holds,

$$S_1 \subset E^{H_0}S_1 \oplus E^{H_1 \ominus H_0}S_1 \subset S_2 \oplus (H_1 \ominus H_0)$$

which is (6.23). The backward version of (6.22) reads

$$\text{if } X_1 \in \mathcal{X}_0, \text{ then } X_1 < X_2 \iff E^{H_0}\bar{S}_2 \subset \bar{S}_1$$

Now, the last statement together with (6.21) and (6.22) covers all the cases of the corollary. □

COROLLARY 6.9. *Let X_1 and X_2 be equivalent minimal Markovian splitting subspaces. Then, if one is internal, $X_1 = X_2$.*

Proof. Suppose that X_1 is internal. Then, by Theorem 6.8, $X_1 < X_2$ implies that $S_1 \subset S_2$ and $X_2 < X_1$ implies that $\bar{S}_1 \subset \bar{S}_2$. Hence, by Theorem 4.1, $X_1 \subset X_2$. But, since X_2 is minimal, we must have $X_1 = X_2$. □

Theorem 6.8 will be instrumental in constructing the greatest lower internal bound and the least upper internal bound for an arbitrary minimal Markovian splitting subspace. First, from statement (i) it is seen that the partial ordering of $\mathcal{X}_0$ is isomorphic to subspace inclusion of the S (or $\bar{S}$) spaces. For any X_1 and X_2 in $\mathcal{X}_0$, define $sup(X_1, X_2)$ to be the least element of $\mathcal{X}_0$ which majorizes both X_1 and X_2, and define $inf(X_1, X_2)$ to be the greatest element of $\mathcal{X}_0$ which is majorized by both X_1 and X_2.

THEOREM 6.10. *The family $\mathcal{X}_0$ is a complete lattice with*

$$sup(X_1, X_2) \sim (S_1 \vee S_2, \bar{S}_1 \cap \bar{S}_2) \tag{6.25a}$$
$$inf(X_1, X_2) \sim (S_1 \cap S_2, \bar{S}_1 \vee \bar{S}_2) \tag{6.25b}$$

i.e. each subfamily of $\mathcal{X}_0$ has a least upper bound and a greatest lower bound.

Proof. First we need to verify that, for any pair X_1 and X_2 in $\mathcal{X}_0$ the subspaces defined in (6.25) also belong to $\mathcal{X}_0$, and that they are the sup and inf as defined above. Set $S := S_1 \vee S_2$ and $\bar{S} := \bar{S} \cap \bar{S}_2$. Then, trivially, $S \supset H^-$ and $\bar{S} \supset H^+$, and S and $\bar{S}$ have the required invariance properties. Moreover, the perpendicular intersection of the pairs $(S_1, \bar{S}_1)$ and $(S_2, \bar{S}_2)$ implies that

$$\bar{S}^\perp = \bar{S}_1^\perp \vee \bar{S}_2^\perp \subset S_1 \vee S_2 = S$$

i.e. $(S, \bar{S})$ is also a perpendicularly intersecting pair. Consequently, by Theorem 4.1, $X \sim (S, \bar{S})$ is an internal Markovian splitting subspace. It remains to show that it is minimal. Constructibility of X_1 and X_2 implies that

$$S = H^- \vee S_1^\perp \vee S_2^\perp = H^- \vee S^\perp$$

i.e. X is constructible (Theorem 4.7). Moreover, from minimality of X_1 and X_2 we have $\bar{S}_1 \perp N^- := H^- \cap (H^+)^\perp$ and $\bar{S}_2 \perp N^-$ (Theorem 4.10) and consequently, $S \perp N^-$ which together with constructibility implies that X is minimal and hence belongs to $\mathscr{X}_0$. Now, it is an immediate consequence of Corollary 6.9 (i) that X is indeed the greatest lower bound of X_1 and X_2. In the same way, we show that (6.25b) belongs to $\mathcal{X}_0$ and is the least upper bound of X_1 and X_2. Finally, the arguments above clearly apply to an arbitrary subfamily of $\mathcal{X}_0$. □

6.4. The tightest internal bounds

We are now in the position to prove a theorem which will be of major importance for what follows. Given, any minimal Markovian splitting subspace X, we would like to bound X from above and below by elements of $\mathscr{X}_0$ in the tightest possible way.

THEOREM 6.11. *Let $X \sim (S, \bar{S})$ be a minimal Markovian splitting subspace and define*

$$S_{0-} := S \cap H_0 \qquad \bar{S}_{0-} := E^{H_0} \bar{S} \tag{6.26a}$$

$$S_{0+} := E^{H_0} S \qquad \bar{S}_{0+} := \bar{S} \cap H_0 \tag{6.26b}$$

Then $X_{0-} \sim (S_{0-}, \bar{S}_{0-})$ and $X_{0+} \sim (S_{0+}, \bar{S}_{0+})$ belong to $\mathcal{X}_0$ and

$$X_{0-} < X < X_{0+} \tag{6.27}$$

Moreover,

$$X_{0-} = sup\{X_0 \in \mathcal{X}_0 | \; X_0 < X\} \tag{6.28a}$$

$$X_{0+} = inf\{X_0 \in \mathcal{X}_0 | \; X_0 > X\} \tag{6.28b}$$

i.e. $X_1 < X_{0-}$ and $X_2 > X_{0+}$ for any X_1 and X_2 in $\mathscr{X}_0$ such that $X_1 < X < X_2$.

Proof. First, we show that $X_{0-} \in \mathcal{X}_0$. Trivially, $S_{0-} \supset H^-$ and $\bar{S}_{0-} \supset H^+$. The required invariance property of S_{0-} follows immediately from that of S. Moreover, since H_0 is doubly invariant under the shift $\{U_t\}$, $U_t \bar{S}_{0-} = E^{H_0} U_t \bar{S}$ so that the right shift invariance of $\bar{S}_{0-}$ follows from that of $\bar{S}$. Since, by perpendicular intersection $\bar{S}^\perp \subset S$,

$$\bar{S}_{0-}^\perp = H_0 \ominus E^{H_0} \bar{S} = H_0 \cap \bar{S}^\perp \subset H_0 \cap S = S_{0-}$$

so that $(S_{0-}, \bar{S}_{0-})$ intersect perpendicularly. (Here we have also used formula (4.25).) Next, we show that the observability of X carries over to X_{0-}. In fact, if $\bar{S} = H^+ \vee S^\perp$, or equivalently, $\bar{S}^\perp = S \cap (H^+)^\perp$, then

$$\bar{S}_{0-}^\perp = H_0 \cap \bar{S}^\perp = H_0 \cap S \cap (H^+)^\perp = S_{0-} \cap (H^+)^\perp$$

i.e. X_{0-} is observable (Theorem 4.7). Moreover, since $S \perp N^+$, we have $S_{0-} \perp N^+$, and consequently, X_{0-} is minimal (Theorem 4.10). In the same way we show that $X_{0+} \in \mathcal{X}_0$. Then, (6.27) follows from Theorem 6.8 (ii) and (iii). Also, if $X_0 \in \mathcal{X}_0$ satisfies $X_0 < X$, then, by Theorem 6.8, $S_0 \subset S$, which implies that $S_0 \subset S_{0-}$, i.e. $X_0 < X_{0-}$. Likewise, if $X < X_0 \in \mathcal{X}_0$, then $E^{H_0} S \subset S_0$ so that $S_{0+} \subset S_0$, i.e. $X_{0+} < X_0$. □

6.5. Ordering and splitting (finite dimensional case)

We shall conclude this section with some useful alternative characterizations of ordering in terms of splitting, valid in the finite-dimensional case.

Proposition 6.12. *Let X_1 and X_2 be minimal Markovian splitting subspaces, at least one of which is internal. Then, $X_1 < X_2$ if and only if*

$$x_1(0) = E^{X_1} x_2(0) \tag{6.29}$$

for any uniform choice of basis (6.10).

Proof. From (6.10) we see that (6.29) is equivalent to

$$E^{X_1}\lambda = E^{X_1} X^{X_2}\lambda \text{ for all } \lambda \in X_+ \tag{6.30}$$

which, due to the fact that X_1 and X_2 are orthogonal to $N_+ := H^+ \ominus X_+$ (Theorem 4.10), can be extended to all $\lambda \in H^+$. This in turn is equivalent to

$$E^{X_1}\lambda = E^{X_1} E^{S_2}\lambda \;\text{ for all }\; \lambda \in H^+ \tag{6.31}$$

because of the splitting property of X_2, i.e. to $X_1 \perp H^+ | S_2$, or equivalently, to $S_1 \perp \bar{S}_2 \mid S_2$, which holds if and only if

$$S_1 \perp H_2 \ominus S_2 \tag{6.32}$$

where H_2 is the ambient space of X_2. Now, first assume that X_1 is internal. Then, (6.32) is equivalent to $S_1 \subset S_2$, i.e. $X_1 < X_2$ (Theorem 6.8). Next, assume that X_2 is internal. The (6.32) is equivalent to $S_1 \subset S_2 \oplus H_0^\perp$, or, equivalently, $E^{H_0} S_1 \subset S_2$, i.e. $X_1 < X_2$ (Theorem 6.8). □

Proposition 6.13. *Let X, X_1, X_2 be minimal Markovian splitting subspaces with X_1 and X_2 internal. Then, if $X_1 < X < X_2$,*

$$X_1 \perp X_2 \mid X$$

Proof. Let $x(0), x_1(0)$ and $x_2(0)$ be a uniform choice of bases in X, X_1 and X_2. Then, applying Proposition 6.12 first to $X_1 < X_2$ and then to $X_1 < X$ and $X < X_2$ we obtain two representations for $x_1(0)$ yielding the equation

$$E^{X_1} x_2(0) = E^{X_1} E^X x_2(0)$$

which is equivalent to $X_1 \perp X_2 \mid X$. □

7. Anderson-Faurre theory and the algebraic Riccati inequality

The classical theory of stochastic realization, initiated by Kalman [20] and developed mainly by Anderson and Faurre [3, 10], deals primarily with the problem of constructing all minimal shaping filters, that is all stable minimal spectral factors $W(s)$ of a given rational spectral density matrix $\Phi(s)$. The family of such W is parametrized by the solutions P of a certain linear matrix inequality which, under certain invertibility conditions, reduces to an algebraic Riccati inequality. To unify the theory and set notations, in this sections we shall give a survey of some of these classical results. But thus will, at least in part, be done in the framework of the geometric theory providing several new insights.

In Section 6 we parametrized the family $\mathcal{X}$ of minimal Markovian splitting subspaces by a set $\mathcal{P}$ of covariance matrices. One of the main results of this section identifies the set $\mathcal{P}$ with the solution set of the linear matrix inequality of Andersson-Faurre theory. This also establishes a one-one correspondence between $\mathcal{X}$ and the family (of equivalence classes) of minimal spectral factors.

7.1. The set $\mathcal{P}$ and the linear matrix inequality

Once a basis $x(0)$ has been fixed in X there is, as explained in Theorem 5.10 and 5.12, a corresponding pair of forward and backward realizations, (5.53) and (5.54) respectively, which are unique modulo right multiplication of $\begin{bmatrix} B \\ D \end{bmatrix}$ and $\begin{bmatrix} \bar{B} \\ \bar{D} \end{bmatrix}$ by constant orthogonal matrices.

Lemma 7.1. *All forward-backward pairs $(\Sigma, \bar{\Sigma})$ of stochastic realizations (5.53)-(5.54) corresponding to a uniform choice of basis (6.10) have the same matrices A, C, and $\bar{C}$. Conversely, for any realization (5.53) [(5.54)] there is a choice of basis $x_+(0)$ in X_+ so that (6.10) holds.*

Proof. Let X be arbitrary (finite-dimensional) minimal Markovian splitting subspace. We want to prove that $(A, C, \bar{C})$ corresponding to X equals $(A_+, C_+, \bar{C}_+)$ corresponding to X_+. First note that (6.10) may be written

$$a' x(0) = \mathcal{O}^* a' x_+(0) \text{ for all } a \in \mathrm{R}^n \tag{7.1}$$

where $\mathcal{O}$ is the restricted observability map (4.29), which in the present setting is a bijection (Corollary 4.12). Moreover, by Corollary 4.13, we have

$$U_t(X)\mathcal{O}^* a' x_+(0) = \mathcal{O}^* U_t(X_+) a' x_+(0)$$

and, since the left member equals $U_t(X)a^{'}x(0)$ because of (7.1), this is, in view of (4.12), equivalent to

$$a^{'}e^{at}x(0) = \mathcal{O}^* a^{'}e^{A_+t}x_+(0)$$

Again applying (7.1), this is seen to be the same as

$$a^{'}e^{At}x(0) = a^{'}e^{A_+t}x(0)$$

yielding $a^{'}e^{At}P = a^{'}e^{A_+t}P$ for all $a \in \mathbb{R}$ and $t \geq 0$, where P, defined by (6.11), is nonsingular. This proves that $A = A_+$. Next, recall from [34] that

$$Cx(0) = \lim_{h \downarrow 0} E^S[y(h) - y(0)]$$

But, since $S \perp N^+$ (Theorem 4.10) and $H_0 = S_+ \oplus N^+$, $E^S = E^S S^{S_+}$, and therefore

$$b^{'}Cx(0) = E^S b^{'}C_+x_+(0) = \mathcal{O}^* b^{'}C_+x_+(0)$$

for all $b \in \mathbb{R}^m$ where the splitting property has been used to obtain the last equality. Then, as above, (7.1) implies that $b^{'}CP = b^{'}C_+P$ for all $b \in \mathbb{R}^m$, and hence the identity $C = C_+$ has been established. In view of Lemma 6.6, we can use the same argument in the backward formulation to show that $\bar{C} = \bar{C}_-$, corresponding to X_-, which then of course also equals $\bar{C}_+$. Finally, the last statement of the lemma follows immediately from the fact that $\mathcal{O}^*$ is a bijection and from Lemma 5.7, and hence $x_+(0)$ can be solved uniquely in terms of $x(0)$ from (6.10). Lemma 6.6 insures that $x(0)$, and hence $x_+(0)$ is uniquely determined by $\bar{x}(0)$. □

Since the parameters $(A, C, \bar{C})$ are invariant, it should be possible to read them off from the covariance description of the process y. To show that this is indeed the case we shall compute the incremental covariance matrix of y. By using both of representations (5.49) of Theorem 5.12 we have

$$\begin{aligned} dy(t)dy(\tau)^{'} &= Cx(t)\bar{x}(\tau)^{'}\bar{C}^{'}dtd\tau + Ddw(t)d\bar{w}(\tau)^{'}D^{'} \\ &\quad + Cx(t)d\bar{w}(\tau)^{'}D^{'}dt + Ddw(t)\bar{x}(\tau)\bar{C}^{'}d\tau \end{aligned} \tag{7.2}$$

which as usually should be understood in the integrated form. Then, for $t \geq \tau$, we have

$$E\{dy(t)dy(\tau)^{'}\} = CE\{x(t)\bar{x}(\tau)^{'}\}\bar{C}^{'}dtd\tau + DE\{dw(t)d\bar{w}(\tau)^{'}\}D^{'} \tag{7.3}$$

since the last two terms of (7.2) vanish because of the orthogonality expressed in (5.55). Consequently, the incremental covariance (7.3) can formally be written as $\Lambda(t-\tau)dtd\tau$, where Λ is an $m \times m$ matrix distribution given by

$$\Lambda(t) = Ce^{At}\bar{C}^{'} + R\delta(t) \quad \text{for } t \geq 0$$

where $R := \Phi(\infty)$. To see this, first note that $E\{x(0)\bar{x}(0)^{'}\} = I$. To obtain the second term in (7.3) invoke (5.47) in Theorem 5.10 and the orthogonality in (5.55). For $t \leq 0, \Lambda(t) = \Lambda(-t)^{'}$, and hence taking the double-sided Laplace transform we obtain the spectral density Φ expressed in the form

$$\Phi(s) = \Phi_+(s) + \Phi_+(-s)^{'} \tag{7.4}$$

where the analytic matrix function Φ_+ has the minimal realization

$$\Phi_+(s) = C(sI - A)^{-1}\bar{C}' + \frac{1}{2}R \tag{7.5}$$

We note that Φ_+ is the positive real part of Φ and can for example be obtained by partial fraction expansion [3, 10].

It follows from Theorems 5.10 and 5.12 that P satisfies

$$AP + PA' + BB' = 0 \tag{7.6a}$$
$$PC' + BD' = \bar{C}' \tag{7.6b}$$
$$DD' = R \tag{7.6c}$$

or equivalently

$$M(P) = -\begin{bmatrix} B \\ D \end{bmatrix} [B', D'] \tag{7.7}$$

$M : \mathrm{R}^{n\times n} \to \mathrm{R}^{(n+m)\times(n+m)}$ being the linear function

$$M(P) = \begin{bmatrix} AP + PA' & PC' - \bar{C}' \\ CP - \bar{C} & R \end{bmatrix} \tag{7.8}$$

Notice that Lemma 7.1 states that the function M is invariant over all realizations. Thus we see that P satisfies the *linear matrix inequality*

$$M(P) \leq 0 \tag{7.9}$$

In fact, the following theorem states that every symmetric solution P of (7.9) is a legitimate state covariance.

THEOREM 7.2. *The set $\mathcal{P}$ of state covariances defined in Section 6 is precisely the set of all symmetric solutions of the linear matrix inequality (7.9).*

Proof. It remains to show that if P satisfies (7.9), then $P \in \mathcal{P}$. To this end, we shall first follow a computation in [3] to identify P with a spectral factor. Let P be a solution of (7.9). Then there is a full rank factorization of $-M(P)$ producing matrices B and D as in (7.7). Define

$$W(s) = C(sI - A)^{-1}B + D \tag{7.10}$$

Then a straightforward application of (7.6), using the standard trick of rewriting the first of these equations as

$$BB' = (sI - A)P + P(-sI - A')$$

yields

$$W(s)W(-s)' = \Phi_+(s) + \Phi_+(-s)' \tag{7.11}$$

with Φ_+ defined by (7.5). Therefore, in view of (7.4), and the fact that A is a stability matrix, W is an analytic spectral factor. By Proposition 3.1, deg $W \geq \frac{1}{2}\Phi = n$, with equality if and only if W is a minimal spectral factor. Since deg $W \leq \dim A = n$, W must be minimal. As a consequence (A, B) is reachable and hence P, solving the Lyapunov equation (7.6a), must be positive definite. Choosing an arbitrary dz of appropriate dimension, (5.13) defines a generating process dw. This defines a minimal realization (5.53) with the preassigned parameters $(A, C, \bar{C})$ having covariance matrix P. By Lemma 7.1, there is a basis $x_+(0) \in X_+$ and a corresponding realization having parameter $(A, C, \bar{C})$ and state process x_+ such that (6.10) holds. Hence $P \in \mathcal{P}$. □

7.2. Spectral factorization and the positive real lemma

From Theorem 7.2 the original result of Anderson [3] on the so-called "inverse problem of stationary covariance generation" follows.

Theorem 7.3 (B.D.O. Anderson). *Given a minimal realization $(A, C, \bar{C}, \frac{1}{2}R)$of the causal part $\Phi_+(s)$ of a rational spectral density matrix $\Phi(s)$ as in (7.5), the family of all analytic minimal spectral factors is parametrized by the solution set of the corresponding linear matrix inequality (7.9) in the following sense. Given a symmetric solution P of (7.9), take $\begin{bmatrix} B \\ D \end{bmatrix}$ to be the unique (modO) full-rank factor of $-M(P)$ as in (7.7) and define $W(s)$ (modO) as in (7.10). Then all such W are minimal spectral factors. Viceversa, given an equivalence class $[W]$ of W as in (7.10) there is a unique symmetric $P > 0$ solving (7.6) and hence (7.9).*

In particular, it follows from this theorem and Theorem 7.2 that two finite-dimensional minimal Markovian splitting subspaces are equivalent (in the sense defined in Section 6.1) if and only if they have the same analytic (coanalytic) spectral factor $W(\bar{W})$ modO.

An analytic matrix function Φ_+ satisfying (7.4) with Φ a spectral density is a so-called *positive real* matrix function. Equations (7.6) are often called the *positive real equations* because of the following classical result due to Yakubovich[55], Kalman[18], and Popov[45].

Theorem 7.4 (Positive Real Lemma). *The rational matrix function Φ_+ with minimal realization (7.5) is positive real if and only if the solution set $\mathcal{P}$ of (7.9) is nonempty.*

Proof. The function Φ_+ is positive real if and only if there is an analytic spectral factor W such that (7.12) holds. However, Theorem 7.3 states that the (equivalence classes of) analytic spectral factors are in one-one correspondence with the set $\mathcal{P}$, and therefore such a W exists if and only if $\mathcal{P} \neq \emptyset$. □

The geometry of the set $\mathcal{P}$ has been studied by Faurre [10]. The following theorem summarizes what is known about the structure of $\mathcal{P}$, as for example reported in [10] and makes connection to the geometric theory presented in Section 6. It will be a basic point of departure for subsequent analysis.

Theorem 7.5 (Faurre). *Let $\mathcal{P}$ be the solution to the linear matrix inequality (7.9). Then $\mathcal{P}$ is a closed, bounded, convex set with a maximal and a minimal element, P_+ and P_-, respectively, equal to the covariance matrices of $x_+(0)$ and of $x_-(0) := E^{X_-}x_+(0)$. Both P_- and P_+ belong to $\mathcal{P}_0$.*

Proof. It follows immediately from the linear matrix inequality (7.9) that $\mathcal{P}$ is closed and convex. Theorem 6.6 states that the partially ordered set $\mathcal{P}$ and $\mathcal{X}$ are isomorphic.

Therefore, since $\mathcal{X}$ has a maximal element, X_+, and a minimal element, X_-, in $\mathcal{X}_0$, given by (6.8) and (6.7) respectively, there are corresponding P_+ and P_- in $\mathcal{P}_0$ having the properties stated. From this it also follows that $\mathcal{P}$ is bounded. ☐

In Faurre's work the existence of the maximal element P_+ of P is proved by considering a so-called dual spectral factorization problem and a dual set $\bar{\mathcal{P}}$ of solutions which turns out to be the family of all inverses P^{-1} of the elements of $\mathcal{P}$, i.e.

$$\bar{\mathcal{P}} = \{P^{-1} \mid M(P) \leq 0\} \tag{7.12}$$

In our context this is actually the set of all state convariances of the backward model (5.54) corresponding to the uniform bases obtained from (6.10) through the transformation (5.44). This is discussed in [25].

Proposition 7.6. *Suppose* $R := \Phi(\infty) > 0$. *Then*

$$\mathcal{P} = \{P \mid P^{'} = P \; ; \; \Lambda(P) \leq 0\} \tag{7.13}$$

where $\Lambda : \mathrm{R}^{n\times n} \to \mathrm{R}^{n\times n}$ *is the quadratic matrix function*

$$\Lambda(P) = AP + PA^{'} + (\bar{C} - CP)^{'} R^{-1} (\bar{C} - CP) \tag{7.14}$$

where $(A, C, \bar{C}, R)$ *are given by (7.5).*

Proof. Since $R > 0$, $M(P)$ can be block diagonalized as

$$\begin{bmatrix} I & T \\ 0 & I \end{bmatrix} M(P) \begin{bmatrix} I & 0 \\ T^{'} & I \end{bmatrix} = \begin{bmatrix} \Lambda(P) & 0 \\ 0 & R \end{bmatrix}$$

where

$$T = (\bar{C} - CP)^{'} R^{-1}$$

From this it follows that (7.9) is equivalent to $\Lambda(P) \leq 0$. ☐

From now on we shall always assume that Φ is *coercive*, i.e. Φ has no zeros on the imaginary axis I including the points at infinity. In particular this implies that $R > 0$. Then the set $\mathcal{P}$ can be identified with the symmetric solutions of the *algebraic Riccati inequality*

$$\Lambda(P) \leq 0 \tag{7.15}$$

as stated in Proposition 7.6.

7.3. Stochastic realizations in standard form

As was done in [25], it is convenient in this situation to fix a representative in each equivalence class of spectral factors by choosing the arbitrary orthogonal transformation in the factorization of (7.7) so that

$$\begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ R^{1/2} & 0 \end{bmatrix} \tag{7.16}$$

where $R^{1/2}$ is the symmetric positive square root of R, and B_2 is a full-rank matrix chosen in some canonical way. Then (7.6b) can be solved for B_1, i.e.

$$B_1 = (\bar{C} - PC)^{'} R^{-1/2}, \tag{7.17}$$

which inserted in (7.6a) yields

$$\Lambda(P) = -B_2 B_2^{'}. \tag{7.18}$$

Now, to each $P \in \mathcal{P}$ there corresponds in a one-to-one fashion an element in $\mathcal{X}$, i.e. an equivalence class of minimal Markovian splitting subspaces with a forward realization

$$\begin{cases} dx = Axdt + B_1 du + B_2 dv \\ dy = Cxdt + R^{1/2} du \end{cases} \tag{7.19}$$

which is uniquely determined except for the arbitrariness of the possible external part of the driving noise $dw = \begin{bmatrix} du \\ dv \end{bmatrix}$ as explained in Section 5. In Section 6, we defined $\mathcal{P}_0$ as the subfamily of $\mathcal{P}$ corresponding to internal X. It should be clear from (7.19) that the internal realization (7.19) are precisely those for which $B_2 = 0$. Consequently, it follows from (7.18) that $\mathcal{P}_0$ is precisely the symmetric solutions of the algebraic Riccati equation

$$\Lambda(P) = 0 \tag{7.20}$$

and that the internal realizations correspond to square spectral factors, as has already been pointed out in Section 5.

7.4. Remarks on Kalman filtering

It is here natural to make contract with Kalman filtering. Given a linear observable (but not necesserily minimal) stochastic system

$$(\Sigma) \begin{cases} dx = Axdt + Bdw \\ dy = Cxdt + Ddw \end{cases} \tag{7.21}$$

with state covariance P, the linear minimum-variance estimate

$$\hat{x}(t) = E^{H^{-}_{[0,t]}(dy)} x(t) \tag{7.22}$$

for $t \geq 0$, where $H^{-}_{[0,t]}(dy)$ is the subspace generated by (the increments of) the observed process y on the finite interval $[0, t]$, is given by the Kalman filter

$$d\hat{x} = A\hat{x}dt + K(t)[dy - C\hat{x}dt] \; ; \quad \hat{x}(0) = 0 \tag{7.23}$$

with the gain

$$K(t) = [Q(t)C^{'} + BD^{'}]R^{-1} \tag{7.24}$$

and the error covariance matrix function

$$Q(t) = E\{[x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]^{'}\} \tag{7.25}$$

satisfying the matrix Riccati equation

$$\begin{cases} \dot{Q} = AQ + QA' - (QC' + BD')R^{-1}(QC' + BD')' + BB' \\ Q(0) = P \end{cases} \tag{7.26}$$

It is well-known that, under the present conditions, $Q(t)$ tends to a limit $Q_\infty \geq 0$ as $t \to \infty$, thus defining a *steady-state Kalman filter*

$$d\hat{x} = A\hat{x}dt + K_\infty[dy - C\hat{x}dt] \tag{7.27}$$

where the gain K_∞ is constant and the system is defined on the whole real line. Let the stationary process represented by this system be denoted $\hat{x}_\infty(t)$. Then, because the innovation process

$$d\nu = R^{1/2}[dy - C\hat{x}_\infty dt] \tag{7.28}$$

is a Wiener process, (7.28) defines a stochastic realization

$$\begin{cases} \hat{x}_\infty = A\hat{x}_\infty dt + K_\infty R^{-1/2} d\nu \\ dy = C\hat{x}_\infty dt + R^{1/2} d\nu \end{cases} \tag{7.29}$$

of dy on the real line; for details, see e.g.[25]. By assumption the Markovian splitting subspace X defined by Σ is observable, and hence Lemma 4.6 and Corollary 4.9 imply that

$$E^{H^-}X = X_- \tag{7.30}$$

Consequently, since

$$E^{H^-}x(t) = \hat{x}_\infty(t), \tag{7.31}$$

$\hat{x}_\infty(0)$ is a generator of X_-. As explained in Section 3, $\hat{x}_\infty(0)$ is a basis if and only if the model (7.29) is reachable. We shall prove that reachability of (7.29) is equivalent to minimality of the underlying model Σ. This is the content of the following "folk theorem".

Proposition 7.7. *An observable system Σ is a minimal realization of y if and only if its steady state Kalman filter (7.29) is reachable.*

Proof. Let the dimension of X_- be n. Then all minimal X have this dimension. (Corollary 4.11). We have already seen above that (7.29) is reachable if and only if the dimension of $\hat{x}_\infty(0)$ is n. However, $\dim X \leq \dim x(0) = \dim \hat{x}_\infty(0)$, and consequently (7.29) is reachable if and only if $\dim X \leq n$, from which the stated result follows. □

Now, suppose that the linear stochastic system Σ, regarded as a realization of y, is minimal. Then, it follows from what has just been discussed that the steady-state Kalman filtering estimate $\hat{x}_\infty$ equals x_-, the (forward) state process corresponding to the predictor space X_- in a uniform basis. To see this, compare (7.31) with (6.29) in Theorem 6.12, remembering that, by splitting, $E^{H^-}\lambda = E^{X_-}\lambda$ for all $\lambda \in \bar{S} \supset X$.

With Σ being an arbitrary minimal stochastic realization, we would like to express the Kalman-filtering equations in terms of the invariant parameters $(A, C, \bar{C}, R)$ determined by the covariance of y. To this end, introduce a change of variables

$$\Pi(t) = P - Q(t) \tag{7.32}$$

and use the positive real lemma equations (7.6) to transform (7.24) and (7.26) into

$$K(t) = [\bar{C} - C\Pi(t)]R^{-1} \tag{7.33}$$

and

$$\dot{\Pi} = \Lambda(\Pi) \;\; ; \;\; \Pi(0) = 0 \tag{7.34}$$

where Λ is the same function, (7.14), as in the characterization of $\mathcal{P}$. The matrix Riccati equation (7.34) is invariant in the sense that it is independent of the particular choice of model Σ, in agreement with the property (7.30). Moreover, the equilibria of the matrix differential equation (7.34) precisely constitute the solution set of the algebraic Riccati equation (7.20), i.e. the set $\mathcal{P}_0$ of state covariances of internal realizations. As $t \to \infty$, $\Pi(t) \to P_- \in \mathcal{P}_0$. To see this, just note that

$$\Pi(t) = E\{\hat{x}(t)\hat{x}(t)'\} \tag{7.35}$$

as is immediate from (7.25) and (7.32). Then $Q_\infty = P - P_-$ and since $Q_\infty \geq 0$, we have an independent verification of the fact that $P \geq P_-$ for all $P \in \mathcal{P}$.

Analogously, starting from a minimal backward realization (5.44), we can define a backward Kalman filter, the steady-state version of which can be identified with the backward realization of X_+. From this and (7.12) we deduce that $P^{-1} \geq P_+^{-1}$, i.e. $P \leq P_+$, for all $P \in \mathcal{P}$, obtaining an independent proof of the ordering $P \leq P_+$. The details of this analysis can be found in [25].

7.5. Summing up

In Section 6 we showed that the partially ordered set $\mathcal{X}$ of (equivalence classes of) minimal Markovian splitting subspaces can be parametrized by the family $\mathcal{P}$ of its state covariance matrices under an arbitrary uniform choice of basis. Moreover, the one-one correspondence between $\mathcal{X}$ and $\mathcal{P}$ is order-preserving so that

$$X_- < X < X_+ \tag{7.36}$$

corresponds to

$$P_- \leq P \leq P_+ \tag{7.37}$$

In this section we have identified $\mathcal{P}$ with the solution set of an algebraic Riccati equation, the steady state Kalman filter with the forward realization (5.53) of X_-, and the backward steady-state Kalman filter with the backward realization (5.54) of X_+. In the next section we shall analyze a noncausal estimation problem, corresponding to a stationary smoothing problem and show that it can be understood in terms of the equilibrium set $\mathcal{P}_0$, giving filtering interpretations to all the elements of $\mathcal{P}_0$.

8. A noncasual estimation problem

Given a minimal noninternal realization

$$(\Sigma) \quad \begin{cases} dx = Axdt + Bdw \\ dy = Cxdt + Ddw \end{cases} \tag{8.1}$$

consider the problem of determining the estimates $\{\hat{x}(t);\ t \in \mathrm{R}\}$ with components

$$\hat{x}_k(t) = E^{H_0} x_k(t) \ \ ; \ \ k = 1, 2, \ldots, n \tag{8.2}$$

and a minimal recursive filter generating it. This is a steady-state smoothing estimate, formed in analogy with the steady state Kalman filter. A theory for finite interval smoothing can be developed using the same principles.

For simplicity, the process dy described by the model (8.1) will be assumed to be *coercive*, i.e. its special density satisfies $\Phi(i\omega) > 0$ for all ω including points at infinity.

8.1. A geometric problem formulation

Geometrically, the problem can be stated as follows. Let $X \sim (S, \bar{S})$ be the minimal Markovian splitting subspace corresponding to (8.1). Then

$$\hat{X} = E^{H_0} X \tag{8.3}$$

is the space spanned by the components of $\hat{x}(0)$, and

$$\mathrm{span}\{\hat{x}_1(t), \hat{x}_2(t), \ldots, \hat{x}_n(t)\} = U_t \hat{X} \tag{8.4}$$

Now, $\hat{X}$ is in general non-Markovian, and consequently there is no stochastic differential equation satisfied by $\hat{x}$. Therefore we need to embed it minimally in a Markovian space. This amounts to determining a subspace X_0 such that

(i) X_0 is an internal Markovian splitting subspace

(ii) $\hat{X} \subset X_0$

(iii) X_0 is minimal, in the sense that if X_1 satisfied (i) and (ii) and $X_1 \subset X_0$ then $X_1 = X_0$

This problem formulation has the following motivation. To such an X_0, which is in general a *nonminimal* Markovian splitting subspace of, say, dimension $n_0 \geq n$, there corresponds a realization

$$\begin{cases} dx_0 = A_0 x_0 dt + B_0 du_0 \\ dy = C_0 x_0 dt + D_0 du_0 \end{cases} \tag{8.5}$$

of y, where $x_0(0)$ is a basis in X_0 (Theorem 5.10). Since X_0 is internal,

$$W_0(s) = C_0 (sI - A_0)^{-1} B_0 + D_0 \tag{8.6}$$

is a square spectral factor with D_0 invertible, and thus

$$du_0 = D_0^{-1}(dy - C_0 x_0 dt) \tag{8.7}$$

so that x_0 is computable by a filter driven by the observed process dy, i.e.

$$dx_0 = (A_0 - B_0 D_0^{-1} C_0) x_0 dt + B_0 D_0^{-1} dy \tag{8.8a}$$

Because of (ii) there is an $n \times n_0$ matrix H such that

$$\hat{x}(t) = Hx_0(t) \tag{8.8b}$$

Equations (8.8) constitute the “recursive” form of the estimator, which in view of (iii) is of smallest possible dimension.

A few comments are in order concerning some of these formulas. First, since in general X_0 is a nonminimal Markovian splitting subspace, we cannot expect (C_0, A_0) to be an observable pair, and hence (A_0, B_0, C_0, D_0) to be a minimal realization of $W_0(s)$. Several procedures for determining the dimension n_0 of X_0 will be given in Sections 9–11. Secondly, since, in general, the state evolution matrix

$$\Gamma_0 = A_0 - B_0 D_0^{-1} C_0 \tag{8.9}$$

of the filter (8.8) has eigenvalues in both the right and the left open half planes, but, due to coercivity, not on the imaginary axis (see Section 10), the state equation (8.8a) requires some interpretation. Let T be a nonsingular matrix such that

$$T^{-1}\Gamma_0 T = \begin{bmatrix} \Lambda_- & 0 \\ 0 & \Lambda_+ \end{bmatrix} \tag{8.10}$$

where all eigenvalues of $\Lambda_- \, (\Lambda_+)$ are in the open left (right) half plane. For example, the Jordan form provides such a decomposition. Moreover, define

$$\begin{bmatrix} \xi_- \\ \xi_+ \end{bmatrix} := T^{-1}x_0, \quad T = [T_-, T_+], \quad \text{and} \quad \begin{bmatrix} L_- \\ L_+ \end{bmatrix} := T^{-1}B_0 D_0^{-1} \tag{8.11}$$

Then

$$x_0(0) = T_-\xi_- + T_+\xi_+ \tag{8.12}$$

where

$$\begin{cases} d\xi_- = \Lambda_-\xi_- dt + L_- dy \\ d\xi_+ = \Lambda_+\xi_+ dt + L_+ dy \end{cases} \tag{8.13}$$

Here the first of equations (8.13) could be integrated over the past and the second over the future so that

$$\hat{x}(t) = \int_{-\infty}^{t} HT_- e^{\Lambda_-(t-\sigma)} L_- dy + \int_{t}^{\infty} HT_+ e^{\Lambda_+(\sigma - t)} L_+ dy \tag{8.14}$$

8.2. The geometric solution

THEOREM 8.1. *There is a unique smallest (in the sense of subspace inclusion) Markovian splitting subspace* X_0 *containing* $\hat{X} := E^{H_0}X$, *which is internal, namely,*

$$X_0 = X_{0-} \vee X_{0+} \tag{8.15}$$

where X_{0-} is the greatest lower internal bound of X and X_{0+} is the least upper internal bound of X, as defined in Theorem 6.11. In particular, if $X \sim (S, \bar{S})$,

$$X_0 \sim (E^{H_0} S, E^{H_0} \bar{S}) \tag{8.16}$$

Moreover, X_{0+} is the observable and X_{0-} the constructible subspace of X_0.

The proof of this theorem is based on the following series of lemmas.

LEMMA 8.2. *Let X_1 and X_2 be internal minimal Markovian splitting subspaces such that $X_1 < X_2$. Then $X_1 \vee X_2$ is a Markovian splitting subspace and*

$$X_1 \vee X_2 \sim (S_2, \bar{S}_1) \tag{8.17}$$

Proof. In view of the decomposition (4.4), $S_1 = X_1 \oplus \bar{S}_1^\perp$ and $\bar{S}_2 = X_2 \oplus S_2^\perp$. Since $S_1 \subset S_2$ [Theorem 6.8 (i)], then $S_1^\perp \supset S_2^\perp$, and consequently, since $X_1 \subset S_1, X_1 \perp S_2^\perp$. Likewise, by a symmetric argument, $X_2 \perp \bar{S}_1^\perp$. Moreover, since $S_1 \supset H^-$ and $\bar{S}_2 \supset H^+, S_1 \vee \bar{S}_2 = H_0$, from which we have

$$H_0 = \bar{S}_1^\perp \oplus (X_1 \vee X_2) \oplus S_2^\perp$$

which is equivalent to $X_1 \vee X_2$ being a Markovian splitting subspace represented by (8.17) (Theorem 4.1). □

From this lemma and Theorem 6.11 it follows that (8.15) has the representation (8.16).

LEMMA 8.3. *Let X_0 be given (8.16). Then $X_0 \supset \hat{X}$. In fact X_0 is the smallest Markovian splitting subspace containing $\hat{X}$.*

Proof. It is easy to see that

$$E^{H_0}(S \cap \bar{S}) \subset (E^{H_0} S) \cap (E^{H_0} \bar{S}), \tag{8.18}$$

which, in view of (8.16), is the same as $\hat{X} \subset X_0$. Next we show that X_0 is the unique smallest Markovian splitting subspace with this property. Using the notation $S_0 := E^{H_0} S$ and $\bar{S}_0 := E^{H_0} \bar{S}$ so that $X_0 \sim (S_0, \bar{S}_0)$, we have

$$S_0 = H^- \vee \hat{X}^- \tag{8.19a}$$

$$\bar{S}_0 = H^+ \vee \hat{X}^+ \tag{8.19b}$$

To see this, note that, since $S = H^- \vee X^-$ and $H^- \subset H_0, E^{H_0} S = H^- \vee E^{H_0} X^-$, and therefore, because of the doubly invariant property of H_0 under the shift $\{U_t\}$, (8.19a) follows. A similar argument yields (8.19b). Let $X_1 \sim (S_1, \bar{S}_1)$ be an arbitrary Markovian splitting subspace containing $\hat{X}$. Then $X_1^- \supset \hat{X}^-$ and $X_1^+ \supset \hat{X}^+$, and consequently (8.19) implies that $S_1 \supset S_0$ and $\bar{S}_1 \supset \bar{S}_0$, i.e. $X_1 = S_1 \cap \bar{S}_1 \supset S_0 \cap \bar{S}_0 = X_0$, which is what we wanted to prove. This also concludes the proof of the theorem. □

LEMMA 8.4. *Let X_0, X_{0-} and X_{0+} be defined as in Theorem 8.1. Then X_{0+} is the observable and X_{0-} the constructible subspace of X_0.*

Proof. If $X_0 \sim (S_0, \bar{S}_0)$, then, by Theorems 6.11 and 8.1, $S_0 = S_{0+}$ and $\bar{S}_0 = \bar{S}_{0-}$. Since X_{0+} is minimal and hence observable, it follows from (4.26) that $X_{0+} = E^{X_{0+}}H^+$. But, in view of the splitting property (4.9), $E^{X_{0+}}H^+ = E^{S_0}H^+$, which in turn equals $E^{X_0}H^+$, the observable subspace of X_0. A symmetric argument shows that X_{0-} is the constructible subspace of X_{0-}. $\square$

8.3. A spectral-domain solution

Now we proceed to a spectral-domain characterization of the solution of the noncausal estimation problem presented in geometric form above. Let us stress at this point that the main purpose of this analysis is not the derivation of estimation algorithms *per se* but that of giving a filtering interpretation to all the solutions of the algebraic Riccati equation $\Lambda(P) = 0$.

Theorem 8.5. *Let*

$$W(s) = C(sI - A)^{-1}B + D \tag{8.20}$$

be the transfer function of the minimal realization (8.1) and let

$$W(s) = W_0(s)U(s)^* \tag{8.21}$$

be the unique (modulo a constant unitary transformation) coprime factorization in H^∞ *[13, p.255]. Then*

(i) $W_0 \in H^\infty_{m\times m}$ *is the transfer function of the internal model (8.5), i.e. the analytic spectral factor of the Markovian splitting subspace* X_0 *defined in Theorem 8.1, and* $U \in H^\infty_{p\times m}$ *is outer and isometric on the imaginary axis.*

(ii) *the noncausal estimator (8.8) is described by the following block diagram*

$$\xrightarrow{dy} \boxed{W_0^{-1}} \xrightarrow{du_0} \boxed{V} \xrightarrow{\hat{x}}$$

where

$$V(s) = (sI - A)^{-1}BU(s) \tag{8.22}$$

(iii) *Let* $\bar{W}$ *be the coanalytic spectral factor (5.51b) formed from (8.1) as in Theorem 5.12, let*

$$\bar{W}(s) = \bar{W}_0(s)\bar{U}(s)^* \tag{8.23}$$

be the unique conjugate coprime factorization obtained in analogy with (8.21) over the conjugate $\bar{H}^\infty$*-spaces and set* $K_0 := \bar{W}_0^{-1}W_0$. *Then* $(W_0, \bar{W}_0, K_0)$ *is the Markovian triplet of* X_0.

(iv) *Let* $(A_0, B_0, \bar{B}_0)$ *be a minimal realization*

$$K_0(s) = I - \bar{B}_0'(sI - A_0)^{-1}B_0 \tag{8.24}$$

of the structural function K_0. *Then there are unique matrices* H, C_0, *and* D_0 *such that*

$$V(s) = H(sI - A_0)^{-1}B_0 \tag{8.25a}$$

$$W_0(s) = C_0(sI - A_0)^{-1}B_0 + D_0 \tag{8.25b}$$

and A_0, B_0, C_0, D_0 and H define via (8.8) a minimal noncausal filter for $\hat{x}$ with a state process x_0 such that $x_0(0)$ is a basis in X_0.

Before proving this theorem we shall give some clarifying remarks and a procedure for solving the factorization problem (8.21). First, we note that W and W_0 are minimal spectral factors, the latter being square. Hence we have the inner - outer factorizations

$$W(s) = W_-(s)Q(s) \quad \text{and} \quad W_0(s) = W_-(s)Q_0(s) \tag{8.26}$$

where W_- is the outer (minimum phase) spectral factor. Then it is seen that factorization (8.21) is actually equivalent to the coprime factorization

$$Q(s) = Q_0(s)U(s)^*$$

which is the form found in [13, p.255]. Secondly, we remark that the decomposition depicted in the block diagram of Theorem 8.5 is very much in the spirit of Wiener-Kolmogorov theory, W_0^{-1} being the whitening filter of the observation process. Note, however, that in the present setting the whitening filter is noncausal. In the same vein, the computation of the estimator is immediately seen to involve the coprime factorization of the *cross-spectral density matrix*

$$\Phi_{xy}(s) = (sI - A)^{-1}BW(-s)'$$

as

$$V(s)W_0(s)^* = \Phi_{xy}(s) \tag{8.27}$$

which is now of the general Wiener-Hopf type.

A procedure for solving the factorization problem (8.21) can be based on the matrix fraction representation

$$W(s) = D(s)^{-1}N(s) \tag{8.28}$$

where D and N are coprime matrix polynomials of dimensions $m \times m$ and $m \times p$ respectively. First reduce N to the *Smith form*

$$[\Theta(s), 0] = T_1(s)N(s)T_2(s) \tag{8.29}$$

where Θ, T_1 and T_2 are square matrix polynomials, T_1 and T_2 being unimodular, i.e. having polynomial inverses; see, e.g., [13]. Then, setting $Z := T_1^{-1}\Theta T_1$ and $\hat{N} := T_1^{-1}[I, 0]T_2^{-1}$, we have the polynomial factorization

$$N(s) = Z(s)\hat{N}(s) \tag{8.30}$$

for N, where Z is a square matrix polynomial having the same zeros as N, and hence as W, and $\hat{N}$ is a rectangular matrix polynomial without zeros. Although $\hat{N}$ has no zeros, the square matrix polynomial $\hat{N}\hat{N}^*$, may have, but due to coercivity of Φ, none lies on the imaginary axis, and $\hat{N}\hat{N}^*$ has full rank.

Proposition 8.6. *Let D, N, Z and $\hat{N}$ be the matrix polynomial defined above, and let $\hat{M}$ be a square matrix-polynomial solution, with all its zeros in the open right half of the complex plane, of the factorization problem*

$$\hat{M}\hat{M}^* = \hat{N}\hat{N}^* \tag{8.31}$$

Moreover, set $M := Z\hat{M}$. *Then*

$$W_0 = D^{-1}M \qquad \text{and} \qquad U^* = \hat{M}^{-1}\hat{N} \tag{8.32}$$

solve the coprime factorization problem (8.21). The matrix fraction representations (8.32) are coprime.

Proof. Clearly, $W_0U^* = W$ if W_0, U and N are given by (8.28), (8.31) and (8.32) respectively. It remains to show that the factorization is coprime. But, since $\hat{N}$ has no zeros, the same is true for U, which is therefore outer [13]. Hence, W_0 and U have no right inner factor in common and are therefore coprime. The coprimeness of the first of the matrix fractions (8.32) follows from the fact that W and W_0, being minimal spectral factors, have the same degree, while coprimeness of the second is immediate. □

For the proof of Theorem 8.4 we need the following lemma which is of independent interest and will be used again below.

Lemma 8.7. *Let* $(K, Q, \bar{Q})$ *be the inner triplet of* X, *let*

$$Q = Q_0U^* \tag{8.33}$$

be the unique coprime factorization for which Q_0 *is inner,* $U \in H^\infty$, *and* Q_0 *and* U *are right coprime, and let*

$$\bar{Q} = \bar{Q}_0\bar{U}^* \tag{8.34}$$

be the corresponding coprime factorization in the conjugate space. Then U *is the outer spectral factor of* Q^*Q, *i.e. the outer function satisfying*

$$UU^* = Q^*Q \tag{8.35}$$

and $\bar{U}$ *is the conjugate outer factor of*

$$\bar{U}\bar{U}^* = \bar{Q}^*\bar{Q} \tag{8.36}$$

Moreover, defining

$$K_0 = \bar{U}^*KU, \tag{8.37}$$

$(K_0, Q_0, \bar{Q}_0)$ *is the inner triplet of* X_0.

Proof. By Theorem 3.5 in [13; p.254] or by the procedure of Proposition 8.6, there are unique factorizations (8.33) and (8.34). Clearly U is outer, because if it were not, it would have an outer-inner factorization U_0U_i, and then U_i would have to be a right inner factor of Q_0, or else Q could not be analytic. But this would contradict coprimeness. Likewise, $\bar{U}$ is seen to be conjugate outer. Now, (8.35) and (8.36) follow from the fact that $Q_0^*Q_0 = I$ and $\bar{Q}_0^*\bar{Q}_0 = I$.

Next we prove that

$$d\hat{u}_0 = U^*d\hat{w} \tag{8.38}$$

To this end, note that $S = \int H_p^2 d\hat{w}$ and that

$$S_0 = E^{H_0}S = \int H_p^2 Q^*Q d\hat{w} \tag{8.39}$$

(see Section 5.3). But, since U is outer, $H^2_p U = H^2_m$ [52; p. 190], and therefore in view of (8.35) we may write (8.39) as

$$S_0 = \int H^2_m U^* d\hat{w} \tag{8.40}$$

This together with the fact that $U^*U = I$ implies (8.38). Then

$$d\hat{u} = Q d\hat{w} = Q_0 U^* d\hat{w} = Q_0 d\hat{u}_0 \tag{8.41}$$

i.e. Q_0 is the inner factor of W_0, the spectral factor corresponding to S_0, as claimed. By symmetry we see that $\bar{Q}_0$ is the (conjugate) inner factor of $\bar{W}_0$. Inserting (8.33) and (8.34) into $T_0 = \bar{Q}KQ^*$, displayed in (5.35) we obtain

$$T_0 = \bar{Q}_0 \bar{U}^* K U Q_0^* \tag{8.42}$$

from which (8.37) follows. Consequently $(Q_0, \bar{Q}_0, K_0)$ is the inner triplet of X_0. □

Proof of Theorem 8.4. A comparison of (8.21) and (8.33) shows that $W_0 = W_- Q_0$, and therefore it follows from Lemma 8.7 that W_0, as defined by the factorization (8.21), is in fact the analytic spectral factor of X_0, and consequently the transfer function of (8.5). The fact that U is outer follows from Lemma 8.7, and, in view of (8.33), $U^*U = I$. Thus we have established statement (i). Next, from (8.1) we have

$$x(0) = \int_{-\infty}^{\infty} (i\omega - A)^{-1} B d\hat{w} \tag{8.43}$$

(cf. (B.18) in Appendix B) and consequently, projecting onto H_0, we obtain

$$\hat{x}(0) = \int_{-\infty}^{\infty} (i\omega - A)^{-1} B Q^*(i\omega) Q(i\omega) d\hat{w} \ , \tag{8.44}$$

as explained in Section 5.3. However, in view of (8.35) and (8.38), $Q^*Q d\hat{w} = U d\hat{u}_0$, and consequently

$$\hat{x}(t) = \int_{-\infty}^{\infty} e^{i\omega t} V(i\omega) d\hat{u}_0 \tag{8.45}$$

where V is defined by (8.22). But $d\hat{u}_0 = W_0 d\hat{y}$ and therefore (ii) follows. By Theorem 5.12, the model (8.1) has the coanalytic spectral factor $\bar{W}$. Then an argument symmetric to that used in proving statement (i) shows that there is a unique coprime factorization (8.23) and that $\bar{W}_0$ is the coanalytic spectral factor of X_0. Since X_0 is internal, it has the structural function $K_0 := \bar{W}_0^{-1} W_0$, and hence statement (iii) has been established. Given (8.24), Theorem 5.10 implies that $x_0(0)$ as defined by (8.5), is a basis in X_0, and from Theorem 5.12 we see that there are unique matrices C_0 and D_0 so that (8.25b) holds. Since $\hat{X} \subset X_0$, there is also a unique matrix H such that $\hat{x}(0) = Hx_0$, i.e. (8.8b) holds, in terms of which we may write V as (8.25a). Consequently, $\hat{x}$ is given by the filter (8.8). Finally, given A_0 and B_0, standard theory of canonical forms show that H, C_0 and D_0 are uniquely determined by (8.24). □

9. The tightest local frame

In this section we provide some further insight into the role played by the nonminimal Markovian splitting subspace X_0, introduced in Section 8. From Theorem 8.1 we see that the complexity of the filter (8.8) depends on the dimension of X_0. We know from (4.20) that

$$X_0 \subset H^\square \tag{9.1}$$

and that the dimension of the frame space $H^\square$ is $2n$. Moreover in [5] $H^\square$ plays the same role as does X_0 in this paper, being the Markovian space of the smoothing estimate. Therefore, if $\dim X_0 < \dim H^\square$, we have reduced the complexity of the corresponding filter. As it turns out, this is the case in many interesting situations.

As we saw in Section 6, there is a partial ordering under which any minimal Markovian splitting subspace X satisfies

$$X_- < X < X_+ \tag{9.2}$$

where X_- is the predictor space and X_+ is the backward predictor space. The frame space

$$H^\square = X_- \vee X_+ \tag{9.3}$$

is the linear convex hull of all internal X, and X_- is the constructible and X_+ the observable subspace of $H^\square$. As we have seen above, the ordering (9.2) induces the ordering

$$P_- \leq P \leq P_+ \tag{9.4}$$

of state covariances under any uniform choice of bases. Here we shall investigate under which conditions the bounds (9.2) and (9.4) can be tightened about X and P respectively while retaining the basic structure of ordering.

To this end we first note that, according to Theorem 6.11, to each $X \in \mathcal{X}$ there corresponds $X_{0-}, X_{0+} \in \mathcal{X}_0$, so that

$$X_{0-} < X < X_{0+} \tag{9.5}$$

is the tightest possible bounding of X. Moreover, we recall from Theorem 8.1 that

$$X_0 = X_{0-} \vee X_{0+} \tag{9.6}$$

is the state space of the noncausal estimator of smallest possible dimensions and that X_{0-} is the constructible and X_{0+} the observable subspace of X_0. This suggests that X_0, X_{0-} and X_{0+} locally play the same role as globally played by $H^\square, X_-$ and X_+. For this reason we shall call X_0 the *local frame space* of X and the subfamily of all $X \in \mathcal{X}$ satisfying (9.5) the *tightest local frame* of X. Isomorphically, we shall call the subfamily of all $P \in \mathcal{P}$ satisfying the inequality

$$P_{0-} \leq P \leq P_{0+} \tag{9.7}$$

where P_{0-} and P_{0+} correspond to X_{0-} and X_{0+} respectively, the *tightest local frame* of P.

Let us now introduce some notations. For any P_1 and P_2 in $\mathcal{P}_0$ let

$$[P_1, P_2] := \{P \in \mathcal{P} \mid P_1 \leq P \leq P_2\} \tag{9.8}$$

Then $[P_1, P_2]$ is nonempty if and only if $P_1 \leq P_2$. Moreover, let (P_1, P_2) be the subset of $[P_1, P_2]$ consisting of those P having the property that, for all nonzero $a \in Im(P_2 - P_1)$, $a^{'}(P - P_1)a > 0$ and $a^{'}(P_2 - P)a > 0$ hold simultaneously. Analogous notations will be used for $X \in \mathcal{X}$.

We shall now state the main results of this section which characterize tightness and provide a formula for the dimension of the local frame space.

THEOREM 9.1 *Let* $P \in [P_1, P_2]$ *and define*

$$\mathcal{V} := Im(P_2 - P_1) \tag{9.9a}$$

$$\mathcal{V}_1 := Im(P - P_1) \tag{9.9b}$$

$$\mathcal{V}_2 := Im(P_2 - P) \tag{9.9c}$$

Then $[P_1, P_2]$ *is the tightest local frame of* P *if and only if* $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}$.

The following is the alternative formulation of this theorem.

THEOREM 9.1$^{'}$. *The family* $[P_1, P_2]$ *is the tightest local frame of* P *if and only if* $P \in (P_1, P_2)$.

The proof of these statements will follow from a series of lemmas which will, at the same time, provide a constructive proof of the following theorem.

THEOREM 9.2. *The dimension of the local frame space of* X *is given by*

$$\dim X_0 = \dim X + \frac{1}{2} \deg Q^* Q \tag{9.10}$$

where $Q := W_-^{-1} W$ *and* W *is the analytic spectral factor of* X. *Equation (9.10) also holds with* Q *replaced by* $\bar{Q} := \bar{W}_+^{-1} \bar{W}$, $\bar{W}$ *being the coanalytic spectral factor.*

Before proceeding with the proofs, a few comments on the last result are in order. Note that $Q^* Q = W^\sharp W$ is the projector mentioned in Section 5.3, i.e. the spectral-domain version of the projector E^{H_0}. In Sections 10 and 11 we shall give alternative formulas for the dimension of X_0 which involve a characterization of the extent to which external noise enter into X as well as a characterization of the zeros of W. The degree of $Q^* Q$ varies between zero, when X is internal and hence $Q^* Q = I$, and n, which correspond to a maximal influence from external noise.

LEMMA 9.3. *Let* $P \in [P_1, P_2]$, *and let* $\mathcal{V}, \mathcal{V}_1, \mathcal{V}_2$ *be defined as in (9.9). Then* $\mathcal{V}_1 \subset \mathcal{V}$ *and* $\mathcal{V}_2 \subset \mathcal{V}$.

Proof: We shall use a standard argument; see e.g. [5]. Let $\hat{x}(0) := E^{X_1 \vee X_2} x(0)$, where $x(0)$ is the basis in X corresponding the uniform choice of basis (6.10). Then there are matrices L_1 and L_2 such that

$$\hat{x}(0) = L_1 x_1(0) + L_2 x_2(0) \tag{9.11}$$

where $x_1(0)$ and $x_2(0)$ are the corresponding bases in X_1 and X_2. Then the components of $[x(0) - \hat{x}(0)]$ are orthogonal to $X_1 \vee X_2$ and hence (i) to X_1 and (ii) to X_2. Statement (i) implies that

$$E\{x(0)x_1(0)^{'}\} = L_1 E\{x_1(0)x_1(0)^{'}\} + L_2 E\{x_2(0)x_1(0)^{'}\} \tag{9.12}$$

But, by Proposition 6.12, $X_1 < X$ implies that $E\{x(0)x_1(0)^{'}\} = E\{x_1(0)x_1(0)^{'}\} = P_1$, and therefore (9.12) is equivalent to $P_1 = L_1 P_1 + L_2 P_1$. But $P_1 > 0$, so

$$L_1 + L_2 = I \tag{9.13}$$

In the same way, statement (ii) is equivalent to

$$P = L_1 P_1 + L_2 P_2 \tag{9.14}$$

which together with (9.13) yields

$$P - P_1 = L_2(P_2 - P_1) \tag{9.15}$$

implying that $\mathcal{V}_1 \subset \mathcal{V}$, and

$$P_2 - P = L_1(P_2 - P_1) \tag{9.16}$$

from which $\mathcal{V}_2 \subset \mathcal{V}$ follows. □

Lemma 9.4. *Let $X_1, X_2 \in \mathcal{X}_0$ satisfy $X_1 < X_2$, and let $\mathcal{V}$ be defined as in (9.9a). Then*

$$\dim(X_1 \vee X_2) = n + \dim \mathcal{V}$$

where $n := \dim X_1 = \dim X_2$.

Proof. By Lemma 8.2, $X_1 \vee X_2 \sim (S_2, \bar{S}_1)$ is a Markovian splitting subspace. Let $(K, Q, \bar{Q})$ denote its inner triplet and let $(K_i, Q_i, \bar{Q}_i), i = 1, 2$, be the corresponding triplets for X_1 and X_2. Then, clearly, $Q = Q_2$ and $\bar{Q} = \bar{Q}_1$, which together with

$$T_0 = \bar{Q} K Q^* = \bar{Q}_1 K_1 Q_1^* \tag{9.17}$$

[see (5.35)] yields $K = K_1 J$, where $J := Q_1^* Q_2$ is unitary on the imaginary axis. Since $X_1 < X_2$, $S_1 \subset S_2$ (Theorem 6.8), and hence $H^2 J \subset H^2$, J is also analytic and therefore inner. From the fact that K, K_1 and J are all inner and hence the factorization $K = K_1 J$ in minimal we deduce that

$$\deg K = \deg K_1 + \deg J \tag{9.18}$$

Since $\dim X_1 = \deg K_1$ and $\dim(X_1 \vee X_2) = \deg K$ (Theorem 5.4), the conclusion of the lemma will follow as soon as we show that $\deg J = \dim \mathcal{V}$. This will be the content of the next lemma. □

Lemma 9.5. *Let $P \in \mathcal{P}$ and $P_0 \in \mathcal{P}_0$ satisfy $P \leq P_0$, and set $\mathcal{V}_0 := Im(P_0 - P)$. Then*

$$\dim \mathcal{V}_0 = \deg Q^* Q_0 \tag{9.19}$$

where $Q := W_-^{-1} W$ and $Q_0 := W_-^{-1} W_0$, W and W_0 being the analytic spectral factors of X and X_0 respectively.

Proof. Since $d\hat{y} = W d\hat{w} = W_0 d\hat{u}_0$, $d\hat{u}_0 = Q_0^* Q d\hat{w}$. We shall construct a stochastic realization of the all-pass filter

$$\xrightarrow{dw} \boxed{Q_0^* Q} \xrightarrow{du_0} \tag{9.20}$$

Subtracting the (forward) realization of X, written, without loss of generality, in the form

$$\begin{cases} dx_0 = Ax_0 dt + B_0 du_0 \\ dy = Cx_0 dt + R^{1/2} du_0 \end{cases} \tag{9.21}$$

we obtain the following representation of (9.20):

$$\begin{cases} dz = \Gamma_0 z dt + \tilde{B} dw \\ du_0 = -R^{-1/2} C z dt + (I, 0) dw \end{cases} \tag{9.22}$$

where $z := x_0 - x$, $\Gamma_0 := A - B_0 R^{-1/2} C$, and $\tilde{B} := -(B_1 - B_0, B_2)$. Using (7.17) we see that

$$\tilde{B}\tilde{B}' = (P_0 - P)C' R^{-1} C (P_0 - P) + B_2 B_2' \tag{9.23}$$

Since (C, A) is observable, and hence also (C, Γ_0), computing the degree of $Q_0^* Q$ (and hence of $Q^* Q_0$) amounts to computing the dimension of the reachable subspace of the realization (9.22). Here it should be noted that, since

$$E|a' z(0)|^2 = a'(P_0 - P)a = 0 \quad \text{for all} \quad a \perp \mathcal{V}_0, \tag{9.24}$$

$a' z$ is nonzero only for $a \in \mathcal{V}_0$, and therefore the representation (9.22) is in general nonminimal. We now proceed to show that $\mathcal{V}_0$ is actually the reachable subspace, thus proving the lemma. To this end, subtract from (7.18) the algebraic Riccati equation $\Lambda(P_0) = 0$, which after some rearranging of terms yields

$$(-\Gamma_0)(P_0 - P) + (P_0 - P)(-\Gamma_0)' + (P_0 - P)C' R^{-1} C (P_0 - P) + B_2 B_2' = 0 \tag{9.25}$$

Using the argument of [42, Lemma A.1], which will also be reported in Lemma 10.2 below, one can show that $Im B_2 \in \mathcal{V}_0$ and $\Gamma_0 \mathcal{V} \subset \mathcal{V}_0$. From (9.23) and (9.25) we obtain the Lyapunov equation

$$(-\Gamma_0) Z + Z(-\Gamma_0)' + \tilde{B}\tilde{B}' = 0 \tag{9.26}$$

for $Z := P_0 - P$. Noting that $Im \tilde{B} \in \mathcal{V}_0$, we can now restrict (9.26) to $\mathcal{V}_0$. Since $Z_{|\mathcal{V}_0} > 0$, it follows that $\mathcal{V}_0$ is indeed the reachable subspace for $(\Gamma_0, \tilde{B})$. □

LEMMA 9.6. *Let $X_1, X_2, X_3, X_4 \in \mathcal{X}_0$ be such that*

$$X_1 < X_2 < X_3 < X_4 \tag{9.27}$$

Then $X_2 \vee X_3$ and $X_1 \vee X_4$ are Markovian splitting subspaces such that

$$X_2 \vee X_3 \subset X_1 \vee X_4 \tag{9.28}$$

with proper inclusion if and only if at least one of the two conditions $X_1 \neq X_2, X_3 \neq X_4$ *holds, in which case*

$$\dim(X_2 \vee X_3) < \dim(X_1 \vee X_4) \tag{9.29}$$

Proof. It follows from Lemma 8.2 that $X_2 \vee X_3$ and $X_1 \vee X_4$ are Markovian splitting subspaces, and that $X_2 \vee X_3 \sim (S_3, \bar{S}_2)$ and $X_1 \vee X_4 \sim (S_4, \bar{S}_1)$. Consequently, since $X_3 < X_4$ and $X_1 < X_2$ imply that $S_3 \subset S_4$ and $\bar{S}_2 \subset \bar{S}_1$ (Theorem 6.8), we have, by Theorem 4.1,

$$X_2 \vee X_3 = S_3 \cap \bar{S}_2 \subset S_4 \cap \bar{S}_1 = X_1 \vee X_4 \tag{9.30}$$

Since the correspondence $X \sim (S, \bar{S})$ is one-one, we will have proper inclusion in (9.30) precisely when at least one of the inclusions $S_3 \subset S_4, \bar{S}_2 \subset \bar{S}_1$ is strict. The last statement, (9.29), then follows from finite dimensionality. ☐

Proof of Theorem 9.1: Suppose $[P_1, P_2]$ is the tightest local frame of P, and let X_0 be the local frame space. Then $X_1 = X_{0-}$ and $X_2 = X_{0+}$. Moreover, in view of (8.15) of Theorem 8.1, Lemma 9.4 implies that

$$\dim X_0 = n + \dim \mathcal{V} \tag{9.31}$$

On the other hand, by construction, $\dim X_0$ equals the degree of the transfer function $V(s)$ of the noncausal estimator in Section 8. In view of (8.22), $\deg V \leq n + \deg U$, and consequently,

$$\dim \mathcal{V} \leq \deg U \tag{9.32}$$

But, by (8.33) in Lemma 8.7, $U = Q^* Q_0$ where $Q_0 := W_-^{-1} W_0$, W_0 being the analytic spectral factor of X_{0+}. Since $P \leq P_{0+} = P_2$, Lemma 9.5 leads us to the conclusion that

$$\deg U = \dim \mathcal{V}_2 \tag{9.33}$$

Hence, $\dim \mathcal{V} \leq \dim \mathcal{V}_2$, which, in view of the fact that $\mathcal{V}_2 \subset \mathcal{V}$ (Lemma 9.3), implies that $\mathcal{V}_2 = \mathcal{V}$.

The same idea of proof applied to the backward setting yields the backward counterparts of (9.32) and (9.33), namely,

$$\dim \mathcal{V} \leq \deg \bar{U} = \operatorname{rank}(\bar{P}_1 - \bar{P}), \tag{9.34}$$

where $\bar{P} = P^{-1}$ and $\bar{P}_1 = P_1^{-1}$; see Theorem 5.9 and (7.12). But

$$\bar{P}_1 - \bar{P} = P_1^{-1}(P - P_1)P^{-1} \tag{9.35}$$

has the same rank as $(P - P_1)$ and therefore $\dim \mathcal{V} \leq \dim \mathcal{V}_1$. But, by Lemma 9.3, $\mathcal{V}_1 \subset \mathcal{V}$, and consequently, $\mathcal{V}_1 = \mathcal{V}$.

Conversely, suppose that $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}$. We want to show that $[P_1, P_2]$ is the tightest local frame of P. Suppose this is not the case and, say, the lower bound is not tight so that there is an $X_1' \in \mathcal{X}_0$ with $X_1' \neq X_1$ such that

$$X_1 < X_1' < X < X_2. \tag{9.36}$$

Then, by Lemma 9.6, $\dim(X_1' \vee X_2) < \dim(X_1 \vee X_2)$, and consequently, by Lemma 9.4, $\dim \mathcal{V}' < \dim \mathcal{V}$, where $\mathcal{V}' := Im(P_2 - P_1')$. However, Lemma 9.3 implies that $\mathcal{V}_2 \subset \mathcal{V}'$, and therefore, since $\mathcal{V}_2 = \mathcal{V}$ by assumption, we have a contradiction. A symmetric argument shows that nontightness of the upper bound leads to a contradiction also. □

Proof of Theorem 9.1$'$. Given that $P \in [P_1, P_2]$, the condition $P \in (P_1, P_2)$ is equivalent to the matrices $(P_2 - P_1), (P - P_1)$ and $(P_2 - P)$ having the same rank, which in turn is equivalent to $\dim \mathcal{V} = \dim \mathcal{V}_1 = \dim \mathcal{V}_2$. But, by Theorem 9.3, $\mathcal{V}_1 \subset \mathcal{V}$ and $\mathcal{V}_2 \subset \mathcal{V}$, and therefore $[P_1, P_2]$ is the tightest local frame if and only if $P \in (P_1, P_2)$. □

Proof of Theorem 9.2. Since $U(s)$ $[\bar{U}(s)]$ is an outer [conjugate outer] spectral factor of Q^*Q $[\bar{Q}^*\bar{Q}]$, we have $\deg U = \frac{1}{2} \deg Q^*Q$ and $\deg \bar{U} = \frac{1}{2} \deg \bar{Q}^*\bar{Q}$. But, it follows from the proof of Theorem 9.1, that

$$\deg U = \dim \mathcal{V} = \deg \bar{U} \tag{9.37}$$

and hence $\deg \bar{Q}^*\bar{Q} = \deg Q^*Q$. Then, (9.10) follows (9.31). □

10. Geometry of the Riccati inequality

Recall from Section 7 that $\mathcal{P}$ is a closed, bounded, convex subset of the vector space $\mathbb{S}^n$ of real symmetric $n \times n$ matrices and the solution set of the algebraic matrix inequality

$$\Lambda(P) \le 0 \tag{10.1}$$

where Λ is defined in terms of the spectral density Φ by (7.14) and (7.5), and that $\mathcal{P}_0 \subset \mathcal{P}$ is the solution set of the algebraic Riccati equation

$$\Lambda(P) = 0. \tag{10.2}$$

In this section we study the local geometric structure of $\mathcal{P}$ and compute the tightest local frame for any $P \in \mathcal{P}$.

10.1. The local structure of $\mathcal{P}$

The following theorem shows, not surprisingly, that locally the subset $[P_1, P_2]$ of $\mathcal{P}$, defined in Section 9, has the same geometric structure as $\mathcal{P}$.

THEOREM 10.1. *For any $P_1, P_2 \in \mathcal{P}_0$ such that $P_1 \le P_2$,*

$$[P_1, P_2] = \mathcal{L} \cap \mathcal{P}, \tag{10.3}$$

where $\mathcal{L}$ is the affine subspace of $\mathbb{S}^n$

$$\mathcal{L} = \{P \mid Im(P - P_1) \subset Im(P_2 - P_1)\}. \tag{10.4}$$

In particular, $[P_1, P_2]$ is a closed, convex, bounded subset of $\mathcal{P}$, which either is $\mathcal{P}$ itself (if $P_1 = P_-$ and $P_2 = P_+$) or lies in the boundary of $\mathcal{P}$.

The study of the relationship between any $P \in \mathcal{P}$ and a $P_0 \in \mathcal{P}_0$ requires some preliminary analysis. Recall that $\Lambda(P) = -B_2 B_2'$, where B_2 is the matrix defined by (7.16) and that $\Lambda(P_0) = 0$. Subtracting the latter of these equations from the former yields the following equation for $Z := P - P_0$:

$$\Gamma(P_0)Z + Z\Gamma(P_0)' + ZC'R^{-1}CZ + B_2B_2' = 0 \tag{10.5}$$

where

$$\Gamma(P) := A - (\bar{C} - CP)'R^{-1}C \tag{10.6}$$

We recall from the literature [10, p.87] that, under the standard coercivity assumption of this paper, there is exactly one $P \in \mathcal{P}_0$ for which $\Gamma(P)$ is a stability matrix, namely the minimal element P_-, and exactly one $P \in \mathcal{P}_0$, namely the maximal element P_+, for which $\Gamma(P)$ is antistable. For any other $P \in \mathcal{P}_0$, the spectrum of $\Gamma(P)$ is contained in the union of the spectra of $\Gamma(P_-)$ and $\Gamma(P_+)$, and hence there are no eigenvalues on the imaginary axis. Moreover, we have the following lemma, adopted from [42].

LEMMA 10.2.(Molinari). *Let* $P \in \mathcal{P}$ *and* $P_0 \in \mathcal{P}_0$, *and set* $\Gamma_0 := \Gamma(P_0)$ *and* $\mathcal{V}_0 := Im(P - P_0)$. *Then,*

(i) $ImB_2 \subset \mathcal{V}_0$

(ii) $\Gamma_0 \mathcal{V}_0 \subset \mathcal{V}_0$

(iii) if $P \geq P_0$, *then* $\Gamma_{0\,|\mathcal{V}_0}$, *the restriction of* Γ_0 *to* $\mathcal{V}_0$, *is asymptotically stable.*

Proof (i) Take $a \perp \mathcal{V}_0$. Then $a'Z = 0$ and, equivalently, $Za = 0$, so that $a'B_2B_2'a = 0$ is obtained from (10.5). Hence, $a \perp ImB_2$. Consequently, we have shown that $\mathcal{V}_0^\perp \subset (ImB_2)^\perp$, which is the same as (i). (ii) Take $a \perp \mathcal{V}_0$. Then, as we have just seen, $Za = 0$ and $B_2'a = 0$, and therefore we have $Z\Gamma_0'a = 0$. Consequently,

$$\Gamma_0'\mathcal{V}_0^\perp \subset \ker Z = \mathcal{V}_0^\perp$$

which is equivalent to (ii). (iii) Because of (i) and (ii) we can now restrict (10.5) to $\mathcal{V}_0$ on which Z is strictly positive definite. The pair (C, Γ_0) is observable, since (C, A) is, and therefore a standard Lyapunov argument yields (iii). □

The following corollary shows that the invariance condition (ii) of Lemma 10.2 can be extended to hold also under $\Gamma(P)$, where $P \in \mathcal{P}_0$, as long as P belongs to the tightest local frame corresponding to the invariant subspace.

COROLLARY 10.3. *Let* $P_1, P_2 \in \mathcal{P}_0$ *be ordered as* $P_1 \leq P_2$, *and set* $\mathcal{V} := Im(P_2 - P_1)$. *Then, if* $P \in (P_1, P_2)$,

$$\Gamma(P)\mathcal{V} \subset \mathcal{V} \tag{10.7}$$

where $\Gamma(P)$ *is defined by (10.6).*

Proof. First note that

$$\Gamma(P) = \Gamma(P_1) + (P - P_1)C'R^{-1}C \tag{10.8}$$

and that $\mathcal{V}_1 := Im(P - P_1) = \mathcal{V}$ (Theorem 9.1 and 9.1$'$). Therefore, since $\Gamma(P_1)\mathcal{V}_1 \subset \mathcal{V}_1$ (Lemma 10.2), (10.7) follows. □

Proof of Theorem 10.1. Let $P \in [P_1, P_2]$, and set $\mathcal{V} := Im(P_2 - P_1)$ and $\mathcal{V}_1 := Im(P - P_1)$. Then, by Lemma 9.3, $\mathcal{V}_1 \subset \mathcal{V}$, and therefore $P \in \mathcal{L}$. Since $P \in \mathcal{P}$ also, $P \in \mathcal{L} \cap \mathcal{P}$. Hence, $[P_1, P_2] \subset \mathcal{L} \cap \mathcal{P}$. Conversely, suppose that $P \in \mathcal{L} \cap \mathcal{P}$ and set $Z := P - P_1$ and $V := P_2 - P_1$. Then Z and V satisfy

$$\Gamma(P_1)Z + Z\Gamma(P_1)' + ZC'R^{-1}CZ + B_2B_2' = 0 \tag{10.9a}$$

$$\Gamma(P_1)V + V\Gamma(P_1)' + VC'R^{-1}CV = 0 \tag{10.9b}$$

Lemma 10.2 applied to (10.9b), shows that $\mathcal{V}$ is invariant for $\Gamma(P_1)$ and that $\hat{\Gamma} := \Gamma(P_1)_{|\mathcal{V}}$ is asymptotically stable. Now, $P \in \mathcal{L}$ is equivalent to $\mathcal{V}_1 \subset \mathcal{V}$ and $P \in \mathcal{P}$ is equivalent to (10.9a). Next we restrict (10.9a) to $\mathcal{V}$, which makes sense because $ImZ = \mathcal{V}_1$ and $ImB_2 \subset \mathcal{V}_1$ (Lemma 10.2(i)). On $\mathcal{V}$ we can thus write (10.9a) as

$$\hat{\Gamma}Z + Z\hat{\Gamma}' + ZC'R^{-1}CZ + B_2B_2' = 0 \tag{10.10}$$

Then, since the sum of the last two terms of (10.10) is nonnegative definite, by Lyapunov theory, the asymptotic stability of $\hat{\Gamma}$ implies that Z is positive definite on $\mathcal{V}$. Therefore, $Z \geq 0$ in $\mathbb{R}^n$, i.e. $P \geq P_1$. A symmetric argument shows that $P_2 \geq P$. Hence $\mathcal{L} \cap \mathcal{P} \subset [P_1, P_2]$. This concludes the proof that $[P_1, P_2] = \mathcal{L} \cap \mathcal{P}$, which clearly is closed, convex and bounded, because $\mathcal{P}$ has these properties and $\mathcal{L}$ is affine. It remains to show that $[P_1, P_2]$ belongs to the boundary of $\mathcal{P}$ whenever $[P_1, P_2]$ is not all of $\mathcal{P}$. To this end, suppose that at least one of the conditions $P_1 = P_-$ and $P_2 = P_+$ is violated. Then rank $(P_2 - P_1) < n$ [10, p.87], and consequently the dimension of $\mathcal{V} := Im(P_2 - P_1)$ is less then n. Now, let $P \in [P_1, P_2]$. Then, since $\mathcal{V}_1 := Im(P - P_1) \subset \mathcal{V}$ (Lemma 9.3), $\dim \mathcal{V}_1 < n$. But, by Lemma 10.2, $ImB_2 \subset \mathcal{V}_1$, and therefore $\Lambda(P) = -B_2B_2'$ does not have full rank, and consequently P belongs to the boundary of $\mathcal{P}$ [10, p.84]. □

10.2. Invariant sets of the Riccati equation and computation of the local frame

The geometric result of Theorem 10.1 suggests that we name $[P_1, P_2]$ the *facet* of $\mathcal{P}$ through P_1 and P_2. The facets are intimately connected to the zero structure of the minimal spectral factors of Φ, as we shall explain in the next section. But, the most important property characterizing them is that they are precisely the invariant sets for the matrix Riccati differential equation

$$\dot{\Pi} = \Lambda(\Pi) \tag{10.11}$$

considered in the positivity region $\mathcal{P}$. The following theorem, which is an amplification of a result in one of our previous papers [25], will make this assertion precise and also as a byproduct will provide an algorithm to compute the extreme points P_{0-} and P_{0+} of the tightest local frame $[P_{0-}, P_{0+}]$ of any $P \in \mathcal{P}$. These two elements of $\mathcal{P}_0$ permit the construction of the noncausal estimator discussed in Section 8. Note that (10.11) is precisely the invariant form of the Riccati equation encountered in Section 7.

THEOREM 10.4. *The facets are precisely the invariant sets of the Riccati differential equation (10.11) in* $\mathcal{P}$. *In particular,*

(i) for every $\Pi(0) := P \in \mathcal{P}$, *the solution exists globally on* R *and* $\Pi(t)$ *belongs to the tightest local frame* $[P_{0-}, P_{0+}]$ *of* P *for all* $t \in \mathrm{R}$

(ii) for any $t_1, t_2 \in \mathrm{R}$, $t_1 \leq t_2$, $\Pi(t_1) \ \geq \ \Pi(t_2)$

(iii) $\lim_{t\to\infty} \Pi(t) = P_{0-}$ *and* $\lim_{t\to-\infty} \Pi(t) \to P_{0+}$

For the proof we shall need the following lemma which is based on a simple computation underlying the work in [16], and less directly also in [22].

LEMMA 10.5. *Any solution of (10.11) satisfies the systems of equations*

$$\begin{aligned} \dot{\Pi} &= U\dot{\Pi}(0)U' && (10.12a) \\ \dot{U} &= \Gamma(\Pi)U\ ; \quad U(0) = I && (10.12b) \end{aligned}$$

where the mapping $P \to \Gamma(P)$ *is defined by (10.6).*

Proof Differentiate (10.11) and order the terms to obtain

$$\ddot{\Pi} = \Gamma(\Pi)\dot{\Pi} + \dot{\Pi}\Gamma(\Pi)' \tag{10.13}$$

from which (10.12) follows after integration. □

Proof of Theorem 10.4: If $\Pi(0) := P \in \mathcal{P}$, then $\dot{\Pi}(0) = \Lambda(P) \leq 0$. Therefore, by Lemma 10.5, $\dot{\Pi}(t) \leq 0$, i.e.

$$\Lambda(\Pi(t)) \leq 0 \tag{10.14}$$

for all $t \in \mathrm{R}$. Hence, the trajectory $\{\Pi(t); t \in \mathrm{R}\}$ stays in $\mathcal{P}$ and cannot escape. From (10.14) we also deduce that $\Pi(t_1) \geq \Pi(t_2)$ for any $t_1, t_2 \in \mathrm{R}$ such that $t_1 \leq t_2$, and consequently, since $\mathcal{P}$ is closed and bounded, $\Pi(t)$ tends monotonically to a limit $\Pi_\infty \in \mathcal{P}$ as $t \to \infty$ and a limit $\Pi_{-\infty} \in \mathcal{P}$ as $t \to -\infty$. Clearly Π_∞ and $\Pi_{-\infty}$ are equilibria for (10.11) and thus belong to $\mathcal{P}_0$. Moreover, $\Pi_\infty \leq \Pi(t) \leq \Pi_{-\infty}$, i.e.

$$\Pi(t) \in [\Pi_\infty, \Pi_{-\infty}] \tag{10.15}$$

for all $t \in \mathrm{R}$. It remains to show that $[\Pi_\infty, \Pi_{-\infty}]$ is the tightest local frame of P. To this end, let $[P_{0-}, P_{0+}]$ denote the tightest local frame of P, and set $\mathcal{V} := Im(P_{0+} - P_{0-})$. We want to show that the trajectory $\{\Pi(t); t \in \mathrm{R}\}$ never leaves the affine space

$$\mathcal{L} = \{P \mid Im(P - P_{0-}) \subset \mathcal{V}\}$$

i.e. that, with $Z(t) := \Pi(t) - P_{0-}$, $ImZ(t) \in \mathcal{V}$ for all $t \in \mathrm{R}$. A calculation similar to the one leading to (10.9a) shows that Z satisfies the differential equation

$$\dot{Z} = \Gamma(P_{0-})Z + Z\Gamma(P_{0-})' + ZC'R^{-1}CZ \tag{10.16}$$

Now, since $Z(0) = P - P_{0-}$, $ImZ(0) \subset \mathcal{V}$ by construction (Theorem 9.1), and $Im\dot{Z}(0) \subset \mathcal{V}$ by (10.16) and Lemma 10.1 (ii). Then, by Nagumo's Theorem [4], the trajectory $Z(t)$

stays inside the closed subset $\mathcal{L}$ of S^n at least locally. But (10.11), and hence (10.16), has a global solution on R, and therefore $\Pi(t) \in \mathcal{L}$, for all $t \in \mathsf{R}$. But, $\Pi(t) \in \mathcal{P}$, and therefore $\Pi(t) \in [P_{0-}, P_{0+}]$ for all $t \in \mathsf{R}$ (Theorem 10.1). Since $[P_{0-}, P_{0+}]$ is closed (Theorem 10.1), Π_∞ and $\Pi_{-\infty}$ belong to $[P_{0-}, P_{0+}]$, and therefore $[\Pi_\infty, \Pi_{-\infty}] \subset [P_{0-}, P_{0+}]$. But, we showed above that $P = \Pi(0) \in [\Pi_\infty, \Pi_{-\infty}]$, and, therefore, since $[P_{0-}, P_{0+}]$ is the tightest local frame, we must have $\Pi_\infty = P_{0-}$ and $\Pi_{-\infty} = P_{0+}$ as required. This concludes the proof. An alternative proof can be constructed by using the method of Lemma 6.3 in [25] restricted to $\mathcal{V}$. □

The following corollary is a slight amplification of Theorem 6.2 in [25].

Corollary 10.6. *Let $P \in \mathcal{P}$, and let (B_1, B_2) satisfy (7.15) and (7.16). Let $\{\Pi(t);\ t \in \mathsf{R}\}$ be the unique trajectory of (10.11) through $\Pi(0) = P$, and, for each $t \in \mathsf{R}$, let $\{(B_1(t), B_2(t));\ t \in \mathsf{R}\}$ be the unique solution of the system of differential equations*

$$\dot{B}_1 = -B_2 B_2' C' R^{-1/2} \qquad B_1(0) = B_1 \tag{10.17a}$$

$$\dot{B}_2 = (A - B_1 R^{-1/2} C) B_2 \qquad B_2(0) = B_2 \tag{10.17b}$$

Then, for each $t \in \mathsf{R}$, $\Pi(t) \in \mathcal{P}$ and

$$B_1(t) = \hat{B}(\Pi(t)) := [\bar{C} - C\Pi(t)]' R^{-1/2} \tag{10.18a}$$

$$B_2(t) B_2(t)' = -\Lambda(\Pi(t)) \tag{10.18b}$$

Moreover, $(B_1(t), B_2(t))$ tends to $(B_{0-}, 0)$, as $t \to \infty$ and to $(B_{0+}, 0)$ as $t \to -\infty$, where $B_{0-} := \hat{B}(P_{0-})$ and $B_{0+} := \hat{B}(P_{0+})$, $[P_{0-}, P_{0+}]$ being the tightest local frame of P.

Proof. Inserting $\dot{\Pi}(0) = \Lambda(P) = -B_2 B_2'$ in (10.12a) yields

$$\dot{\Pi} = -B_2(t) B_2(t)' \tag{10.19}$$

where $B_2(t) := U(t) B_2$. Therefore, defining $B_1(t)$ by (10.18a), $(B_1(t), B_2(t))$ is immediately seen to satisfy (10.17) (Lemma 10.5). Moreover, (10.18b) follows from (10.19) and (10.11). Since (10.11) has a unique solution, then so does (10.17). The convergence, finally, is an immediate consequence of the corresponding statement of Theorem 10.3, recalling that $B_2 = 0$ if and only if $P \in \mathcal{P}_0$ (Section 7). □

Consequently, given any minimal analytic spectral factor

$$W(s) = C(sI - A)^{-1}(B_1, B_2) + (R^{1/2}, 0) \tag{10.20}$$

the system of differential equations (10.17) in Corollary 10.6 generates a family $\{W_t(s);\ t \in \mathsf{R}\}$ of minimal spectral factors

$$W_t(s) = C(sI - A)^{-1}(B_1(t), B_2(t)) + (R^{1/2}, 0) \tag{10.21}$$

the corresponding Markovian splitting subspaces of which are totally ordered between X_{0-} and X_{0+}. As we shall see in the next section, all these spectral factors have the same zeros.

11. The zero structure of minimal spectral factors

Recall that $\mathcal{P}$ is not only a parameter set for $\mathcal{X}$ (Theorem 6.5) but also for the (equivalence classes) of minimal spectral factors (Theorem 7.3). In this section we shall show that the geometric structure of $\mathcal{P}$ is reflected in the zero structure of the family of minimal spectral factors. The main results are that the dimension of the local frame space X_0 is determined by the number of zeros and also by the dimension of internal part $X \cap H_0$ of the corresponding Markovian splitting subspace X.

11.1. Zeros and facets

As is well-known, the zeros of any $m \times p$ spectral factor

$$W(s) = C(sI - A)^{-1}B + D, \tag{11.1}$$

with (A, B, C, D) minimal, are precisely the complex numbers λ for which the rank of the system matrix

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \tag{11.2}$$

drops below its normal rank. For square spectral factors with D invertible, which correspond to internal realizations, the zeros are just the poles of the inverse

$$W(s)^{-1} = -D^{-1}C(sI - A - BD^{-1}C)^{-1}BD^{-1} + D^{-1} \tag{11.3}$$

and consequently the eigenvalues of the feedback matrix $\Gamma := A - BD^{-1}C$. In general, when W is not necessarily square, setting as usual,

$$\begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ R^{1/2} & 0 \end{bmatrix} \tag{11.4}$$

in the standard form of Section 7.3, and recalling from (10.6) and (7.17) that the feedback matrix is

$$\Gamma := A - B_1 R^{-1/2} C \tag{11.5}$$

we have the following result.

Theorem 11.1. *Let $P \in \mathcal{P}$, let $\mathcal{V} := Im(P_{0+} - P_{0-})$ where $[P_{0+}, P_{0+}]$ is the tightest local frame of P, let*

$$W(s) = C(sI - A)^{-1}(B_1, B_2) + (R^{1/2}, 0) \tag{11.6}$$

be the corresponding minimal spectral factor in standard form, and let Γ be the feedback matrix (11.5) corresponding to P. Then $\mathcal{V}$ equals the reachability space of the pair (Γ, B_2), i.e.

$$\mathcal{V} = \langle \Gamma | B_2 \rangle := Im(B_2, \Gamma B_2, \Gamma^2 B_2, \ldots) \tag{11.7}$$

and the zeros of W are precisely the eigenvalues of the restricted matrix

$$\Gamma'_{|\mathcal{V}^\perp} : \mathcal{V}^\perp \to \mathcal{V}^\perp \tag{11.8}$$

counted with multiplicity. The dimension of the corresponding local frame space X_0 equals $2n$ minus the number of zeros.

Proof. The zeros of $W(s)$ are the λ for which

$$(a^*, b^*) \begin{bmatrix} A - \lambda I & B_1 & B_2 \\ C & R^{1/2} & 0 \end{bmatrix} = 0 \tag{11.9}$$

for some non-zero $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^{n+m}$, i.e.

$$\begin{cases} a^*(\lambda I - A) - b^* C = 0 \\ a^* B_1 + b^* R^{1/2} = 0 \\ a^* B_2 = 0 \end{cases} \tag{11.10}$$

Eliminating b in (11.10) we see that λ is a zero of W if and only if

$$a^*[\lambda I - \Gamma, B_2] = 0 \tag{11.11}$$

for some nonzero $a \in \mathbb{C}^n$, which is equivalent to

$$a \perp \langle \Gamma | B_2 \rangle \quad \text{and} \quad a^* \Gamma = \lambda a^* \tag{11.12}$$

From this we see that the zeros of W are precisely the eigenvalues of Γ' with generalized eigenspace orthogonal to $\langle \Gamma | B_2 \rangle$. Therefore the number of zeros of W (counted with multiplicity) equals $n - \dim \langle \Gamma | B_2 \rangle$. It remains to show that $\langle \Gamma | B_2 \rangle = \mathcal{V}$. By Corollary 10.3, $\Gamma \mathcal{V} \subset \mathcal{V}$ and therefore, since $\text{Im} B_2 \subset \mathcal{V}$ (Theorem 9.1 and Lemma 10.2), $\langle \Gamma | B_2 \rangle \in \mathcal{V}$. We shall show that $\langle \Gamma | B_2 \rangle$ and $\mathcal{V}$ have the same dimensions and hence are equal. By Lemma 9.5 and Theorem 9.1, $\dim \mathcal{V} = \deg U$, where $U = Q^* Q_0$ (Lemma 8.6). From Proposition 8.6 we see that U^* has the coprime factorization $U^* = \hat{M}^{-1} \hat{N}$ and therefore $\dim \mathcal{V}$ equals the degree of the polynomial $\det \hat{M}$. However, by Proposition 8.6, $M = Z \hat{M}$, which implies that

$$\det M = \det Z \, \det \hat{M},$$

and therefore dim $\mathcal{V}$ equals the degree of det M (which is n, since $W_0 = D^{-1} M$ is a square spectral factor) minus the degree of det Z (which equals the number of zeros of W). But, as shown above, this is precisely the dimension of $\langle \Gamma | B_2 \rangle$. Hence we have shown that $\langle \Gamma | B_2 \rangle = \mathcal{V}$. Moreover, since $\mathcal{V}$ is invariant for Γ, $\mathcal{V}^\perp$ is invariant for Γ' and hence it follows from the discussion above that the zeros of W are precisely the eigenvalues of the restricted map (11.8). The statement concerning the local frame space X_0 now follows from (9.31). □

This result could be described by using the language of geometric control theory [54]. In fact, it can be shown that $\mathcal{V}$ is identical to the maximal reachability space $\mathcal{R}^*$ for the realization $[A, (B_1, B_2), C, (R^{1/2}, 0)]$ and that $\mathcal{V}^\perp$ is a particular version of the quotient space $\mathcal{V}^* / \mathcal{R}^*$. For definitions refer to [54, p.125; Problem 5.9]. In this context notice that $\mathcal{V}^* = \mathbb{R}^n$.

As a simple corollary we have yet another proof of he fact that the zero structure of a minimal stochastic realization is carried over to its backward counterpart.

COROLLARY 11.2. *Let W and $\bar{W}$ be the analytic and coanalytic spectral factors of a minimal Markovian representation. Then W and $\bar{W}$ have the same zeros.*

Proof. Choose coordinates such that the forward and the backward realization have the same state process. Then the backward version of (A, B, C) will be $(-PA'P^{-1}, B, \bar{C}P^{-1})$ where $\bar{C}$ is given by (7.6b) [25]. Consequently, using the Lyapunov equation (7.6a), the corresponding backward feedback matrix is seen to be

$$\bar{\Gamma} = \Gamma - B_2 B_2' P^{-1} \tag{11.13}$$

from which it follows that $\langle \bar{\Gamma} | \bar{B}_2 \rangle = \langle \Gamma | B_2 \rangle$, i.e. $\bar{\mathcal{V}} = \mathcal{V}$. Since moreover $Im B_2 \subset \mathcal{V}$, we have

$$\bar{\Gamma}'_{|\bar{\mathcal{V}}^\perp} = \Gamma'_{|\mathcal{V}^\perp} \tag{11.14}$$

and therefore the statement of the corollary is a consequence of the theorem. □

11.2. Zeros as invariants of tightest local frames

Since there is a one-to-one correspondence between $\mathcal{P}$ and the equivalence classes of analytic minimal spectral factors W (Theorem 7.3), under which $\mathcal{P}_0$ corresponds to the square minimal spectral factors, we shall denote by $[W_1, W_2]$ and (W_1, W_2) the subfamilies of spectral factors which correspond to P in $[P_1, P_2]$ and (P_1, P_2) respectively, as defined in Section 8, where of course P_1, P_2 correspond to W_1, W_2. Accordingly, we shall say that $[W_1, W_2]$ is the *tightest local frame* of W if $W \in (W_1, W_2)$; cf. Theorem 9.1$'$.

PROPOSITION 11.3. *Let W be an arbitrary minimal spectral factor, and let $[W_{0-}, W_{0+}]$ be its tightest local frame. Let Γ, Γ_{0-} and Γ_{0+} be the corresponding feedback matrices. Then*

$$\Gamma'_{0-|\mathcal{V}^\perp} = \Gamma'_{|\mathcal{V}^\perp} = \Gamma'_{0+|\mathcal{V}^\perp} \tag{11.15}$$

where $\mathcal{V}$ is defined as in Theorem 11.1.

Proof. By Lemma 10.2 and Corollary 10.3, $\mathcal{V}$ is invariant under Γ_{0-}, Γ and Γ_{0+}. Moreover,

$$\Gamma = \Gamma_{0-} + (P - P_{0-})C'R^{-1}C, \tag{11.16}$$

and, since the image of the second term belongs to $\mathcal{V}$ (Lemma 9.3)

$$a'\Gamma = a'\Gamma_{0-}$$

for each $a \perp \mathcal{V}$, and hence the first equation in (11.15) follows. The second equation follows from a symmetric argument. □

Recall that the feedback matrix Γ can be written

$$\Gamma = A - (\bar{C} - CP)'R^{-1}C \tag{11.17}$$

Proposition 11.3 shows that when P varies over (P_{0-}, P_{0+}), the eigenvalues of Γ corresponding to $\mathcal{V}^{\perp}$ are fixed, and those corresponding to $\mathcal{V}$ vary arbitrarily in a certain subset of the complex plane. This situation corresponds in geometric control theory to the eigenvalues of the feedback matrix being arbitrary in $\mathcal{R}^*$ and fixed in the quotient space $\mathcal{V}^*/\mathcal{R}^*$.

From Theorem 11.1 and Proposition 11.3 we see that all W in (W_{0-}, W_{0+}) have the same zeros and that these belong to the set of common zeros of W_{0-} and W_{0+}. The following theorem is an amplification of this observation.

Theorem 11.4 *The zeros of any W for which $[W_{0-}, W_{0+}]$ is the tightest local frame are precisely the common zeros of W_{0-} and W_{0+}.*

Proof. Let $\bar{W}$ be the unique (modO) coanalytic spectral factor which together with W defines a minimal Markovian triplet. Let $W = D^{-1}N$ and $\bar{W} = \bar{D}^{-1}\bar{N}$ be coprime matrix fraction representations. Then from Proposition 8.6 (and its backward counterpart) , $N = Z\hat{N}$ and $\bar{N} = \bar{Z}\hat{\bar{N}}$ where Z and $\bar{Z}$ have the same zeros (Corollary 11.2). By the same construction, $W_{0+} = W_0 = D^{-1}Z\hat{M}$, where $\hat{M}$ has all its zeros in the right half plane, and $\bar{W}_{0-} = \bar{W}_0 = D^{-1}\bar{Z}\hat{\bar{M}}$, where $\hat{\bar{M}}$ has all its zeros in the left half plane (Proposition 8.6). However, by Corollary 11.2, W_{0-} has the same zeros as $\bar{W}_{0-}$. Consequently, since $\hat{M}$ and $\hat{\bar{M}}$ cannot have common zeros, the common zeros of W_{0-} and W_{0+} are those of Z, which by construction are the zeros of W. □

11.3. Zeros and the internal subspace of X

The zero structure described above is reflected in the splitting geometry through the decomposition

$$X = E^X H_0^{\perp} \oplus X \cap H_0 \tag{11.18}$$

immediately obtained by using formula (4.25). The two components in (11.18) will be called the *external* and the *internal subspace* of X respectively. We recall that the internal minimal realizations can be parametrized by their zero structure, as for example represented by the inner parts Q of their spectral factors. In the noninternal case the zero structure is connected to the internal part of the splitting subspace only. As can be seen from the following theorem, the internal subspace is invariant as X varies over a tightest frame.

Theorem 11.5. *Let X be a minimal Markovian splitting subspace with tightest frame $[X_{0-}, X_{0+}]$. Then its internal subspace is given by*

$$\begin{aligned} X \cap H_0 &= X_{0-} \cap X_{0+} && (11.19)\\ &= \{a' x(0) \mid a \in \mathcal{V}^{\perp}\}, && (11.20) \end{aligned}$$

where $\mathcal{V}$ is defined in Theorem 11.1 and x is the state process corresponding to the choice of coordinates in X under which $\mathcal{V}$ is computed, and its external subspace by

$$\begin{aligned} E^X H_0^{\perp} &= E^X[(H_0 \vee X) \cap H_0^{\perp}] && (11.21)\\ &= \{a' \bar{x}(0) \mid a \in \mathcal{V}\} && (11.22) \end{aligned}$$

where $\bar{x} := P^{-1}x$ *is the state process of the corresponding backward realization. In particular, the dimension of the internal subspace equals the number of zeros* ν *of the corresponding spectral factor. Moreover, the external subspace has the same dimension, namely* $n - \nu$, as $(H_0 \vee X) \cap H_0^{\perp}$.

In the language of geometric control theory the internal subspace of X corresponds to the quotient space $\mathcal{V}^*/\mathcal{R}^*$, while the external subspace corresponds to $\mathcal{R}^*$. Consequently, the maximal reachability space $\mathcal{R}^*$ is also isomorphic to

$$(H_0 \vee X) \cap H_0^{\perp} \tag{11.23}$$

showing that it corresponds to the part of X which "sticks out" from the output-induced subspace H_0.

Proof of Theorem 11.5. We first prove that $X \cap H_0 = X_{0-} \cap X_{0+}$. Theorem 6.10 establishes the connection between $X \sim (S, \bar{S})$, $X_{0-} \sim (S_{0-}, \bar{S}_{0-})$ and $X_{0+} \sim (S_{0+}, \bar{S}_{0+})$. In particular, $S_{0-} = S \cap H_0$ and $\bar{S}_{0+} = \bar{S} \cap H_0$. Therefore, since $X = S \cap \bar{S}$,

$$X \cap H_0 = S_{0-} \cap \bar{S}_{0+} \tag{11.24}$$

However, since

$$\begin{cases} S_{0-} = X_{0-} \oplus \bar{S}_{0-}^{\perp} \\ \bar{S}_{0+} = X_{0+} \oplus S_{0+}^{\perp} \end{cases} \tag{11.25}$$

where $\perp$ is taken with respect to H_0, and since

$$\bar{S}_{0-}^{\perp} \subset S_{0-} \subset S_{0+} \perp S_{0+} \tag{11.26}$$

by perpendicular intersection and the ordering $X_{0-} < X_{0+}$, we have

$$S_{0-} \cap \bar{S}_{0+} = X_{0-} \cap X_{0+} \tag{11.27}$$

and consequently (11.19) follows.

Next, we prove that

$$a^{'}x(0) \in H_0 \iff a \perp \mathcal{V} \tag{11.28}$$

To this end, note that

$$a^{'}x(0) = a^{'}[x(0) - x_{0-}(0)] + a^{'}x_{0-}(0) \tag{11.29}$$

where the two terms are orthogonal (Proposition 6.12) and therefore

$$E|a^{'}[x(0) - x_{0-}(0)]|^2 = a^{'}(P - P_{0-})a. \tag{11.30}$$

If $a \perp \mathcal{V} = Im(P - P_{0-})$, the right member of (11.30) is zero, and consequently, $a^{'}x(0) = a^{'}x_{0-}(0) \in H_0$. Conversely, if $a^{'}x(0) \in H_0$, then by (11.19), $a^{'}x(0) \in X_{0-}$. However, from the ordering $X_{0-} < X$, we have

$$a^{'}x_{0-}(0) = E^{X_{0-}}a^{'}x(0) \tag{11.31}$$

(Proposition 6.12), and consequently $a^{'}x(0) = a^{'}x_{0-}(0)$. From this and (11.30), we have $a'(P - P_{0-})a = 0$, from which it follows that $a \perp \mathcal{V}$. To see this recall that $P - P_{0-}$ is semidefinite and symmetric, and hence it has a rank factorization $P - P_{0-} = VV'$ with the columns of V spanning $\mathcal{V}$. Since $X = \{a'x(0) \mid a \in R^n\}$, this establishes (11.20).

Furthermore, set $T := E^X_{|H_0^\perp}$. Then the nullspace of T is

$$\ker T = H_0^\perp \cap X^\perp = (H_0 \vee X)^\perp, \tag{11.32}$$

and hence T can be restricted to

$$H_0^\perp \ominus \ker T = (H_0 \vee X) \cap H_0^\perp \tag{11.33}$$

making it injective. This proves (11.21) and establishes that $(H_0 \vee X) \cap H_0^\perp$ has the same dimension as the external subspace. To prove (11.22), note that the external subspace is the orthogonal complement of (11.20) in

$$X = \{b'\bar{x}(0) \mid b \in R^n\} \tag{11.34}$$

Since $E\{\bar{x}(0)x(0)'\} = I$, this complement is generated by all b such that $b'a = 0$ for all $a \perp \mathcal{V}$. Consequently (11.22) holds.

Finally, the statements about the connection between dimensions and the number of zeros follow from Theorem 11.1, since $\nu = \dim \mathcal{V}^\perp$. □

Appendices

In these appendices we shall collect some basic facts about stationary increment processes and Hardy spaces which will be used in the geometric theory of stochastic models. A detailed account of this topics can be found in [46], [7] and [34].

A. Stationary increments processes and the continuous-time Wold representation

Let $\{z(t)\}$ be an $m-$dimensional second order process defined on some probability space $\{\Omega, \mathcal{F}, P\}$, continuous in mean square and with stationary increments. Generally speaking, processes with stationary increments are "integrated versions" of the random signals which are being modelled, and the only thing of interest are the increments, so $\{z(t)\}$ is viewed as an equivalence class defined up to an additive fixed random vector z_0. This equivalence class is denoted by dz. Under a very mild *conditional Lipschitz condition,* which is discussed in detail in [34], a stationary increments process admits representations of the type

$$dz(t) = s(t)dt + Ddw(t) \tag{A.1}$$

where $\{s(t)\}$ is stationary, D is a constant $m \times p$ matrix and dw is a $p-$dimensional (wide-sense) *Wiener process*, that is a process with stationary orthogonal increments,

$$E\{[w_i(t) - w_i(s)][w_j(\tau) - w_j(\sigma)]\} = \delta_{ij}|(s,t) \cap (\sigma, \tau)| \tag{A.2}$$

where δ_{ij} is the Kronecker delta and $|\cdot|$ denotes Lebesgue measure on R. We shall write this

$$E\{dwdw'\} = Idt \tag{A.3}$$

for short. Such a process is commonly referred to as (integrated) "white noise". Let $H(dz)$ be, as defined in Section 2, i.e. the Hilbert space [46] generated by the increments of $\{z(t)\}$, i.e. the closure in $L^2(\Omega, \mathcal{F}, P)$ of the linear manifold $\{\Sigma\alpha_{ij}^{'}[z(t_i) - z(t_j)];\ t_i, t_j \in \mathrm{R},\ \alpha_{ij} \in \mathrm{R}^m\}$, where prime denotes transpose. In general, given any subspace K of $H(dz)$, we define the stationary family of translates $\{K_t\}$, of K, by setting $K_t := U_t K,\ t \in \mathrm{R}$ and introduce the *past* and *future* (at the time zero) of the family $\{K_t\}$ by

$$K^- := \vee_{t \leq 0} K_t, \qquad K^+ := \vee_{t \geq 0} K_t \tag{A.4}$$

where the symbol $\vee$ denotes closed vector sum. Clearly, $K_t^- := U_t K^-$ and $K_t^+ := U_t K^+$ form an increasing, respectively, a decreasing family of subspaces of $H(dz)$.

Subspaces K for which $K_t = K_t^-$ or $K_t = K_t^+$ can be characterized in the following way. Introduce the *forward* and *backward shift semigroups* $\{U_t;\ t \geq 0\}$ and $\{U_t^*;\ t \geq 0\}$ acting on $H(dz)$, where U_t is the shift induced by dz, defined in Section 2. It is then easy to check that a subspace K generates an *increasing* stationary family of translates $\{K_t\}$ if and only if

$$U_t^* K \subset K \qquad \text{for all } t \geq 0. \tag{A.5}$$

Similarly, K generates a *decreasing* family of translates $\{K_t\}$ if and only if

$$U_t K \subset K \qquad \text{for all } t \geq 0 \tag{A.6}$$

i.e. K is a *forward shift invariant* subspaces. A subspace satisfying both conditions (A.5), (A.6) will be called a *doubly invariant*.

We shall say that an increasing family $\{K_t\}$ is *purely nondeterministic* (p.n.d) if the "remote past" $K_{-\infty} := \cap_{t \in \mathrm{R}} K_t$ contains only the zero random variable. The property of being p.n.d. depends on the structure of the backward shift invariant subspace K alone. Dually, for a decreasing family $\{K_t\}$ in $H(dz)$, define the "remote future" $\bar{K}_\infty := \cap_{t \in \mathrm{R}} \bar{K}_t$. If $\bar{K}_\infty$ is trivial we say that $\{\bar{K}_t\}$ is p.n.d. or that $\bar{K}$ is a p.n.d. (forward shift) invariant subspace. A stationary increment process dz will be called p.n.d. whenever *both* $H^-(dz)$ and $H^+(dz)$ are p.n.d.

The following representation theorem is essentially a continuous-time version of the Wold representation theorem [21, 38].

Theorem A.1. *A necessary and sufficient condition for a subspace $S \subset H(dz)$ to be backward shift-invariant and p.n.d. is that there is a vector Wiener process dw such that*

$$S = H^-(dw) \tag{A.7}$$

Similarly, a necessary and sufficient condition for a subspace $\bar{S} \subset H(dz)$ to be forward shift-invariant and p.n.d. is that there is a vector Wiener process $d\bar{w}$ such that

$$\bar{S} = H^+(d\bar{w}) \tag{A.8}$$

Both dw and $d\bar{w}$ are uniquely determined by S and $\bar{S}$ modulo multiplication by a constant orthogonal matrix. The dimension of dw is called the multiplicity of S or $H(dw)$ and the dimension of $d\bar{w}$ the multiplicity of $\bar{S}$ or of $H(d\bar{w})$.

Note that whenever $\vee_{t\in \mathrm{R}} S_t = H(dz)$, in which case S is said to be of *full range* , we have a representation of the space $H(dz)$ as

$$H(dz) = H(dw) \tag{A.9}$$

An analogous representation of $H(dz)$ is obtained in the case $\bar{S}$ is full range.

B. Spectral representation of stationary increment processes

Given a p-dimensional Wiener process dw, any $\eta \in H(dw)$ has a unique representation

$$\eta = \int_{-\infty}^{\infty} f(-t)dw(t) \tag{B.1}$$

where $f \in L^2_p(\mathrm{R})$ is row-vector valued and the integral (B.1) is defined in quadratic mean [7, 46]. This is immediately seen by first observing that the η corresponding to the step functions in $L^2_p(\mathrm{R})$ are the finite linear combinations of the increments of dw and hence are dense in $H(dw)$, and then noting that

$$\langle \eta_1, \eta_2 \rangle_{H(dw)} = \langle f_1, f_2 \rangle_{L^2_p(\mathrm{R})} \tag{B.2}$$

under this correspondence. Then taking the closure, we see that (B.1) defines an isometric isomorphism between $L^2_p(\mathrm{R})$ and $H(dw)$.

Moreover, let $\mathcal{F}$ be the unitary map from $L^2_p(\mathrm{R})$ to $L^2_p(\mathrm{I})$, the space of square-integrable p-vector valued functions on the imaginary axis I with Lebesgue measure $d\omega/2\pi$, defined on $L^2 \cap L^1$ by the Fourier integral

$$(\mathcal{F}f)(i\omega) = \hat{f}(i\omega) := \int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt \tag{B.3}$$

Also, the inverse transform

$$(\mathcal{F}^{-1}\hat{f})(t) = \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(i\omega)\frac{d\omega}{2\pi} \tag{B.4}$$

holds on $L^2 \cap L^1$. Then the Plancherel formula

$$\langle f_1, f_2 \rangle_{L^2_p(\mathrm{R})} = \langle \hat{f}_1, \hat{f}_2 \rangle_{L^2_p(\mathrm{I})} \tag{B.5}$$

establishes the isometric isomorphism between $L^2_p(\mathrm{R})$ and $L^2_p(\mathrm{I})$; see, e.g., [9].

Next, define a process $d\hat{w}$ on I with increments

$$\hat{w}(i\omega_2) - \hat{w}(i\omega_1) = \int_{-\infty}^{\infty} \frac{e^{-i\omega_2 t} - e^{-i\omega_1 t}}{2\pi i t} dw(t) \tag{B.6}$$

Then, since

$$\frac{e^{-i\omega_2 t} - e^{-i\omega_1 t}}{2\pi i t} = (\mathcal{F}^{-1} 1_{[\omega_1,\omega_2]})(-t) \tag{B.7}$$

where $1_{[\omega_1,\omega_2]}(i\omega)$ is the indicator function equal to one for $\omega \in [\omega_1, \omega_2]$ and zero otherwise, (B.2) and (B.5) imply that the process $\hat{w}$ has orthogonal increments. In fact,

$$E\{d\hat{w}d\hat{w}^*\} = I\frac{d\omega}{2\pi} \tag{B.8}$$

where * denotes transpose and conjugation. Hence, $d\hat{w}$ is a p-dimensional Wiener process on the imaginary axis. Now, (B.6) may be written

$$\int_{-\infty}^{\infty} 1_{[\omega_1,\omega_2]}(i\omega)d\hat{w}(i\omega) = \int_{-\infty}^{\infty} (\mathcal{F}^{-1}1_{[\omega_1,\omega_2]})(-t)dw(t)$$

and consequently, since the indicator functions are dense in L^2,

$$\int_{-\infty}^{\infty} \hat{f}(i\omega)d\hat{w} = \int_{-\infty}^{\infty} f(-t)dw \tag{B.9}$$

Let $I_w : L^2_p(\mathrm{I}) \to H(dw)$ be the unitary map defined by

$$I_w\hat{f} = \int_{-\infty}^{\infty} \hat{f}(i\omega)d\hat{w}(i\omega) \tag{B.10}$$

Then applying the shift U_t to (B.1) we observe that

$$U_t\eta = \int_{-\infty}^{\infty} f(t-\tau)dw(\tau) \tag{B.11}$$

which together with (B.9) shows that

$$U_tI_w\hat{f} = I_we^{i\omega t}\hat{f} \tag{B.12}$$

Moreover, choosing f to be the indicator function of the interval $[t_1, t_2]$, (B.9) yields

$$w(t_2) - w(t_1) = \int_{-\infty}^{\infty} \frac{e^{i\omega t_2} - e^{i\omega t_1}}{i\omega} d\hat{w}(i\omega) \tag{B.13}$$

which is the spectral representation of dw [7, 46]. More generally, it is known [7], [34], that every R^m-valued process with finite second moments and continuous stationary increments dz admits a *spectral representation*

$$z(t) - z(s) = \int_{-\infty}^{+\infty} \frac{e^{i\omega t} - e^{i\omega s}}{i\omega} d\hat{z}(i\omega) \ \ , \ t, s \in \mathrm{R} \tag{B.14}$$

where $d\hat{z}$ is an n-dimensional orthogonal increments process on the imaginary axis I, called the *spectral measure* of dz, with

$$E\{d\hat{z}(i\omega)d\hat{z}(i\omega)^*\} = dZ(i\omega) \tag{B.15}$$

dZ being a nonnegative definite Hermitian matrix measure on the Borel sets of the imaginary axis (not necesserily finite) called the *spectral distribution* of dz. The spectral measure $d\hat{z}$ is uniquely determined by dz.

As an example consider the process dy defined as the output of the linear stochastic system (3.1). In the time domain (3.1) has the following solution

$$x(t) = \int_{-\infty}^{t} e^{A(t-\tau)} B dw \tag{B.16}$$

$$y(t) - y(s) = \int_{s}^{t} Cx(\tau)d\tau + D[w(t) - w(s)] \tag{B.17}$$

Applying (B.9) to the first of these equations, we obtain

$$x(t) = \int_{-\infty}^{\infty} e^{i\omega t}(i\omega I - A)^{-1} B d\hat{w} \tag{B.18}$$

which then inserted into (B.17) together with (B.13) yields the spectral representation

$$y(t) - y(s) = \int_{-\infty}^{\infty} \frac{e^{i\omega t} - e^{i\omega s}}{i\omega} d\hat{y}(i\omega) \tag{B.19}$$

where W is given by (3.2). Hence dy has a spectral measure

$$d\hat{y} = W(i\omega)d\hat{w}(i\omega) \tag{B.20}$$

with an absolutely continuous spectral distribution

$$E\{d\hat{y}d\hat{y}^*\} = \Phi(i\omega)\frac{d\omega}{2\pi} \tag{B.21}$$

where Φ is the spectral density given by (3.3). This leads to the next topic, namely spectral factorization.

C. Hardy spaces and spectral factorization

The subspaces S and $\bar{S}$, defined by (A.7) and (A.8), consist of random variables with stochastic-integral representations of the type (B.1) in which, in the case of S, f is a *casual* function in $L_p^2(\mathsf{R})$, i.e. $f(t) = 0$ a.e. for $t < 0$ or, in case of $\bar{S}$, an *anticausal* function, for which $f(t) = 0$ a.e. for $t > 0$. Causal and anticausal functions form orthogonal complementary subspaces of L_p^2. In this context it is useful to introduce the *Hardy spaces* H_p^2, $\bar{H}_p^2$ which are the orthogonal subspaces in $L_p^2(\mathsf{I})$ obtained as L^2-Fourier transforms of the causal, respectively anticausal, functions in $L_p^2(\mathsf{R})$. It is well known (see e.g. [15]) that the functions in H_p^2 $[\bar{H}_p^2]$, which we shall always write as p-dimensional *row* vector functions, are the boundary values of analytic functions in the right [left] half of the complex plane. Since there is a unitary isomorphism between analytic (coanalytic) functions and these boundary values [15] it is common usage to refer to functions in H_p^2 as *analytic* and to those in $\bar{H}_p^2$ as *coanalytic*. From this it follows that the subspaces S and $\bar{S}$ in (A.7) , (A.8) naturally correspond to the Hardy spaces H_p^2 and $\bar{H}_p^2$ under the appropriate representation maps (B.10), namely,

$$S = H^-(dw) = I_w H_p^2 \ , \ \bar{S} = H^+(d\bar{w}) = I_{\bar{w}} H_{\bar{p}}^2 \tag{C.1}$$

where p and $\bar{p}$ are the respective multiplicities.

Assume now that the stationary-increment process dz is *purely non-deterministic* in the sense defined in Appendix A. Then, by Theorem A.1 applied to the subspaces $S = H^-(dz)$ and $\bar{S} = H^+(dz)$, there are two Wiener processes, which throughout this paper are denoted $d\bar{u}_-$ and $d\bar{u}_+$, called the *forward* and, respectively, *backward innovation* processes of dz, such that $H^-(dz) = H^-(du_-)$ and $H^+(dz) = H^+(d\bar{u}_+)$. Note that this implies that $H(du_-) = H(dz)$ so that the two Wiener processes have the same dimensions which is called the *multiplicity*, or *rank*, of the process dz. (A stationary increments process is *full rank* if its multiplicity equals its dimensions).

Now, for any $h > 0$, $z(-h) - z(0) \in H^-(du_-)$, and $z(h) - z(0) \in H^+(d\bar{u}_+)$ so that, by (C.1),

$$z(-h) - z(0) = \int_{-\infty}^{+\infty} W_h(i\omega) d\hat{u}_-(i\omega) \tag{C.2}$$

and

$$z(h) - z(0) = \int_{-\infty}^{+\infty} \bar{W}_h(i\omega) d\hat{\bar{u}}_+(i\omega) \tag{C.3}$$

where $d\hat{u}_-$, $d\hat{\bar{u}}_+$ are the spectral measures of $du_-, d\bar{u}_+$ [compare (B.6), (B.11)] and W_h, $\bar{W}_h$ are $m \times r$ *analytic* and, respectively, *coanalytic* matrix functions, i.e. with rows in H^2_r and $\bar{H}^2_r$ respectively. Letting $\chi(i\omega) := \frac{e^{i\omega h}-1}{i\omega}$, $\bar{\chi}(i\omega) := \chi(-i\omega)$ and rewriting (C.2), (C.3) in terms of the new functions

$$W_- := \bar{\chi}_h^{-1} W_h \tag{C.4}$$

$$\bar{W}_+ := \chi_h^{-1} \bar{W}_h \tag{C.5}$$

it follows, by comparison with the spectral representation (B.12), that

$$d\hat{z} = W_- d\hat{u}_- = \bar{W}_+ d\hat{\bar{u}}_+ \tag{C.6}$$

the relations holding by uniqueness of the spectral measure $d\hat{z}$. From this it is easily seen that W_- and $\bar{W}_+$ do not depend on h. It follows from (C.6) that the spectral distribution dZ of a purely nondeterministic stationary increments processes must be absolutely contionuous with a (matrix) spectral density $\Phi := dZ/d(\omega/2\pi)$ satisfying

$$\Phi(i\omega) = W_-(i\omega)W_-(i\omega)^* = \bar{W}_+(i\omega)\bar{W}_+(i\omega)^* \tag{C.7}$$

(almost everywhere) on the imaginary axis. Thus, W_- and $\bar{W}_+$ are *spectral factors* of Φ, i.e. they satisfy the factorization equation

$$\Phi(i\omega) = W(i\omega)W(i\omega)^* \tag{C.8}$$

on I. In fact, W_- and $\bar{W}_+$ are the unique (modO) *outer* and *conjugate outer* spectral factors of Φ. To justify this terminology, recall that vector-valued functions g [$\bar{g}$] on I with the property that $\bar{\chi}_h g \in H^2_r$ [$\chi_h \bar{g} \in \bar{H}^2_r$] belong to the "modified Hardy space" $\mathcal{W}^2_r$ [$\bar{\mathcal{W}}^2_r$], defined in [15] and Section 6 of [34], for which an essentially identical collection of results as in H^2-theory applies. In particular, there is a notion of *analytic* ($\mathcal{W}^2_r$) and *coanalytic* ($\bar{\mathcal{W}}^2_r$) functions which retains, *mutatis mutandis,* the same meaning as for the ordinary H^2-functions. That $W_- \in \mathcal{W}^2_r$ and deserves to be called *outer* and, similarly, $\bar{W}_t \in \bar{\mathcal{W}}^2_r$ is *conjugate outer*, follows from the identities

$$\overline{\text{span}}\{\bar{\chi}_h W_- \; ; \; h > 0\} = H^2_r \tag{C.9}$$

$$\overline{\text{span}}\{\chi_h \bar{W}_+ \; ; \; h > 0\} = \bar{H}^2_r \tag{C.10}$$

which are an immediate consequence of (C.2), (C.3) and the definitions (C.4), (C.5). So, in particular, the spectral density Φ of a purely nondeterministic process admits *analytic* (i.e. with rows in W^2) and *coanalytic* (with rows in $\bar{\mathcal{W}}^2$) spectral factors. A spectral density matrix with this property is called *factorizable.*

The previous argument can be reversed yielding the following characterization of purely nondeterministic stationary-increment processes, a proof of which can be found in [34].

THEOREM C.1. *A continuous stationary-increments process dz is purely nondeterministic if and only if its spectral distribution dZ is absolutely continuous and admits a factorizable density.*

In case dz has a rational spectral density Φ, the factorizability condition is automatically satisfied [56].

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