The Circulant Rational Covariance Extension Problem: The Complete Solution

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***Abstract*—The rational covariance extension problem to determine a rational spectral density given a finite number of covariance lags can be seen as a matrix completion problem to construct an infinite-dimensional positive-definite Toeplitz matrix the northwest corner of which is given. The circulant rational covariance extension problem considered in this paper is a modification of this problem to partial stochastic realization of periodic stationary process, which are better represented on the discrete unit circle $\mathbb{Z}_{2N}$ rather than on the discrete real line $\mathbb{Z}$. The corresponding matrix completion problem then amounts to completing a finite-dimensional Toeplitz matrix that is circulant. Another important motivation for this problem is that it provides a natural approximation, involving only computations based on the fast Fourier transform, for the ordinary rational covariance extension problem, potentially leading to an efficient numerical procedure for the latter. The circulant rational covariance extension problem is an inverse problem with infinitely many solutions in general, each corresponding to a bilateral ARMA representation of the underlying periodic process. In this paper, we present a complete smooth parameterization of all solutions and convex optimization procedures for determining them. A procedure to determine which solution that best matches additional data in the form of logarithmic moments is also presented.**

***Index Terms*—Bilateral ARMA models, circulant matrices, covariance extension, generalized entropy, moment problems, periodic processes, reciprocal processes.**

I. Introduction

THE rational covariance extension problem or the *partial stochastic realization problem* has been studied in various degrees of detail in a long series of papers [1]–[6], [17], [21], [22], [30], [41]. In a formulation suitable for this paper, it can be stated as follows. Given a sequence $(c_0, c_1, \ldots, c_n)$ of numbers, with c_0 real and the rest possibly complex, such that the Toeplitz matrix

$$\mathbf{T}_n = \begin{bmatrix} c_0 & \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_n \\ c_1 & c_0 & \bar{c}_1 & \cdots & \bar{c}_{n-1} \\ c_2 & c_1 & c_0 & \cdots & \bar{c}_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_0 \end{bmatrix} \tag{1}$$

is positive definite, find an infinite extension $c_{n+1}, c_{n+2}, c_{n+3}, \ldots$ such that, with $c_{-k} = \bar{c}_k$, $k = 1, 2, \ldots$, the series expansion

$$\Phi(e^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k e^{-ik\theta} \tag{2}$$

converges for all $\theta \in [-\pi, \pi]$ to a positive spectral density that takes the rational form

$$\Phi(z) = \frac{P(z)}{Q(z)} \tag{3}$$

where P and Q are symmetric trigonometric polynomial of the form

$$P(e^{i\theta}) = \sum_{k=-n}^{n} p_k e^{-ik\theta}, \quad p_{-k} = \bar{p}_k \tag{4}$$

of degree at most n. In [21] and [22], it was shown that there exists a Q for each assignment of P, and in [1] it was finally proved that this assignment is unique and smooth, yielding a complete parameterization suitable for tuning. Consequently, the rational covariance extension problem reduces to a trigonometric moment problem, where, for each P, the remaining problem is to determine a unique Q such that

$$\int_{-\pi}^{\pi} e^{ik\theta} \frac{P(e^{i\theta})}{Q(e^{i\theta})} \frac{d\theta}{2\pi} = c_k, \quad k = 0, 1, 2, \ldots, n. \tag{5}$$

In [3] and [4], a convex optimization procedure to determine these Q was introduced, a result that has then been generalized in several directions [5]–[7], [9], [10], [17], [19], [23], [24].

The rational covariance extension problem can be seen as a *matrix completion problem* to construct an infinite-dimensional positive-definite Toeplitz matrix $\mathbf{T}_\infty$ with $\mathbf{T}_n$ in its northwest corner, which moreover satisfies the rationality constraint (3). There is an large literature on band extension of positive-definite Toeplitz matrices, starting with the work of Dym and Gohberg [16] and surveyed in the books [25], [43], dealing with the maximum-entropy solution corresponding to $P \equiv 1$.

The *circulant rational covariance extension problem*, considered in this paper, is a modification of this problem to partial stochastic realization of periodic stationary process, which, as we shall explain in detail below, are better represented on the discrete unit circle $\mathbb{Z}_{2N}$ (the integers modulo $2N$) than on the discrete real line $\mathbb{Z}$. The corresponding matrix completion problem then amounts to completing a finite-dimensional Toeplitz matrix that is *circulant*. An important motivation for this problem is that its solution is a natural approximation of the solution to the ordinary rational covariance extension problem that turns out to

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involve only computations based on the fast Fourier transform and seems to lead to an efficient numerical procedure.

Circulant matrices are Toeplitz matrices with a special circulant structure

$$\mathrm{Circ}\{\gamma_0,\gamma_1,\ldots,\gamma_\nu\} = \begin{bmatrix} \gamma_0 & \gamma_\nu & \gamma_{\nu-1} & \cdots & \gamma_1 \\ \gamma_1 & \gamma_0 & \gamma_\nu & \cdots & \gamma_2 \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_\nu & \gamma_{\nu-1} & \gamma_{\nu-2} & \cdots & \gamma_0 \end{bmatrix} \tag{6}$$

where the columns (or, equivalently, rows) are shifted cyclically, and where $\gamma_0,\gamma_1,\ldots,\gamma_\nu$ here are taken to be complex numbers [14], [26]. In the circulant rational covariance extension problem we consider *Hermitian* circulant matrices

$$\mathbf{M} := \mathrm{Circ}\{m_0, m_1, m_2, \ldots, m_N, \bar{m}_{N-1}, \ldots, \bar{m}_2, \bar{m}_1\} \tag{7}$$

which can be represented in form

$$\mathbf{M} = \sum_{k=-N+1}^{N} m_k \mathbf{S}^{-k}, \quad m_{-k} = \bar{m}_k \tag{8}$$

where $\mathbf{S}$ is the nonsingular $2N \times 2N$ cyclic shift matrix

$$\mathbf{S} := \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{9}$$

The pseudo-polynomial

$$M(\zeta) = \sum_{k=-N+1}^{N} m_k \zeta^{-k}, \quad m_{-k} = \bar{m}_k \tag{10}$$

called the *symbol* of $\mathbf{M}$, will play important role the following analysis.

Hermitian circulant matrices appear naturally in the context of periodic stationary stochastic processes. To see this consider a zero-mean stationary process $\{y(t)\}$, in general complex-valued, defined on a finite interval $[-N+1,\ N]$ of the integer line $\mathbb{Z}$ and extended to all of $\mathbb{Z}$ as a periodic stationary process with period $2N$; i.e., such that $y(t+2kN) = y(t)$ almost surely. Processes of this kind can naturally be defined on the group $\mathbb{Z}_{2N}$ of the integers with arithmetics modulo $2N$, and in this setting stationarity can be seen as propagation in time of random variables under the action of a (finite) unitary group. We shall write the string $\{y(-N+1),\ldots,y(0),\ldots,y(N)\}$ as a $2N$-dimensional column vector $\mathbf{y}$ and only consider stationary processes of *full rank*, whose covariance matrix

$$\boldsymbol{\Sigma} := \mathbb{E}\{\mathbf{y}\mathbf{y}^*\} \tag{11}$$

is positive definite, where * denotes transpose conjugate. Let $\hat{\mathbb{E}}\{y(t) \mid y(s),\ s \neq t\}$ be the wide sense conditional mean of $y(t)$ given all $\{y(s),\ s \neq t\}$. Then the error process

$$d(t) := y(t) - \hat{\mathbb{E}}\{y(t) \mid y(s),\ s \neq t\} \tag{12}$$

is orthogonal to all random variables $\{y(s),\ s \neq t\}$; i.e., $\mathbb{E}\{y(t)\,\overline{d(s)}\} = \sigma^2\,\delta_{ts},\ t,s \in \mathbb{Z}_{2N}$, where δ is the Kronecker function and σ^2 is a positive number, or, equivalently,

$$\mathbb{E}\{\mathbf{y}\mathbf{d}^*\} = \sigma^2 \mathbf{I} \tag{13}$$

where $\mathbf{I}$ denotes the $2N \times 2N$ identity matrix. Interpreting (12) in the $\bmod 2N$ arithmetics of $\mathbb{Z}_{2N}$, $\mathbf{y}$ admits a linear representation of the form $\mathbf{F}\,\mathbf{y} = \mathbf{d}$, where $\mathbf{F}$ is a $2N \times 2N$ circulant matrix with ones on the main diagonal. Following Masani [38], d is called the (unnormalized) *conjugate process* of y. Therefore, setting $\mathbf{e} := (1/\sigma^2)\mathbf{d}$ and $\mathbf{A} := (1/\sigma^2)\mathbf{F}$, we see that a full-rank stationary periodic process admits a normalized representation

$$\mathbf{A}\mathbf{y} = \mathbf{e}, \quad \mathbb{E}\{\mathbf{e}\mathbf{y}^*\} = \mathbf{I} \tag{14}$$

where $\mathbf{A}$ is Hermitian and circulant. Since $\mathbf{A}\mathbb{E}\{\mathbf{y}\mathbf{y}^*\} = \mathbb{E}\{\mathbf{e}\mathbf{y}^*\} = \mathbf{I}$, $\mathbf{A}$ is also positive definite and the covariance matrix (11) is given by

$$\boldsymbol{\Sigma} = \mathbf{A}^{-1} \tag{15}$$

which is circulant, since the inverse of a circulant matrix is itself circulant. Therefore, if

$$c_k := \mathbb{E}\{y(t+k)\overline{y(t)}\}, \quad k = 0,1,2,\ldots,N \tag{16}$$

$\boldsymbol{\Sigma}$ is precisely the Hermitian circulant matrix

$$\boldsymbol{\Sigma} = \mathrm{Circ}\{c_0, c_1, c_2, \ldots, c_N, \bar{c}_{N-1}, \ldots, \bar{c}_2, \bar{c}_1\}. \tag{17}$$

In fact, a stationary process $\mathbf{y}$ is full-rank periodic in $\mathbb{Z}_{2N}$, if and only if $\boldsymbol{\Sigma}$ is a Hermitian positive definite circulant matrix [11].

We are now in a position to state the main problem of this paper. Supposing that only the covariance lags $c_0, c_1, \ldots, c_n$ are available for $n < N$, how do we complete the matrix (17) with the entries $c_{n+1}, c_{n+2}, \ldots, c_N$ so that it is circulant and the covariance matrix (11) of a stationary periodic process $\mathbf{y}$ with a spectral density of the rational form (3). We would like to parametrize the set of all solutions to this problem.

This can be seen as a generalization of modeling of *reciprocal processes* about which there is a large and important literature [20], [28], [29], [32], [33], [35]–[37]. A first step in this direction was taken in [11], where the circulant matrix $\mathbf{A}$ in (15) is required to be *banded of order* n; i.e.,

$$\mathbf{A} = \mathrm{Circ}\{a_0, a_1, \ldots, a_n, 0, \ldots, 0, \bar{a}_n, \bar{a}_{n-1}, \ldots, \bar{a}_1\}. \tag{18}$$

For example, a banded matrix of order $n = 2$ takes the form

$$\mathbf{A} = \begin{bmatrix} a_0 & \bar{a}_1 & \bar{a}_2 & 0 & \cdots & 0 & a_2 & a_1 \\ a_1 & a_0 & \bar{a}_1 & \bar{a}_2 & 0 & \cdots & 0 & a_2 \\ a_2 & a_1 & a_0 & \bar{a}_1 & \bar{a}_2 & 0 & \cdots & 0 \\ 0 & a_2 & a_1 & a_0 & \bar{a}_1 & \bar{a}_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & a_2 & a_1 & a_0 & \bar{a}_1 & \bar{a}_2 \\ \bar{a}_2 & 0 & \ddots & 0 & a_2 & a_1 & a_0 & \bar{a}_1 \\ \bar{a}_1 & \bar{a}_2 & 0 & \cdots & 0 & a_2 & a_1 & a_0 \end{bmatrix}. \tag{19}$$

In this case, it follows from (14) that $\mathbf{y}$ admits a bilateral AR representation

$$\sum_{k=-n}^{n} a_k y(t-k) = e(t), \quad a_{-k} = \bar{a}_k \tag{20}$$

for all $t \in \mathbb{Z}_{2N}$, which can also be written

$$A(\zeta)y(t) = e(t) \tag{21}$$

in terms of the *symbol*

$$A(\zeta) = \sum_{k=-n}^{n} a_k \zeta^{-k}, \quad a_{-k} = \bar{a}_k \tag{22}$$

of $\mathbf{A}$, where ζ represents the forward shift. These models are representations of stationary *reciprocal processes of order* n; see [11], where they are determined by solving a *maximum-entropy* problem.

In this paper, we show that for each choice of banded circulant matrix $\mathbf{P}$ of order at most n, there is an unique banded circulant matrix $\mathbf{Q}$ of order n such that

$$\mathbf{\Sigma} = \mathbf{Q}^{-1}\mathbf{P}. \tag{23}$$

If the corresponding symbols are $P(\zeta)$ and $Q(\zeta)$, respectively, then, by (14) and (15), such a solution corresponds to a bilateral ARMA representation

$$Q(\zeta)y(t) = P(\zeta)e(t) \tag{24}$$

or, equivalently,

$$\sum_{k=-n}^{n} q_k y(t-k) = \sum_{k=-n}^{n} p_k e(t-k). \tag{25}$$

We have therefore a complete parameterization of such representations, and hence of the completions of $\mathbf{\Sigma}$, in terms of the $\mathbf{P}$. However, as explained in Remark 5, (23) has to be interpreted with some care. If $\mathbf{Q}$ is singular, then so is $\mathbf{P}$, and there is zero cancellation between $P(\zeta)$ and $Q(\zeta)$, leading to a model of lower order.

In Section II, we review basic facts about circulant matrices and harmonic analysis on $\mathbb{Z}_{2N}$ and set up notations. The main results on the complete parameterization of the circulant rational covariance extension problem are presented in Section III, where we also consider the circulant rational covariance extension problem as an approximation procedure for the ordinary rational covariance extension problem. In Section IV, following [5], [6], [17], [18], [24], we show how the parameter $\mathbf{P}$ can be determined from logarithmic moments computed from data.

II. Preliminaries

Most of the harmonic analysis of stationary processes on $\mathbb{Z}$ carries over naturally, provided the Fourier transform is understood as a mapping from functions defined on $\mathbb{Z}_{2N}$ onto complex-valued functions on the unit circle of the complex plane, regularly sampled at intervals of length $\Delta := \pi/N$. We shall call this object the *discrete unit circle* and denote it by $\mathbb{T}_{2N}$. This Fourier map is usually called the *discrete Fourier transform (DFT)*. Next we shall review some pertinent facts and at the same time setup notations.

A. Harmonic Analysis on $\mathbb{Z}_{2N}$

Let $\zeta_1 := e^{i\Delta}$ be the primitive $2N$th root of unity; i.e., $\Delta = \pi/N$, and define the discrete variable ζ taking the $2N$ values $\zeta_k \equiv \zeta_1^k = e^{i\Delta k}$; $k = -N+1, \ldots, 0, \ldots, N$ running counterclockwise on the discrete unit circle $\mathbb{T}_{2N}$. In particular, we have $\zeta_{-k} = \overline{\zeta_k}$ (complex conjugate).

The discrete Fourier transform $\mathcal{F}$ maps a finite signal $g = \{g_k;\ k = -N+1, \ldots, N\}$, into a sequence of complex numbers

$$G(\zeta_j) := \sum_{k=-N+1}^{N} g_k \zeta_j^{-k}, \qquad j = -N+1, -N+2, \ldots, N. \tag{26}$$

It is well known that the signal g can be recovered from its DFT G by the formula

$$g_k = \sum_{j=-N+1}^{N} \zeta_j^k G(\zeta_j)\frac{\Delta}{2\pi}, \quad k = -N+1, -N+2, \ldots, N \tag{27}$$

where $\Delta/2\pi = 1/2N$ plays the role of a uniform discrete measure with total mass one on the discrete unit circle $\mathbb{T}_{2N}$. In the sequel, it will be useful to write (27) as an integral

$$g_k = \int_{-\pi}^{\pi} e^{ik\theta} G(e^{i\theta}) d\nu(\theta), \quad k = -N+1, -N+2, \ldots, N \tag{28}$$

where ν is a step function with steps $1/2N$ at each ζ_k; i.e.,

$$d\nu(\theta) = \sum_{j=-N+1}^{N} \delta(e^{i\theta} - \zeta_j)\frac{d\theta}{2N}. \tag{29}$$

In particular, we have

$$\int_{-\pi}^{\pi} e^{ik\theta} d\nu(\theta) = \delta_{k0} \tag{30}$$

where δ_{k0} equals one for $k = 0$ and zero otherwise. To see this, note that, for $k \neq 0$

$$\begin{aligned}(1-\zeta_k)\int_{-\pi}^{\pi} e^{ik\theta} d\nu &= \frac{1}{2N}\sum_{j=-N+1}^{N}\left(\zeta_k^j - \zeta_k^{j+1}\right)\\ &= \frac{1}{2N}\left(\zeta_k^{-N+1} - \zeta_k^{N+1}\right) = 0.\end{aligned}$$

In particular, if F is the DFT of $\{f_k\}$,

$$\begin{aligned}\sum_{k=-N+1}^{N} f_k \bar{g}_k &= \frac{1}{2N}\sum_{k=-N+1}^{N} F(\zeta_k)G(\zeta_{-k})\\ &= \int_{-\pi}^{\pi} F(e^{i\theta})G(e^{i\theta})^* d\nu(\theta).\end{aligned} \tag{31}$$

This is *Plancherel's Theorem* for DFT.

It is sometimes convenient to write the discrete Fourier transform (26) in the matrix form

$$\hat{\mathbf{g}} = \mathbf{F}\mathbf{g} \tag{32}$$

where $\hat{\mathbf{g}} := \big(G(\zeta_{-N+1}), G(\zeta_{-N+2}), \dots, G(\zeta_N)\big)^{\mathsf{T}}$, $\mathbf{g} := (g_{-N+1}, g_{-N+2}, \dots, g_N)^{\mathsf{T}}$, and $\mathbf{F}$ is the nonsingular $2N \times 2N$ Vandermonde matrix

$$\mathbf{F} = \begin{bmatrix} \zeta_{-N+1}^{N-1} & \zeta_{-N+1}^{N-2} & \cdots & \zeta_{-N+1}^{-N} \\ \vdots & \vdots & \cdots & \vdots \\ \zeta_0^{N-1} & \zeta_0^{N-2} & \cdots & \zeta_0^{-N} \\ \vdots & \vdots & \cdots & \vdots \\ \zeta_N^{N-1} & \zeta_N^{N-2} & \cdots & \zeta_N^{-N} \end{bmatrix}. \tag{33}$$

Likewise, it follows from (27) that

$$\mathbf{g} = \frac{1}{2N}\mathbf{F}^*\hat{\mathbf{g}} \tag{34}$$

i.e., $\mathcal{F}^{-1}$ corresponds to $(1/2N)\mathbf{F}^*$. Consequently, $\mathbf{F}\mathbf{F}^* = 2N\,\mathbf{I}$, and hence $\mathbf{F}^{-1} = (1/2N)\mathbf{F}^*$ and $(\mathbf{F}^*)^{-1} = (1/2N)\mathbf{F}$.

B. Circulant Matrices

As pointed out above, a Hermitian circulant matrix (7) can be represented in the form (8), where $\mathbf{S}$ is the nonsingular $2N\times 2N$ cyclic shift matrix (9), which is itself a circulant matrix with symbol $S(\zeta) = \zeta$. Clearly, $\mathbf{S}^{2N} = \mathbf{S}^0 = \mathbf{I}$, and

$$\mathbf{S}^{k+2N} = \mathbf{S}^k, \qquad \mathbf{S}^{2N-k} = \mathbf{S}^{-k} = (\mathbf{S}^k)^{\mathsf{T}}. \tag{35}$$

Consequently,

$$\mathbf{S}\mathbf{M}\mathbf{S}^* = \mathbf{M} \tag{36}$$

and the condition (36) is both necessary and sufficient for $\mathbf{M}$ to be circulant. (Clearly, what is said in this section holds for circulant matrices in general, but in this paper we are only interested in the Hermitian ones.)

As before setting $\mathbf{g} := (g_{-N+1}, g_{-N+2}, \dots, g_N)^{\mathsf{T}}$, we have

$$[\mathbf{S}\mathbf{g}]_k = g_{k+1}, \quad k \in \mathbb{Z}_{2N}. \tag{37}$$

In view of (26), it then follows that $\zeta\mathcal{F}(\mathbf{g})(\zeta) = \mathcal{F}(\mathbf{S}\mathbf{g})(\zeta)$, from which we have

$$\mathcal{F}(\mathbf{M}\mathbf{g})(\zeta) = M(\zeta)\mathcal{F}(\mathbf{g})(\zeta) \tag{38}$$

where $M(\zeta)$ is the symbol (10) of the circulant matrix $\mathbf{M}$.

An important property of circulant matrices is that they are diagonalized by the discrete Fourier transform. More precisely, it follows from (38) that

$$\mathbf{M} = \frac{1}{2N}\mathbf{F}^*\mathrm{diag}\big(M(\zeta_{-N+1}), M(\zeta_{-N+2}), \dots, M(\zeta_N)\big)\mathbf{F} \tag{39}$$

i.e., the circulant matrices are simultaneously diagonalizable by the unitary matrix $(1/\sqrt{2N})\mathbf{F}$. Hence, the inverse $\mathbf{M}^{-1}$ is

$$\mathbf{M}^{-1} = \frac{1}{2N}\mathbf{F}^*\mathrm{diag}\big(M(\zeta_{-N+1})^{-1}, \dots, M(\zeta_N)^{-1}\big)\mathbf{F} \tag{40}$$

and, since

$$\mathbf{S} = \frac{1}{2N}\mathbf{F}^*\mathrm{diag}\big(\zeta_{-N+1}, \dots, \zeta_N\big)\mathbf{F}$$
$$\mathbf{S}^* = \frac{1}{2N}\mathbf{F}^*\mathrm{diag}\big(\zeta_{-N+1}^{-1}, \dots, \zeta_N^{-1}\big)\mathbf{F}$$

we have

$$\mathbf{S}\mathbf{M}^{-1}\mathbf{S}^* = \mathbf{M}^{-1}.$$

Consequently, $\mathbf{M}^{-1}$ is also a circulant matrix with symbol $M(\zeta)^{-1}$. In general, in view of the circulant property (8) and (35), quotients of symbols are themselves pseudo-polynomials of degree at most N and hence symbols. The coefficients of the corresponding pseudo-polynomial $M(\zeta)^{-1}$ can be determined by Lagrange interpolation. More generally, if $\mathbf{A}$ and $\mathbf{B}$ are circulant matrices of the same dimension with symbols $A(\zeta)$ and $B(\zeta)$, respectively, then $\mathbf{AB}$ and $\mathbf{A}+\mathbf{B}$ are circulant matrices with symbols $A(\zeta)B(\zeta)$ and $A(\zeta)+B(\zeta)$, respectively. In fact, the circulant matrices of a fixed dimension form an algebra – more precisely, a commutative *-algebra with the involution * being the conjugate transpose – and the DFT is an *algebra homomorphism* of the set of circulant matrices onto the pseudo-polynomials of degree at most N in the variable $\zeta \in \mathbb{T}_{2N}$.

C. Spectral Representation of Periodic Stationary Stochastic Processes

Let $\{y(t)\}$ be a zero-mean stationary process defined on $\mathbb{Z}_{2N}$. Let $c_{-N+1}, c_{-N+2}, \dots, c_N$ be the covariance lags $c_k := \mathbb{E}\{y(t+k)\overline{y(t)}\}$, and define its discrete Fourier transformation

$$\Phi(\zeta_j) := \sum_{k=-N+1}^{N} c_k\zeta_j^{-k}, \qquad j = -N+1, \dots, N \tag{41}$$

which is a positive real-valued function of ζ. Then, as seen from (27) and (28)

$$c_k = \sum_{j=-N+1}^{N} \zeta_j^k\Phi(\zeta_j)\frac{\Delta}{2\pi} = \int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta})d\nu(\theta)$$
$$k = -N+1, \dots, N. \tag{42}$$

The function Φ is the *spectral density* of the process y. In fact, let

$$\hat{y}(\zeta_k) := \sum_{t=-N+1}^{N} y(t)\zeta_k^{-t}, \quad k = -N+1, \dots, N \tag{43}$$

be the discrete Fourier transformation of the process y. The random variables (43) turn out to be uncorrelated, and

$$\frac{1}{2N}\mathbb{E}\{\hat{y}(\zeta_k)\hat{y}(\zeta_\ell)^*\} = \Phi(\zeta_k)\delta_{k\ell}. \tag{44}$$

This can be seen by a straightforward calculation noting that

$$\frac{1}{2N}\sum_{t=-N+1}^{N}(\zeta_k\zeta_\ell^*)^t = \delta_{k\ell}. \tag{45}$$

Then, we obtain a spectral representation of $\{y(t)\}$ analogous to the usual one, namely

$$y(t) = \sum_{k=-N+1}^{N}\zeta_k^t\,\hat{y}(\zeta_k)\frac{1}{2N} = \int_{-\pi}^{\pi} e^{ik\theta}d\hat{y}(\theta) \tag{46}$$

where

$$d\hat{y}(\theta) := \hat{y}(e^{i\theta})d\nu(\theta). \tag{47}$$

It is interesting to note that

$$\Phi(\zeta_k) = \frac{1}{2N}\mathbb{E}\{\hat{y}(\zeta_k)\hat{y}(\zeta_k)^*\}, \quad k = -N+1, \ldots, N \tag{48}$$

obtained from (44) is just the expected value of the *periodogram*

$$\hat{\Phi}(\zeta_k) = \frac{1}{2N}\hat{y}(\zeta_k)\hat{y}(\zeta_k)^*, \quad k = -N+1, \ldots, N \tag{49}$$

widely used in statistics.

D. Modifications for Skew-Periodic Stochastic Processes

A process satisfying the condition

$$y(t+N) = -y(t), \quad t \in \mathbb{Z} \tag{50}$$

is called a *skew-periodic stochastic processes* of period N. Such processes occur in the theory of reciprocal processes [34]. It is still periodic of period $2N$, but to enforce the property (50) we need to exchange (29) for

$$d\nu(\theta) = \sum_{j=1}^{N} \delta(e^{i\theta} - \zeta_{2j-1})\frac{d\theta}{N}. \tag{51}$$

Then, since $(\zeta_{2j-1})^N = -1$

$$\begin{aligned} y(t+N) &= \frac{1}{N}\sum_{j=1}^{N}(\zeta_{2j-1})^t(\zeta_{2j-1})^N\hat{y}(\zeta_{2j-1}) \\ &= -\frac{1}{N}\sum_{j=1}^{N}(\zeta_{2j-1})^t\hat{y}(\zeta_{2j-1}) \\ &= -y(t), \\ \text{and} \quad c_{k+N} &= \frac{1}{N}\sum_{j=1}^{N}(\zeta_{2j-1})^k(\zeta_{2j-1})^N\Phi(\zeta_{2j-1}) = -c_k \end{aligned}$$

which introduces the extra constraint $\bar{c}_k = -c_{N-k}$. Consequently, a circulant covariance extension theory of skew-periodic stochastic processes would require exchanging (29) for (51) in the derivations below. Since then extra linear constraints will be introduced, the theory would have to be modified to a considerable extent. This will be left for another paper.

III. Complete Solution to the Circulant Rational Covariance Extension Problem

Given $n < N$ and $\mathbf{c} := (c_0, c_1, \ldots, c_n)$ with a positive definite Toeplitz matrix (1), find a spectral density Φ of the rational form (3) satisfying the moment conditions

$$\int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta})d\nu = c_k, \quad k = 0, 1, 2, \ldots, n. \tag{52}$$

This moment condition can also be written as an underdetermined system of linear equations

$$\frac{1}{2N}\sum_{j=-N+1}^{N} \zeta_j^k\,\Phi(\zeta_j) = c_k \quad k = 0, 1, 2, \ldots, n \tag{53}$$

in the variables $x_j = \Phi(\zeta_j)$, $j = -N+1, -N+2, \ldots, N$, where the coefficient matrix is a Vandermonde matrix of full rank. Note that it is consistent with (52) to define negative moments by setting $c_{-k} = \bar{c}_k$, so that the pseudo-polynomial

$$C(\zeta) = \sum_{k=-n}^{n} c_k\zeta^{-k} \tag{54}$$

is the symbol of a banded Hermitian circulant matrix

$$\mathbf{C} = \text{Circ}\{c_0, c_1, \ldots, c_n, 0, \ldots, 0, \bar{c}_n, \bar{c}_{n-1}, \ldots, \bar{c}_1\} \tag{55}$$

of order n. We would like to find a rational extension $c_{n+1}, c_{n+2}, \ldots, c_N$ replacing the zeros in $\mathbf{C}$ to obtain a Hermitian circulant matrix

$$\boldsymbol{\Sigma} := \text{Circ}\{c_0, c_1, c_2, \ldots, c_N, \bar{c}_{N-1}, \ldots, \bar{c}_2, \bar{c}_1\} \tag{56}$$

that is positive definite. In terms of stationary periodic processes, this corresponds to the covariance matrix (11).

If for the moment we forsake the rationality condition, we see that the class of positive semidefinite Hermitian completions $\boldsymbol{\Sigma}$ of $\mathbf{C}$ is completely parameterized by the pseudo-polynomials

$$M_\Sigma(\zeta) = \sum_{j=-N+1}^{N} \sigma_j\zeta^{-j}, \quad \sigma_{-k} = \bar{\sigma}_k \tag{57a}$$

$$\text{such that} \quad \sigma_k = c_k, \quad k = 0, 1, \ldots, n \tag{57b}$$

$$\text{and} \quad M_\Sigma(\zeta_j) \geq 0, \quad j = -N+1, \ldots, N. \tag{57c}$$

Indeed, this is immediate from (39), from which we also see that $\boldsymbol{\Sigma}$ is singular if and only if $M_\Sigma(\zeta_j) = 0$ for some j.

However, in the *circulant rational covariance extension problem* we are *not* interested in parameterizing all circulant extensions (56) but only those which have a rational symbol

$$\Phi(\zeta) = \frac{P(\zeta)}{Q(\zeta)}, \quad \deg P, \deg Q \leq n \tag{58}$$

or, equivalently, are of the form (23) with $\mathbf{P}$ and $\mathbf{Q}$ *banded* circulant matrices of order at most n. This is a complexity constraint of the same type as in the original rational covariance extension problem. In fact, if $n \ll N$, such representations are more parsimonious containing at most $2n+1$ parameters. This is a proper subclass of (57), since, in view of (8) and (35), rational symbols (58) can also be written as pseudo-polynomials (57a). Note that the number $n+1$ of parameters in P is equal to the number of given covariance data. The significance of this will be clear from Theorem 1 below.

We now proceed to solve the circulant rational covariance extension problem in terms of the symbols, and then interpret the results in terms of matrices. We begin with positive definite extensions and then turn to the boundary case where $\boldsymbol{\Sigma}$ is singular.

A. Circulant Rational Covariance Extension in Terms of Symbols

Let $\mathfrak{P}$ be the finite-dimensional space of symmetric trigonometric polynomials (4), and define $\mathfrak{P}_+$ to be the positive cone

$$\mathfrak{P}_+ = \{P \in \mathfrak{P} \mid P(e^{i\theta}) > 0 \quad \text{for all } \theta \in [-\pi, \pi]\}.$$

Moreover, let $\mathfrak{C}_+$ be the *interior* of the dual cone of all $\mathbf{c} = (c_0, c_1, \dots, c_n)$ such that

$$\langle C, P\rangle := \sum_{k=-n}^{n} c_k \bar{p}_k \geq 0 \quad \text{for all } P \in \overline{\mathfrak{P}_+} \tag{59}$$

[31], where the notation $\langle C, P\rangle$ is motivated by the fact that, by (31),

$$\langle C, P\rangle = \int_{-\pi}^{\pi} C(e^{i\theta})P(e^{i\theta})^* d\nu. \tag{60}$$

It can be shown that $\mathbf{c} \in \mathfrak{C}_+$ if and only if $\mathbf{T}_n > 0$. In fact, if $a(z) = a_0 z^n + \ldots + a_{n-1}z + a_n$ is a polynomial spectral factor of $P(z)$, i.e., $a(z)a(z)^* = P(z)$, then it is easy to see that

$$\langle C, P\rangle = \mathbf{a}^* \mathbf{T}_n \mathbf{a} \tag{61}$$

where $\mathbf{T}_n$ is the Toeplitz matrix (1) of $c_0, c_1, \dots, c_n$ and $\mathbf{a} = (a_0, a_1, \dots, a_n)^{\mathsf{T}}$.

Next, define the cone

$$\mathfrak{P}_+(N) = \left\{P \in \mathfrak{P} \mid P(\zeta_k) > 0,\ k = -N+1, \dots, N\right\}. \tag{62}$$

Clearly, $\mathfrak{P}_+(N) \supset \mathfrak{P}_+(2N) \supset \mathfrak{P}_+(4N) \supset \ldots \supset \mathfrak{P}_+$, and the corresponding open dual cones satisfy

$$\mathfrak{C}_+(N) \subset \mathfrak{C}_+(2N) \subset \mathfrak{C}_+(4N) \subset \cdots \subset \mathfrak{C}_+. \tag{63}$$

Theorem 1: Let $\mathbf{c} \in \mathfrak{C}_+(N)$. Then, for each $P \in \mathfrak{P}_+(N)$, there is a unique $Q \in \mathfrak{P}_+(N)$ such that

$$\Phi = \frac{P}{Q} \tag{64}$$

satisfies the moment conditions (52).

For the proof, which is given in the appendix, we need to consider a dual pair of optimization problems. First consider the primal problem to maximize the generalized entropy

$$\mathbb{I}_P(\Phi) = \int_{-\pi}^{\pi} P(e^{i\theta}) \log \Phi(e^{i\theta}) d\nu \tag{65}$$

subject to the moment conditions (52). The corresponding Lagrangian is then given by

$$\begin{aligned} L(\Phi, Q) &= \mathbb{I}_P(\Phi) + \sum_{k=-n}^{n} \bar{q}_k \left(c_k - \int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta})d\nu\right) \\ &= \int_{-\pi}^{\pi} P(e^{i\theta}) \log \Phi(e^{i\theta}) d\nu + \langle C, Q\rangle \\ &\quad - \int_{-\pi}^{\pi} Q(e^{i\theta})\Phi(e^{i\theta}) d\nu \end{aligned} \tag{66}$$

where $q_0, q_1, \dots, q_n$ are Lagrange multipliers, and where Q is defined as in (4) with $q_{-k} = \bar{q}_k$. Since the dual functional $\sup_\Phi L(\Phi, Q)$ is finite only if $Q \in \overline{\mathfrak{P}_+(N)}$, where $\overline{\mathfrak{P}_+(N)}$ denotes the closure of $\mathfrak{P}_+(N)$, we may restrict the Lagrange multipliers to that set. Therefore, for each $Q \in \overline{\mathfrak{P}_+(N)}$, consider the directional derivative

$$\delta L(\Phi, Q; \delta\Phi) = \int_{-\pi}^{\pi} \left(\frac{P}{\Phi} - Q\right) \delta\Phi d\nu$$

which equals zero for all variations $\delta\Phi$ if and only if

$$\Phi = \frac{P}{Q}.$$

Inserting this into (66) we obtain

$$\sup_\Phi L(\Phi, Q) = \mathbb{J}_P(Q) + \int_{-\pi}^{\pi} P(e^{i\theta}) \left[\log P(e^{i\theta}) - 1\right] d\nu$$

where

$$\mathbb{J}_P(Q) = \langle C, Q\rangle - \int_{-\pi}^{\pi} P(e^{i\theta}) \log Q(e^{i\theta}) d\nu \tag{67}$$

and the last term is constant. Hence, we may take (67) as the dual functional.

It will be shown below that $\mathbb{J}_P$ is strictly convex, so a stationary point in $\mathfrak{P}_+$, if it exists, would have to be a unique minimizer of $\mathbb{J}_P$. For $k = 1, 2, \dots, n$, we write $q_k = x_k + iy_k$ as a sum of real and imaginary parts and define the partial differential operators

$$\frac{\partial}{\partial q_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right) \tag{68a}$$

$$\frac{\partial}{\partial \bar{q}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right) \tag{68b}$$

in the standard way; see, e.g., [27, p. 1]. It is immediately seen that

$$\frac{\partial q_k}{\partial \bar{q}_k} = 0 \quad \text{and} \quad \frac{\partial \bar{q}_k}{\partial q_k} = 0. \tag{69}$$

(The differential operators (68) are often called *Wirtinger derivatives*.) From this, we readily obtain

$$\frac{\partial \mathbb{J}_P}{\partial \bar{q}_k} = c_k - \int_{-\pi}^{\pi} e^{ik\theta} \frac{P(e^{i\theta})}{Q(e^{i\theta})} d\nu, \quad k = 1, 2, \dots, n. \tag{70}$$

Setting (70) equal to zero yields the moment conditions (52). Then the proof of the following theorem follows directly from Theorem 1.

Theorem 2: Let $\mathbf{c} \in \mathfrak{C}_+(N)$ and $P \in \mathfrak{P}_+(N)$. Then the problem to maximize (65) subject to the moment conditions (52) has a unique solution, namely (64), where Q is the unique optimal solution of the problem to minimize (67) over all $Q \in \mathfrak{P}_+(N)$.

From (70) we have the Hessian

$$\frac{\partial^2 \mathbb{J}_P}{\partial \bar{q}_k \partial q_\ell} = \int_{-\pi}^{\pi} e^{i(k-\ell)\theta} \frac{P(e^{i\theta})}{Q(e^{i\theta})^2} d\nu, \quad k, \ell = 0, 1, \dots, n \tag{71}$$

which is Hermitian and positive definite, showing that $\mathbb{J}_P$ is strictly convex.

Next, we establish that the solution to the circulant rational covariance extension problem depends smoothly on the parameters $\mathbf{c}$ and P. To this end, for each fixed $P \in \mathfrak{P}_+(N)$, we define the moment map F^P componentwise given by

$$F_k^P(Q) = \int_{-\pi}^{\pi} e^{ik\theta} \frac{P(e^{i\theta})}{Q(e^{i\theta})} d\nu, \quad k = 0, 1, \dots, n \tag{72}$$

and, for each $c \in \mathfrak{C}_+$, the map G^c sending $P \in \mathfrak{P}_+(N)$ to $Q \in \mathfrak{Q}_+(N) := (F^P)^{-1}(\mathfrak{C}_+(N))$.

The proof of the following theorem is given in the Appendix.

Theorem 3: The maps F^P and G^c are homeomorphisms.

In particular, we have established a complete smooth parameterization of all solutions Q to the circulant rational covariance extension problem in terms of $P \in \mathfrak{P}_+(N)$.

Finally, we show that the map F^P can be continuously extended to the boundary of $\mathfrak{P}_+(N)$, as can be seen from the following extension, proved in the Appendix, of the family of dual solutions.

Theorem 4: Let $\mathbf{c} \in \mathfrak{C}_+(N)$. Then, for each $P \in \overline{\mathfrak{P}_+(N)} \setminus \{0\}$, the dual problem to minimize (67) over $Q \in \overline{\mathfrak{P}_+(N)} \setminus \{0\}$ has a unique minimizer $\hat{Q}$, and $P/\hat{Q}$ satisfies the moment conditions (52). If $\hat{Q}$ belongs to the boundary of $\mathfrak{P}_+(N)$, then so does P.

Consequently, we have established a complete parameterization of all solutions to the circulant rational covariance extension problem in terms of $P \in \overline{\mathfrak{P}_+(N)} \setminus \{0\}$.

B. Circulant Rational Covariance Extension in Terms of Matrices

Next, we reformulate the optimization problems in terms of circulant matrices. To this end, we define the circulant matrix

$$\mathbf{\Sigma} = \frac{1}{2N}\mathbf{F}^*\text{diag}\big(\Phi(\zeta_{-N+1}), \ldots, \Phi(\zeta_N)\big)\mathbf{F} \tag{73}$$

with symbol (64) and the banded numerator matrix

$$\mathbf{P} = \frac{1}{2N}\mathbf{F}^*\text{diag}\big(P(\zeta_{-N+1}), \ldots, P(\zeta_N)\big)\mathbf{F} \tag{74}$$

of degree at most n with symbol (4). Since $\Phi(\zeta_k) > 0$ for all k and $\log \Phi(\zeta)$ is analytic in the neighborhood of each $\Phi(\zeta_k) > 0$, by the spectral mapping theorem [15, p. 557] the eigenvalues of $\log \mathbf{\Sigma}$ are just the real numbers $\log \Phi(\zeta_k)$, $k = -N+1, \ldots, N$, and hence

$$\log \mathbf{\Sigma} = \frac{1}{2N}\mathbf{F}^*\text{diag}\big(\log \Phi(\zeta_{-N+1}), \ldots, \log \Phi(\zeta_N)\big)\mathbf{F}. \tag{75}$$

Consequently, the primal functional (65) may be written

$$\begin{aligned}\int_{-\pi}^{\pi} P(e^{i\theta}) \log \Phi(e^{i\theta}) d\nu &= \frac{1}{2N}\sum_{j=-N+1}^{N} P(\zeta_j) \log \Phi(\zeta_j)\\ &= \frac{1}{2N}\text{tr}\{\mathbf{P} \log \mathbf{\Sigma}\}\end{aligned} \tag{76}$$

and the moment conditions (52) as

$$\frac{1}{2N}\text{tr}\{\mathbf{S}^k\mathbf{\Sigma}\} = c_k, \quad k = 0, 1, \ldots, n \tag{77}$$

or, equivalently, as

$$\mathbf{E}_n^{\mathsf{T}}\mathbf{\Sigma}\mathbf{E}_n = \mathbf{T}_n, \quad \text{where } \mathbf{E}_n = \begin{bmatrix}\mathbf{I}_n\\ \mathbf{0}\end{bmatrix}. \tag{78}$$

Consequently, the primal problem amounts to maximizing $\text{tr}\{\mathbf{P} \log \mathbf{\Sigma}\}$ over all Hermitian, positive definite $2N \times 2N$ matrices subject to (77) or (78). For the special case $P \equiv 1$ this reduces to the primal problem presented in [11], except that in [11] there is an extra condition insuring that $\mathbf{\Sigma}$ is circulant. However, it was shown in [12] that this condition is automatically satisfied and is therefore not needed.

In the same way, by (60) the dual functional (67) can be written as

$$\begin{aligned}\int_{-\pi}^{\pi} C(e^{i\theta})Q(e^{i\theta})d\nu - \int_{-\pi}^{\pi} P(e^{i\theta}) \log Q(e^{i\theta})d\nu\\ = \frac{1}{2N}\text{tr}\{\mathbf{CQ}\} - \frac{1}{2N}\text{tr}\{\mathbf{P} \log \mathbf{Q}\}\end{aligned} \tag{79}$$

where

$$\mathbf{Q} = \frac{1}{2N}\mathbf{F}^*\text{diag}\big(Q(\zeta_{-N+1}), \ldots, Q(\zeta_N)\big)\mathbf{F} \tag{80}$$

and $\mathbf{C}$ is the banded circulant matrix (55) formed from $c_0, c_1, \ldots, c_n$.

Consequently, given $\mathbf{c} \in \mathfrak{C}_+(N)$, it follows from Theorem 1 that, for each Hermitian, positive-definite circulant matrix $\mathbf{P}$ that is banded of degree at most n, there is a unique $\mathbf{\Sigma}$ given by

$$\mathbf{\Sigma} = \mathbf{Q}^{-1}\mathbf{P} \tag{81}$$

where $\mathbf{Q}$ is the unique solution of the problem to minimize

$$\mathbb{J}_{\mathbf{P}}(\mathbf{q}) = \frac{1}{2N}\text{tr}\{\mathbf{CQ}\} - \frac{1}{2N}\text{tr}\{\mathbf{P} \log \mathbf{Q}\} \tag{82}$$

over all $\mathbf{q} := (q_0, q_1, \ldots, q_n)$ such that the Hermitian, circulant matrix

$$\mathbf{Q} = \text{Circ}\{q_0, q_1, \ldots, q_n, 0, \ldots, 0, \bar{q}_n, \bar{q}_{n-1}, \ldots, \bar{q}_1\}$$

is positive definite. For the maximum-entropy solution corresponding to $\mathbf{P} = \mathbf{I}$ this reduces to an optimization problem that is different from the one presented in [11].

Since

$$\mathbf{Q} = \sum_{k=-n}^{n} q_k\mathbf{S}^{-k}, \quad q_{-k} = \bar{q}_k$$

we have

$$\frac{\partial \mathbf{Q}}{\partial q_k} = \mathbf{S}^{-k} \quad \text{and} \quad \frac{\partial \mathbf{Q}}{\partial \bar{q}_k} = \mathbf{S}^k$$

for $k = 0, 1, \ldots, n$, and therefore

$$\begin{aligned}\frac{\partial \mathbf{J}_{\mathbf{P}}}{\partial \bar{q}_k} &= \frac{1}{2N}\text{tr}\{\mathbf{S}^k\mathbf{C}\} - \frac{1}{2N}\text{tr}\{\mathbf{S}^k\mathbf{P}\mathbf{Q}^{-1}\}\\ &= c_k - \frac{1}{2N}\text{tr}\{\mathbf{S}^k\mathbf{P}\mathbf{Q}^{-1}\}\end{aligned} \tag{83}$$

where we have used the fact that

$$\text{tr}\{\mathbf{S}^k\mathbf{C}\} = \sum_{j=-n}^{n} c_j\text{tr}\{\mathbf{S}^{k-j}\} = 2Nc_k$$

as $\text{tr}\{\mathbf{S}^k\} \neq 0$ only for $\mathbf{S}^0 = \mathbf{I}$. Setting (83) equal to zero yields the moment conditions. Likewise,

$$\begin{aligned}\frac{\partial^2 \mathbf{J}_{\mathbf{P}}}{\partial \bar{q}_k \partial q_k} &= \frac{1}{2N}\text{tr}\{\mathbf{S}^k\mathbf{P}\mathbf{Q}^{-2}\mathbf{S}^{-\ell}\}\\ &= \frac{1}{2N}\text{tr}\{\mathbf{S}^{k-\ell}\mathbf{P}\mathbf{Q}^{-2}\}\end{aligned} \tag{84}$$

showing that the Hessian is a Toeplitz matrix. This is the matrix version of (71).

Remark 5: We note that all solutions (81) corresponding to positive definite $\mathbf{P}$ are positive definite. To parameterize all solutions for which $\boldsymbol{\Sigma} \geq 0$ we need to apply Theorem 4. Indeed, there is a complete parameterization of all circulant rational covariance extensions in terms of nonzero positive-semidefinite $\mathbf{P}$. If $\mathbf{P}$ is singular and the corresponding optimal solution $\mathbf{Q}$ of the dual optimization problem (82) is positive definite, then $\boldsymbol{\Sigma}$ is singular. If $\mathbf{Q}$ is singular, then so is $\mathbf{P}$, and then there is a cancellation of roots in the pseudo-polynomials P and Q leading to a rational function P/Q of lower degree and hence a solution (81) with $\mathbf{P}$ and $\mathbf{Q}$ banded of order at most $n-1$.

In [11], it was observed that the condition that the Toeplitz matrix $\mathbf{T}_n$, defined by (1), is positive definite is a necessary, but not a sufficient, condition for feasibility of the circulant banded covariance extension problem. This can now be understood in the more general setting of moment problems discussed above. In fact, the Toeplitz condition $\mathbf{T}_n > 0$ is equivalent to $\mathbf{c} \in \mathfrak{C}_+$, whereas, by Theorem 2, $\mathbf{c} \in \mathfrak{C}_+(N)$ is required for feasibility. Since $\mathfrak{C}_+(N) \subset \mathfrak{C}_+$, it follows that the Toeplitz condition cannot be sufficient in general. However, as proved in [11], feasibility is achieved for a sufficiently large N. This can also be seen from the following result.

Proposition 6: The feasibility set $\mathfrak{C}_+(N) \to \mathfrak{C}_+$ as $N \to \infty$. In particular, for any $\mathbf{c} \in \mathfrak{C}_+$, there is an N_0 such that $\mathbf{c} \in \mathfrak{C}_+(N)$ for $N \geq N_0$.

Proof: As $N \to \infty$, the set $\{\zeta_j;\ j = -N+1, \dots, N\}$ becomes dense on the unit circle, and therefore $\mathfrak{P}_+(N) \to \mathfrak{P}_+$. Consequently, $\mathfrak{C}_+(N) \to \mathfrak{C}_+$, and the convergence is monotone in the sense of (63). Therefore, since $\mathfrak{C}_+$ is an open set, there is an N_0 such that any $\mathbf{c} \in \mathfrak{C}_+$ will sooner or later end up in $\mathfrak{C}_+(N)$ and remain there as N increases. ■

C. Some Computational Considerations

For each choice of P, the Hessian of $\mathbb{J}_P$ can be computed explicitly in terms of Q as the Toeplitz matrix

$$\mathbf{H}(\mathbf{q}) = \begin{bmatrix} h_0 & \bar{h}_1 & \bar{h}_2 & \cdots & \bar{h}_n \\ h_1 & h_0 & \bar{h}_1 & \cdots & \bar{h}_{n-1} \\ h_2 & h_1 & h_0 & \cdots & \bar{h}_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_n & h_{n-1} & h_{n-2} & \cdots & h_0 \end{bmatrix} \quad (85)$$

where $\mathbf{q} = (q_0, q_1, \dots, q_n)$ are the coefficients in the pseudo-polynomial Q and

$$h_k = \int_{-\pi}^{\pi} e^{ik\theta} \frac{P(e^{i\theta})}{Q(e^{i\theta})^2} d\nu = \frac{1}{2N} \sum_{j=-N+1}^{N} \zeta_j^k \frac{P(\zeta_j)}{Q(\zeta_j)^2} \quad (86)$$

as can be seen from (71) or (84). Therefore Newton's method can be used to find the unique minimizer of the dual problem. The gradient (70) at the point $\mathbf{q}$ is $(\mathbf{c} - \bar{\mathbf{c}}(\mathbf{q}))$, where

$$\bar{\mathbf{c}}(\mathbf{q}) = \frac{1}{2N} \sum_{j=-N+1}^{N} \zeta_j^k \frac{P(\zeta_j)}{Q(\zeta_j)} \quad (87)$$

and consequently a Newton step amounts to solving the Toeplitz system

$$\mathbf{H}(\mathbf{q}^{(k)}) \left(\mathbf{q}^{(k+1)} - \mathbf{q}^{(k)} \right) = \mathbf{c} - \bar{\mathbf{c}}(\mathbf{q}^{(k)}). \quad (88)$$

Clearly, $\mathbf{H}(\mathbf{q})$ and $\bar{\mathbf{c}}$ can be computed by the discrete Fourier transform.

D. Approximation Procedure for the Ordinary Rational Covariance Extension Problem

Given a $\mathbf{c} \in \mathfrak{C}_+$ and a $P \in \mathfrak{P}_+$, the *ordinary* rational covariance extension problem amounts to finding the unique $Q \in \mathfrak{P}_+$ satisfying the moment conditions

$$c_k = \int_{-\pi}^{\pi} e^{ik\theta} \frac{P(e^{i\theta})}{Q(e^{i\theta})} \frac{d\theta}{2\pi}, \quad k = 0, 1, \dots, n. \quad (89)$$

We would like to approximate the solution Q of this problem by the unique solution Q_N of the circulant rational covariance extension problem

$$c_k = \int_{-\pi}^{\pi} e^{ik\theta} \frac{P(e^{i\theta})}{Q_N(e^{i\theta})} d\nu_N, \quad k = 0, 1, \dots, n \quad (90)$$

where $d\nu_N$ is the measure (29) corresponding to N.

Theorem 7: Let $\mathbf{c} \in \mathfrak{C}_+$ and a $P \in \mathfrak{P}_+$. Moreover, for any $N \geq N_0$, where N_0 is defined as in Proposition 6, let Q_N be the unique solution of (90), and let Q be the unique solution of (89). Then $Q_N \to Q$ as $N \to \infty$.

Proof: Let $F : \mathfrak{P}_+ \to \mathfrak{C}_+$ be the map sending Q to $\mathbf{c}$ as in (89); i.e., $\mathbf{c} = F(Q)$. Given Q_N, define $\mathbf{c}^{(N)} := F(Q_N)$ with components

$$c_k^{(N)} = \int_{-\pi}^{\pi} e^{ik\theta} \frac{P(e^{i\theta})}{Q_N(e^{i\theta})} \frac{d\theta}{2\pi}, \quad k = 0, 1, \dots, n \quad (91)$$

for each $N \geq N_0$. Since (90) is a Riemann sum converging to (91) as N in the measure $d\nu_N$ tends to ∞ but Q_N is kept fixed, there is for each $\epsilon > 0$ an $N_1 \geq N_0$ such that $\|\mathbf{c}^{(N)} - \mathbf{c}\| < \epsilon$ for all $N \geq N_1$. Consequently, since $F(Q_N) = \mathbf{c}^{(N)}$, $F(Q) = \mathbf{c}$ and the map F is a diffeomorphism [7, Th. 1.3], $Q_N \to Q$ as $N \to \infty$. ■

IV. Determining P From Logarithmic Moments

We have shown that the solutions of the circulant rational covariance extension problem are completely parameterized in a smooth manner by the numerator trigonometric polynomials $P \in \mathfrak{P}_+(N)$, or, equivalently, by their corresponding banded circulant matrices $\mathbf{P}$. Next, we show how P can be determined from the logarithmic moments

$$m_k = \int_{-\pi}^{\pi} e^{ik\theta} \log \Phi(e^{i\theta}) d\nu, \quad k = 1, 2, \dots, n. \quad (92)$$

In the setting of the classical trigonometric moment problem such moments are known as *cepstral coefficients*, and in speech processing, for example, they are estimated for design purposes.

Now consider the problem of finding the spectral density Φ, or, equivalently, the circulant matrix $\boldsymbol{\Sigma}$, that maximizes the entropy gain

$$\mathbb{I}(\Phi) = \int_{-\pi}^{\pi} \log \Phi(e^{i\theta}) d\nu = \frac{1}{2N} \operatorname{tr} \log \boldsymbol{\Sigma} \quad (93)$$

subject to the two sets of moment conditions (52) and (92). Such a problem was apparently first considered in the usual trigonometric moment setting in an unpublished technical report [40] and then, independently and in a more elaborate form, in [5], [6], [17], [18].

Defining

$$M(\zeta) = \sum_{k=-n}^{n} m_k \zeta^{-k} \tag{94}$$

where $m_{-k} = \bar{m}_k$, $k = 1, 2, \ldots, n$, and m_0 is real, the Langrangian for this optimization problem can be written

$$\begin{aligned} L(\Phi, P, Q) &= \mathbb{I}(\Phi) + \sum_{k=-n}^{n} \bar{q}_k \left(c_k - \int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) d\nu \right) \\ &\quad - \sum_{k=-n}^{n} \bar{p}_k \left(m_k - \int_{-\pi}^{\pi} e^{ik\theta} \log \Phi(e^{i\theta}) d\nu \right) \\ &= \langle C, Q \rangle - \langle M, P \rangle - \int_{-\pi}^{\pi} Q(e^{i\theta}) \Phi(e^{i\theta}) d\nu \\ &\quad + \int_{-\pi}^{\pi} P(e^{i\theta}) \log \Phi(e^{i\theta}) d\nu \end{aligned} \tag{95}$$

where $p_1, \ldots, p_n, q_0, q_1, \ldots, q_n$ are Lagrange multipliers and $p_0 := 1$, and P and Q are the corresponding trigonometric polynomials (4). For the dual functional $(P, Q) \mapsto \sup_\Phi L(\Phi, P, Q)$ to be finite, P and Q must obviously be restricted to the closure of the cone $\mathfrak{P}_+(N)$. Therefore, for each such choice of (P, Q), we have the directional derivative

$$\delta L(\Phi, P, Q; \delta\Phi) = \int_{-\pi}^{\pi} \left(\frac{P}{\Phi} - Q \right) \delta\Phi d\nu \tag{96}$$

and hence a stationary point must satisfy

$$\Phi = \frac{P}{Q} \tag{97}$$

which inserted into (95) yields

$$\sup_{\Phi} L(\Phi, P, Q) = \mathbb{J}(P, Q) - 1$$

where

$$\mathbb{J}(P, Q) = \langle C, Q \rangle - \langle M, P \rangle + \int_{-\pi}^{\pi} P(e^{i\theta}) \log \frac{P(e^{i\theta})}{Q(e^{i\theta})} d\nu \tag{98}$$

and where we have used the fact that $\int P d\nu = p_0 = 1$. Accordingly, we define the bounded subset

$$\mathfrak{P}_+^{\mathrm{o}}(N) := \{ P \in \mathfrak{P}_+(N) \mid p_0 = 1 \} \tag{99}$$

of the cone $\mathfrak{P}_+(N)$. Note that, for $k = 1, 2, \ldots, n$

$$\frac{\partial \mathbb{J}}{\partial \bar{q}_k} = c_k - \int_{-\pi}^{\pi} e^{ik\theta} \frac{P(e^{i\theta})}{Q(e^{i\theta})} d\nu \tag{100a}$$

$$\frac{\partial \mathbb{J}}{\partial \bar{p}_k} = \int_{-\pi}^{\pi} e^{ik\theta} \log \frac{P(e^{i\theta})}{Q(e^{i\theta})} d\nu - m_k. \tag{100b}$$

Consequently, if there exists a stationary point $(P, Q) \in \mathfrak{P}_+^{\mathrm{o}}(N) \times \mathfrak{P}_+(N)$, (97) will satisfy both the moment conditions (52) and (92).

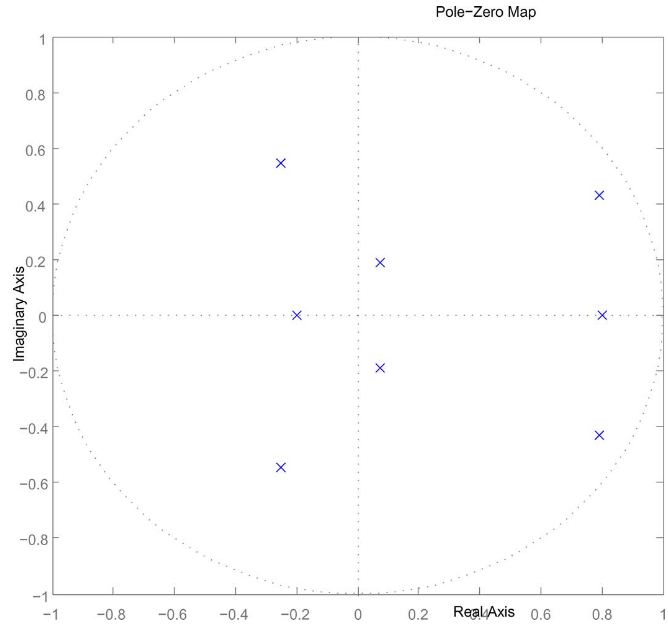


Fig. 1. Poles of true AR model.

A proof of the following theorem, which is a circulant version of Theorem 5.3 in [6], will be given in the Appendix.

Theorem 8: Suppose that $\mathbf{c} \in \mathfrak{C}_+(N)$ and $m_1, \ldots, m_n$ are complex numbers. Then there exists a solution $(\hat{P}, \hat{Q})$ that minimizes $\mathbb{J}(P, Q)$ over all $(P, Q) \in \overline{\mathfrak{P}_+^{\mathrm{o}}(N)} \times \overline{\mathfrak{P}_+(N)}$, and, for any such solution

$$\hat{\Phi} = \frac{\hat{P}}{\hat{Q}} \tag{101}$$

satisfies the covariance moment conditions (52). If, in addition, $\hat{P} \in \mathfrak{P}_+(N)$, (101) also satisfies the logarithmic moment conditions (92) and is an optimal solution of the primal problem to maximize the entropy gain (93) given (52) and (92). Then $\hat{Q} \in \mathfrak{P}_+(N)$, and the solution is unique. In fact, $\mathbb{J}$ is strictly convex on $\mathfrak{P}_+^{\mathrm{o}}(N) \times \mathfrak{P}_+(N)$.

Consequently, solving these optimization problems will always lead to a spectral density with the prescribed covariance lags $c_0, c_1, \ldots, c_n$, provided $\mathbf{c} \in \mathfrak{C}_+(N)$. However, we have not prescribed any condition on the logarithmic moments $m_1, \ldots, m_n$, as such a condition is hard to find and would depend on $\mathbf{c}$. If the moments $c_0, c_1, \ldots, c_n$ and $m_1, \ldots, m_n$ come from the same theoretical spectral density, the optimal solution (101) will also match the cepstral coefficients. In practice, however, $c_0, c_1, \ldots, c_n$ and $m_1, \ldots, m_n$ will be estimated from different data sets, so there is no guarantee that $\hat{P}$ does not end up on the boundary of $\mathfrak{P}_+(N)$ without satisfying the logarithmic moment conditions. Then the problem needs to be regularized, leading to adjusted values of $m_1, \ldots, m_n$ consistent with the covariances $c_0, c_1, \ldots, c_n$.

We shall use a regularization term proposed by Enqvist [17] in the context of the usual rational covariance extension problem. More precisely, we consider the regularized dual problem to find a pair $(P, Q) \in \mathfrak{P}_+^{\mathrm{o}}(N) \times \mathfrak{P}_+(N)$ minimizing

$$\mathbf{J}_\lambda(P, Q) = \mathbb{J}(P, Q) - \lambda \int_{-\pi}^{\pi} \log P(e^{i\theta}) d\nu \tag{102}$$

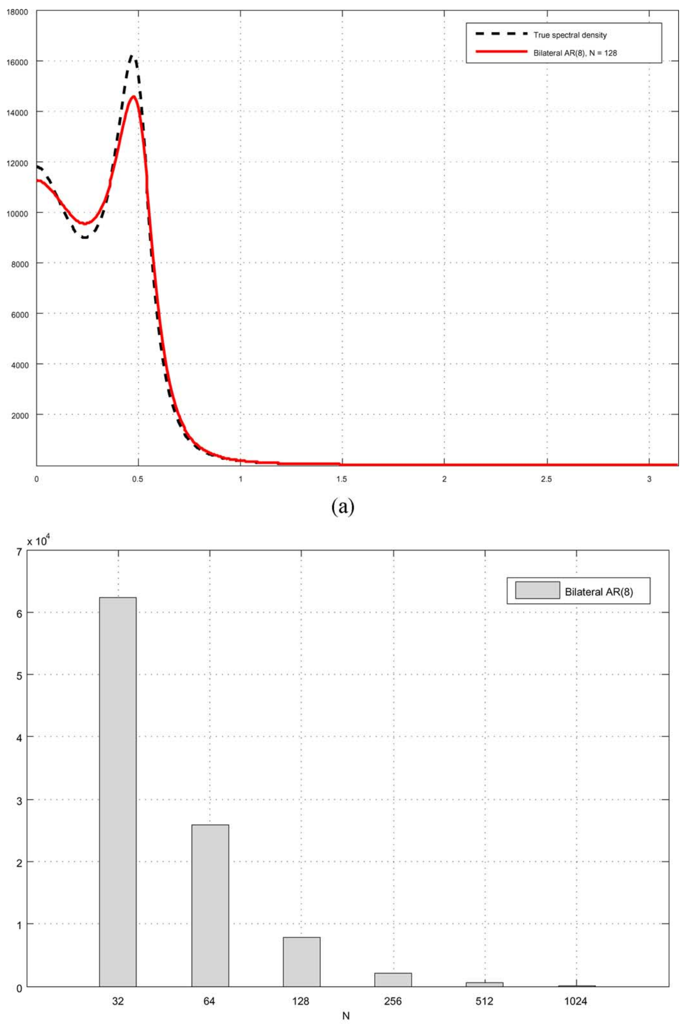


Fig. 2. Bilateral AR approximation: (top) spectrum for $\mathrm{N} = 128$ and true spectrum (dashed); (bottom) errors for $\mathrm{N} = 32, 64, 128, 256, 512$ and 1024.

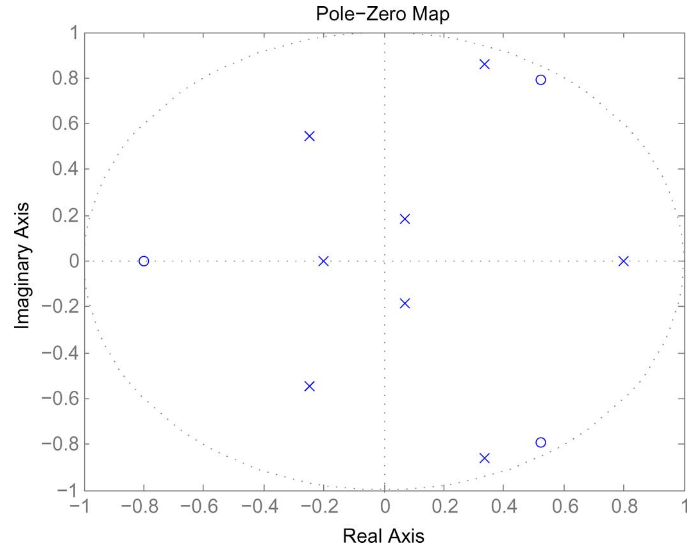


Fig. 3. Poles and zeros of true ARMA model.

for some $\lambda > 0$, or in circulant matrix form

$$\mathbf{J}_\lambda(P,Q) = \frac{1}{2N}\mathrm{tr}\{\mathbf{CQ}\} - \frac{1}{2N}\mathrm{tr}\{\mathbf{MP}\} + \frac{1}{2N}\mathrm{tr}\{\mathbf{P}\log\mathbf{PQ}^{-1}\} - \frac{\lambda}{2N}\mathrm{tr}\{\log\mathbf{P}\}. \quad (103)$$

This functional will take an infinite value for $P \in \partial\mathfrak{P}_+(N)$, since then $P(\zeta_k) = 0$ for some k, and hence the minimum will be in the interior. Then, for $k = 1, 2, \dots, n$

$$\frac{\partial \mathbf{J}_\lambda}{\partial \bar{p}_k} = \int_{-\pi}^{\pi} e^{ik\theta}\log\frac{P(e^{i\theta})}{Q(e^{i\theta})}d\nu - m_k - \varepsilon_k = 0 \quad (104)$$

at the minimum, where

$$\varepsilon_k = \int_{-\pi}^{\pi} e^{ik\theta}\frac{\lambda}{P(e^{i\theta})}d\nu = \frac{\lambda}{2N}\sum_{j=-N+1}^{N}\frac{\zeta_j^k}{P(\zeta_j)} = \frac{\lambda}{2N}\mathrm{tr}\{\mathbf{S}^k\mathbf{P}^{-1}\} \quad (105)$$

and hence the moments (52) and (92) are matched provided one adjusts the logarithmic moments $m_1, m_2, \dots, m_n$ to $m_1 + \varepsilon_1, m_2 + \varepsilon_2, \dots, m_n + \varepsilon_n$, the latter of which are consistent with $c_0, c_1, \dots, c_n$. Modifying the analysis in [17, p. 188–196] to the present setting it is easy to see that (103) is a monotonically nonincreasing function of λ, and that the solution tends as $\lambda \to \infty$ to a $(\hat{P}, \hat{Q})$ where $\hat{P} \equiv 1$, i.e., the maximum entropy solution.

Computing the Hessian of $\mathbf{J}_\lambda$, we notice that

$$\frac{\partial^2\mathbf{J}_\lambda}{\partial\bar{q}_k\partial q_\ell} = \frac{1}{2N}\sum_{j=-N+1}^{N}\zeta_j^{k-\ell}\frac{P(\zeta_j)}{Q(\zeta_j)^2} \quad (106a)$$

is the same as (71). Moreover,

$$\frac{\partial^2\mathbf{J}_\lambda}{\partial\bar{q}_k\partial p_\ell} = -\frac{1}{2N}\sum_{j=-N+1}^{N}\zeta_j^{k-\ell}\frac{1}{Q(\zeta_j)} \quad (106b)$$

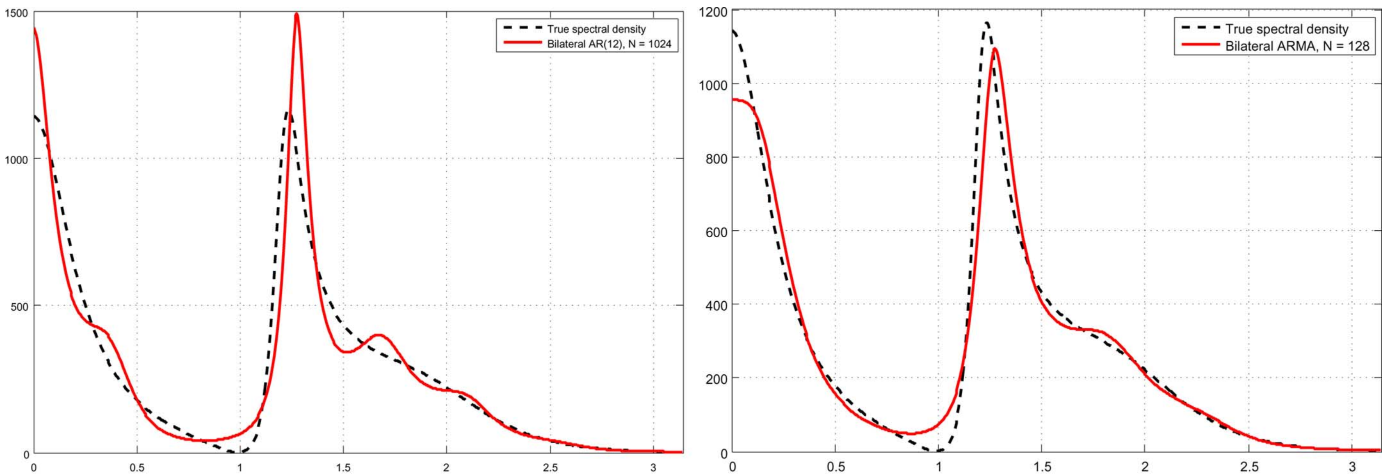
$$\frac{\partial^2\mathbf{J}_\lambda}{\partial\bar{p}_k\partial p_\ell} = \frac{1}{2N}\sum_{j=-N+1}^{N}\zeta_j^{k-\ell}\frac{1}{P(\zeta_j)} + \frac{1}{2N}\sum_{j=-N+1}^{N}\zeta_j^{k-\ell}\frac{\lambda}{P(\zeta_j)^2}. \quad (106c)$$

Since $\mathbb{J}$ is strictly convex (Theorem 8), then so is $\mathbf{J}_\lambda$, so the Hessian is positive definite. Newton's method can then be used as in Section III-C to determine the unique minimizer.

V. Numerical Examples

To illustrate our results we include some numerical examples generously provided by Masiero; for more details and further examples, see [39]. We shall apply our methods to covariance and logarithmic moments computed from true models, and compare the spectra thus obtained with the true spectra.

Our first example illustrates the use of circulant rational covariance extensions to approximate ordinary rational covariance extensions as proposed in Section III-D. Given an AR model of



(a) (b)

Fig. 4. Bilateral approximations with true spectrum (dashed): (left) bilateral AR with $n = 12$ and $N = 1024$; (right) bilateral ARMA with $n = 8$ and $N = 128$ using both covariance and logarithmic moment estimates.

order $n = 8$ with poles as depicted in Fig. 1, we consider bilateral approximations of degree eight for various choices of N. In Fig. 2, the top picture depicts the spectral density for $N = 128$ together with the true spectral density (dashed line), and the bottom picture illustrates how the estimation error decreases with increasing N.

In our second example we start from an ARMA model with $n = 8$ poles and three zeros distributed as in Fig. 3. Then, for various N, we approximate this model by circulant maximum entropy solutions of degree $n = 12$ and compare them to a bilateral ARMA representations of degree $n = 8$ computed by the combined covariance and cepstral procedure of Section IV. In Fig. 4, the left plot depicts the bilateral AR solution for $N = 1024$ and the right plot the bilateral ARMA solution for $N = 128$, both together with the true spectrum. This illustrates the advantage of bilateral ARMA modeling as compared to bilateral AR modeling, as a much lower value on N provides a better approximation, although n is smaller.

VI. Conclusion

In this paper, we have presented a complete parameterization of all solutions to the circulant covariance extension problem. We have shown that determining these solutions involves only computations based on the fast Fourier transform, potentially leading to efficient numerical procedures. This also provides a natural approximation for the ordinary rational covariance extension problem.

The circulant rational covariance extension problem is an inverse problem with infinitely many solutions in general, but by matching additional data in the form of logarithmic moments a unique solution can be determined.

For many applications, like image processing [13], [42], it will be important to generalize these results to the multivariable case. For scalar P this should be straightforward, but we have chosen to consider only the scalar case in this paper in order to keep notations reasonably simple and not blur the picture.

Appendix

Proof of Theorem 1: Consider the moment map $F^P : \mathfrak{P}_+(N) \to \mathfrak{C}_+(N)$ defined by (72) for an arbitrary $P \in \mathfrak{P}_+(N)$. This is a continuous map between connected spaces of the same (finite) dimension. Therefore, if we can prove that F^P is injective and proper, i.e., for any compact $K \subset \mathfrak{C}_+(N)$ the inverse image $(F^P)^{-1}(K)$ is compact—then, by Theorem 2.6 in [8], it is a homeomorphism, implying in particular that the system of moment equations $F^P(Q) = \mathbf{c}$ has a unique solution in $\mathfrak{P}_+(N)$.

Lemma 9: The moment map $F^P : \mathfrak{P}_+(N) \to \mathfrak{C}_+(N)$ is injective.

Proof: From (67) we have the gradient (70) and the Hessian (71), which is positive definite. Therefore, $\mathbb{J}_P$ is strictly convex, and any stationary point is a solution to the moment equations (52), which must be a unique if it exists. Hence, F^P is injective. ■

It remains to show that there exists a solution to the moment equations (52).

Lemma 10: Suppose the Toeplitz matrix $\mathbf{T}_n$ is positive definite; i.e., $\mathbf{c} \in \mathfrak{C}_+$. Then, for any compact $K \subset \mathfrak{C}_+(N)$, the inverse image $(F^P)^{-1}(K)$ is bounded.

Proof: Suppose Q satisfies the moment equations $F^P(Q) = \mathbf{c}$ for some $\mathbf{c} \in \mathfrak{C}_+(N)$. Then

$$\langle C, Q\rangle = \sum_{k=-n}^{n} c_k \bar{q}_k = \int_{-\pi}^{\pi} P(e^{i\theta}) d\nu =: \kappa$$

where κ is a constant. Now, let $a(z) = a_0 z^n + \ldots + a_{n-1} z + a_n$ be the stable polynomial spectral factor of $Q(z)$, i.e., $a(z)a(z)^* = Q(z)$. Then $\kappa = \langle C, Q\rangle = \mathbf{a}^* T_n \mathbf{a}$, where $\mathbf{T}_n$ is the Toeplitz matrix of $\mathbf{c}$ and $\mathbf{a} = (a_0, a_1, \ldots, a_n)^{\mathsf{T}}$. If $\mathbf{c}$ is restricted to the compact subset $K \in \mathfrak{C}_+$, the eigenvalues of T_n are bounded away from zero. Hence, $T_n \geq \varepsilon I$ for some $\epsilon > 0$, and consequently

$$\|\mathbf{a}\|^2 \leq \frac{1}{\varepsilon} \mathbf{a}^* T_n \mathbf{a} = \frac{\kappa}{\varepsilon}.$$

Consequently, $\|\mathbf{q}\|$, where $\mathbf{q} = (q_0, q_1, \dots, q_n)$, is also bounded, and hence so is $(F^P)^{-1}(K)$. ■

Lemma 11: The moment map $F^P : \mathfrak{P}_+(N) \to \mathfrak{C}_+(N)$ is proper.

Proof: Let K be a compact subset of $\mathfrak{C}_+(N)$, and let $\mathbf{c}^{(k)}$ be a sequence in K converging to $\hat{\mathbf{c}} \in K$. Since $(F^P)^{-1}(K)$ is bounded (Lemma 10), there is a convergent sequence $Q^{(k)}$ in the preimage of the sequence $\mathbf{c}^{(k)}$ converging to some limit $\hat{Q}$. We want to show that $\hat{Q} \in (F^P)^{-1}(K)$. The only way this can fail is that $\hat{Q}$ belongs to $\partial\mathfrak{P}_+(N)$, the boundary of $\mathfrak{P}_+(N)$. We observe that

$$\langle C^{(k)}, P\rangle = \int_{-\pi}^{\pi} \frac{P^2}{Q^{(k)}} d\nu$$

and consequently, since $P \in \mathfrak{P}_+(N)$

$$\sum_{j=-N+1}^{N} \frac{P(\zeta_j)^2}{\hat{Q}(\zeta_j)} = \langle \hat{C}, P\rangle$$

which requires that $\hat{Q}(\zeta_j) \neq 0$ for all j. However, $\hat{Q}$ can only belong to $\partial\mathfrak{P}_+(N)$ if some $\hat{Q}(\zeta_j)$ equals zero. Hence, $\hat{Q} \notin \partial\mathfrak{P}_+(N)$, as required. ■

This concludes the proof of Theorem 1.

Proof of Theorem 3: We have already proven above that F^P is a homeomorphism. It remains to prove that G^c is. For this we need two more lemmas.

Lemma 12: For each fixed $\mathbf{c} \in \mathfrak{C}_+(N)$, the map $G^c : \mathfrak{Q}_+(N) \to \mathfrak{P}_+(N)$ is injective.

Proof: Suppose that $G^c(Q_1) = G^c(Q_2) = P$ for some $Q_1, Q_2 \in \mathfrak{Q}_+(N)$. We want to show that $Q_1 = Q_2$. To this end, since

$$\int_{-\pi}^{\pi} e^{ik\theta} \frac{(Q_1 - Q_2)P}{Q_1 Q_2} d\nu = 0, \quad k = 0, 1, \dots, n$$

we have

$$\int_{-\pi}^{\pi} \frac{(Q_1 - Q_2)^2 P}{Q_1 Q_2} d\nu = \sum_{k=-n}^{n} \left(\left[q_k^{(1)} - q_k^{(2)} \right] \int_{-\pi}^{\pi} e^{ik\theta} \frac{(Q_1 - Q_2)P}{Q_1 Q_2} d\nu \right) = 0,$$

where

$$Q_\ell(z) = \sum_{k=-n}^{n} q_k^{(\ell)} z^k, \quad \ell = 1, 2$$

and, consequently, $Q_1(\zeta_j) = Q_2(\zeta_j)$ for all j, as claimed. ■

Lemma 13: For each fixed $\mathbf{c} \in \mathfrak{C}_+(N)$, the map $G^c : \mathfrak{Q}_+(N) \to \mathfrak{P}_+(N)$ is proper.

Proof: The proof follows the same pattern as that of Lemma 11. Hence, let K be a compact subset of $\mathfrak{P}_+(N)$, and let $P^{(k)}$ be a sequence in K converging to $\hat{P} \in K$. Since $(G^c)^{-1}(K) \subset \mathfrak{Q}_+(N)$ is bounded (Lemma 10), there is a convergent sequence $Q^{(k)}$ in the preimage of the sequence $P^{(k)}$ converging to some limit $\hat{Q}$. In order to ensure that $\hat{Q} \in (G^c)^{-1}(K)$, we must demonstrate that $\hat{Q} \notin \partial\mathfrak{P}_+(N)$. To this end, note that

$$\langle C, P^{(k)}\rangle = \int_{-\pi}^{\pi} \frac{(P^{(k)})^2}{Q^{(k)}} d\nu$$

and consequently, since $c \in \mathfrak{C}_+(N)$ and $\hat{P} \in K \subset \mathfrak{P}_+(N)$

$$\sum_{j=-N+1}^{N} \frac{\hat{P}(\zeta_j)^2}{\hat{Q}(\zeta_j)} = \langle C, \hat{P}\rangle > 0.$$

Since $\hat{P}(\zeta_j) > 0$ for all j, this requires that $\hat{Q}(\zeta_j) > 0$ for all j. Hence, $\hat{Q} \notin \partial\mathfrak{P}_+(N)$, as required. ■

The map G^c is a continuous map between connected spaces of the same dimension $n+1$. Noting that (71) is positive definite, the continuity follows from the inverse function theorem applied to the equation $F^P(Q) = \mathbf{c}$. Then, since G^c is injective and proper, it follows from Theorem 2.6 in [8], that it is a homeomorphism.

Proof of Theorem 4: Let (P_ℓ) be a sequence in $\mathfrak{P}_+(N)$ converging to $P \in \overline{\mathfrak{P}_+(N)} \setminus \{0\}$. Then there is a positive constant K such that $P_\ell(\zeta_j) \leq K$ for $\ell = 1, 2, 3, \dots$ and $j = -N+1, \dots, N$. For each ℓ, let Q_ℓ be the unique minimizer of

$$\mathbb{J}_{P_\ell}(Q) = \langle C, Q\rangle - \int_{-\pi}^{\pi} P_\ell(e^{i\theta}) \log Q(e^{i\theta}) d\nu$$

as prescribed by Theorem 2. Then

$$\int_{-\pi}^{\pi} e^{ik\theta} \frac{P_\ell(e^{i\theta})}{Q_\ell(e^{i\theta})} d\nu = c_k, \quad k = 1, 2, \dots, n \tag{107}$$

which, in particular, can be written

$$\frac{1}{2N} \sum_{j=-N+1}^{N} \Phi_\ell(\zeta_j) = c_0 \tag{108}$$

for $k = 0$, where $\Phi_\ell := P_\ell / Q_\ell$. Now suppose that the sequence Q_ℓ is unbounded. Then there is a subsequence, which we shall also denote Q_ℓ, for which $\|Q_\ell\|_\infty > 1$ and $\|Q_\ell\|_\infty \to \infty$. For each such Q, there is an $\varepsilon > 0$ such that

$$\mathbb{J}_{P_\ell}(Q) \geq \varepsilon \|Q\|_\infty - K \log \|Q\|_\infty. \tag{109}$$

To see this, we follow the lines of the proof of the Proposition 2.1 in [7] to note that, since $\langle C, Q/\|Q\|_\infty\rangle$ has a minimum $\varepsilon > 0$ on the compact set $\{Q \in \overline{\mathfrak{P}_+(N)} \mid \|Q\|_\infty = 1\}$ due to the fact that $T_n > 0$, we have $\langle C, Q\rangle \geq \varepsilon \|Q\|_\infty$. Then

$$\mathbb{J}_{P_\ell}(Q) \geq \varepsilon \|Q\|_\infty - \int_{-\pi}^{\pi} P_\ell \log \left(\frac{Q}{\|Q\|_\infty} \right) d\nu - K \log \|Q\|_\infty$$

where the second term is nonnegative and can be deleted.

Next, let $\tilde{Q} \in \mathfrak{P}_+(N)$ be arbitrary. Then, by optimality, $\mathbb{J}_{P_\ell}(\tilde{Q}) \geq \mathbb{J}_{P_\ell}(Q_\ell)$. Since $\mathbb{J}_{P_\ell}(\tilde{Q}) \to \mathbb{J}_P(\tilde{Q})$ as $\ell \to \infty$, there is a positive constant L such that

$$L \geq \mathbb{J}_{P_\ell}(\tilde{Q}) \geq \mathbb{J}_{P_\ell}(Q_\ell), \quad \ell = 1, 2, 3, \ldots$$

which together with (109) yields

$$L \geq \varepsilon \|Q_\ell\|_\infty - K \log \|Q_\ell\|_\infty, \quad \ell = 1, 2, 3, \ldots. \tag{110}$$

Then, comparing linear and logarithmic growth, we see that the sequence (Q_ℓ) is bounded, contrary to hypothesis. Consequently, there is a convergent subsequence (for convenience also indexed by ℓ) such that $Q_\ell \to \hat{Q}$, and, since $P_\ell \to P \neq 0$, (107) implies that $\hat{Q} \neq 0$. Hence, setting $\hat{\Phi} := P/\hat{Q}$, $\Phi_\ell \to \hat{\Phi}$. Since $\Phi_\ell(\zeta_j) \geq 0$ for $j = -N+1, \ldots, N$, it follows from (108) that $\hat{\Phi}(\zeta_j) \geq 0$ is finite for all j. Therefore, if $\hat{Q}(\zeta_j) = 0$ for some j, then $P(\zeta_j)$ must also be zero, so the roots cancel. Hence, taking limits in (107), we obtain

$$\int_{-\pi}^{\pi} e^{ik\theta} \hat{\Phi}(e^{i\theta}) d\nu = c_k, \quad k = 1, 2, \ldots, n$$

showing that $\hat{Q}$ is the required minimizer satisfying the moment conditions.

Proof of Theorem 8: We begin by showing that the sublevel set $\mathbb{J}^{-1}(\infty, r]$ is compact for each $r \in \mathbb{R}$. The sublevel set consists of those $(P, Q) \in \overline{\mathfrak{P}_+^{\mathrm{o}}(N)} \times \overline{\mathfrak{P}_+(N)}$ for which

$$r \geq \mathbb{J}_1(P, Q) + \mathbb{J}_2(P),$$

$$\text{where} \quad \mathbb{J}_1(P, Q) = \langle C, Q \rangle - \int_{-\pi}^{\pi} P(e^{i\theta}) \log Q(e^{i\theta}) d\nu$$

$$\mathbb{J}_2(P) = -\langle M, P \rangle + \int_{-\pi}^{\pi} P(e^{i\theta}) \log P(e^{i\theta}) d\nu.$$

Since $\mathfrak{P}_+^{\mathrm{o}}(N)$ is a bounded set that is bounded away from zero, there is a positive constant K such that $\|P\|_\infty \leq K$ and a $\rho \in \mathbb{R}$ such that $\mathbb{J}_2(P) \geq \rho$ for all $P \in \mathfrak{P}_+^{\mathrm{o}}(N)$. Hence, in view of the estimates leading to (109)

$$r - \rho \geq \mathbb{J}_1(P, Q) \geq \varepsilon \|Q\|_\infty - K \log \|Q\|_\infty$$

and therefore, comparing linear and logarithmic growth, it follows that the sublevel set $\mathbb{J}^{-1}(\infty, r]$ is bounded. Since it is also closed, it is compact, as claimed.

Since $\mathbb{J}$ thus has compact sublevel sets, there is a minimizer $(\hat{P}, \hat{Q})$. Then clearly $\hat{Q}$ is a minimizer of $\mathbb{J}_{\hat{P}}$, and hence, by Theorem 4, $\hat{\Phi} := \hat{P}/\hat{Q}$ satisfies the moment conditions (52). If $\hat{P} \in \mathfrak{P}_+^{\mathrm{o}}(N)$, then the minimizer must satisfy the stationarity condition $\partial \mathbb{J}/\partial \bar{p}_k = 0$, $k = 1, 2, \ldots, N$, and hence, by (100b), $\hat{\Phi}$ also satisfies the logarithmic moment conditions (92). Since

$$\mathbb{I}(\hat{P}, \hat{Q}) = L(\hat{P}, \hat{Q}) \geq L(P, Q) \quad \text{for all } (P, Q)$$

and $L(P, Q) = \mathbb{I}(P, Q)$ for all (P, Q) satisfying the moment conditions (52), $(\hat{P}, \hat{Q})$ solves the primal problem. By Theorem 1, $\hat{Q} \in \mathbb{P}_+(N)$.

It remains to prove that the optimal solution is unique if $(\hat{P}, \hat{Q}) \in \mathfrak{P}_+^{\mathrm{o}}(N) \times \mathfrak{P}_+(N)$. To this end, we form the directional derivative

$$\delta \mathbb{J}(\hat{P}, \hat{Q}; \delta P, \delta Q) = \langle C - \hat{P}\hat{Q}^{-1}, \delta Q \rangle + \langle \log(\hat{P}\hat{Q}^{-1}) - M, \delta P \rangle$$

and the second directional derivative

$$\delta^2 \mathbb{J}(\hat{P}, \hat{Q}; \delta P, \delta Q) = \langle (\delta P - \hat{P}\hat{Q}^{-1}\delta Q), \hat{P}^{-1}(\delta P - \hat{P}\hat{Q}^{-1}\delta Q) \rangle \geq 0$$

with equality if and only if $\delta P - \hat{P}\hat{Q}^{-1}\delta Q = 0$. Then, however,

$$\int_{-\pi}^{\pi} \hat{P}\hat{Q}^{-1} \delta Q \, d\nu = \int_{-\pi}^{\pi} \delta P \, d\nu = 0$$

since the pseudo-polynomial δP has no constant term, as $P(0) = 1$. Therefore, choosing $\delta Q = 1$, it follows from Theorem 1 that

$$c_0 = \int_{-\pi}^{\pi} \hat{P}\hat{Q}^{-1} \, d\nu = 0$$

which is a contradiction. Consequently,

$$\delta^2 \mathbb{J}(\hat{P}, \hat{Q}; \delta P, \delta Q) > 0$$

for all $\delta P, \delta Q$; i.e., the Hessian of $\mathbb{J}$ is positive definite, and hence $\mathbb{J}$ is strictly convex. Therefore, uniqueness follows.

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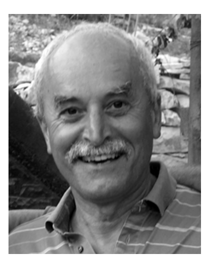

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