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A Theorem on Duality Between Estimation and Control for Linear Stochastic Systems with Time Delay

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Consider the following system of stochastic functional differential equations:

$$\begin{aligned} dx(t) &= \int_0^t d_sA(t, s)\, x(s)\, dt + dv(t), \\ x(0) &= x_0, \\ dz(t) &= \int_0^t d_sH(t, s)\, x(s)\, dt + dw(t), \\ z(0) &= 0, \end{aligned}$$

where $x(t) \in R^n$, $z(t) \in R^m$, the integrals are defined in the Stieltjes sense v and w are (vector) Wiener processes with incremental covariances $R_1(t)\, dt$ and $R_2(t)\, dt$, respectively, and x_0 is a stochastic variable with covariance R_0. The problem to determine the (linear) least-squares estimate of $x(\tau)$, where $0 \leqslant \tau \leqslant T$, given the observations $\{z(t);\ 0 \leqslant t \leqslant T\}$ is shown to be in a certain sense equivalent to the following problem of control (* stands for transposition):

$$\text{minimize } y^*(0)\, R_0\, y(0) + \int_0^T [y^*(t)\, R_1(t)\, y(t) + u^*(t)\, R_2(t)\, u(t)]\, dt,$$

when

$$y(t) + \int_t^T A^*(s, t)\, y(s)\, ds = \theta(\tau - t) b + \int_t^T H^*(s, t)\, u(s)\, ds,$$

where $\theta(t)$ is the Heaviside step function.

This is an extension of the well-known duality theorem of Kalman and Bucy to systems with time delay.

Finally, problems with sampled observations are briefly discussed.

1. FORMULATION OF THE PROBLEM

In their well-known paper [1], Kalman and Bucy formulated an interesting principle of duality between linear least-squares filtering and control. The problem to estimate the state at time t of an ordinary linear stochastic system, given linear incomplete observations in additive white noise up to time t, is in a certain sense equivalent to a classical problem in control theory, viz., the linear-quadratic regulator problem. This duality principle has been extended by Zachrisson [2] who formulated the dual control problem of the smoothing (interpolation) problem corresponding to the class of systems studied by Kalman and Bucy. Other aspects on duality for such systems can be found in Refs. [3] and [4].

The duality theorem of this paper is an extension of that of [1] and [2] (filtering and smoothing). The generalization consists in introducing time delays in both the system and the observation process. We shall give the dual control problem of the most general estimation problem of this type. Kwakernaak [5] has solved a special problem of this kind (discrete time delays). Solutions are also given (although they are not very explicit) by Kailath [6] and Lindquist [7]. However, none of these papers discusses duality between estimation and control—an approach by which new insight can be gained also for problems (like those of [5]) which are already explicitly solved. The approach of this paper has been inspired by the methods of Zachrisson [2] and [8] (the latter reference contains a derivation akin to that of [2] of the original result [1]) but also by the (nonrandom) theory of linear functional differential equations such as it appears in for example Halanay [9], Banks [10], and Hale [11].

Consider the following system of linear stochastic functional differential equations:

$$dx(t) = \int_0^t d_s A(t, s)\, x(s)\, dt + dv(t), \tag{1.1}$$

$$x(0) = x_0 ,$$

where $x(t) \in R^n$ is a state vector. A is an $n \times n$-matrix function, such that $A(t, s) \equiv 0$ for $s \geqslant t$. We further assume that there is an L_1 function $m(t)$ such that

$$\operatorname*{var}_{s\in[0,t]} |A(t, s)| \leqslant m(t)$$

($|\cdot|$ is Euclidean norm and var stands for total variation), in order to secure that the Stieltjes integral in (1.1) exists a.e. and is (Lebesgue) integrable. Finally, $v(t)$ is a (weighted) vector Wiener process with zero mean ($Ev(t) = 0$) satisfying

$$E\{v(s)\, v^*(t)\} = \int_0^{\min(t,s)} R_1(\tau)\, d\tau,$$

where R_1 is a symmetric, positive semidefinite $n \times n$-matrix function which we assume is locally bounded (* stands for transposition), and x_0 is a Gaussian stochastic variable with zero mean and a covariance matrix R_0 which is symmetric and positive semidefinite.

Since (1.1) is not an ordinary functional differential equation (almost all sample functions of v are nondifferentiable in almost every point) we shall interpret (1.1) in the following way[1]:

$$x(t, \omega) = x_0(\omega) + \int_0^t \int_0^s d_\tau A(s, \tau)\, x(\tau, \omega)\, ds + v(t, \omega). \tag{1.2}$$

Indeed, from Theorem 2.1 we see that there exists a unique sample continuous solution to (1.2) and thus (1.1), and this solution is given by Theorem 2.3. (In the sequel, we shall suppress ω from notation, whenever there is no cause of misunderstanding.)

Equation (1.1) can also be expressed in a more intuitive way:

$$\dot{x}(t) = \int_0^t d_s A(t, s)\, x(s) + \dot{v}(t), \tag{1.3}$$

where $\dot{v}$ is Gaussian white noise with zero mean, and

$$E\{\dot{v}(s)\, \dot{v}^*(t)\} = R_1 \delta(s - t)$$

(δ is the Dirac function); but of course we shall mean precisely (1.2). Now, we have the following *observation process*:

$$\begin{aligned} dz(t) &= \int_0^t d_s H(t, s)\, x(s)\, dt + dw(t), \\ z(0) &= 0, \end{aligned} \tag{1.4}$$

the solution of which is defined analogously to x. The observation $z(t) \in R^m$, H is an $m \times n$-matrix function, such that $H(t, s) \equiv 0$ for $s \geqslant t$, and such that

$$\operatorname*{var}_{s \in [0,t]} | H(t, s)| \leqslant k(t),$$

where k is an L_2 function. Moreover, $w(t)$ is a vector Wiener process with zero mean, satisfying

$$E\{w(s)\, w^*(t)\} = \int_0^{\min(t,s)} R_2(\tau)\, d\tau.$$

[1] We assume an underlying probability space $(\Omega, \mathscr{B}, P)$, where Ω is the sample space (elements: ω), $\mathscr{B}$ is a σ-algebra of events sufficiently large for our problem and P is the probability measure. Furthermore, all Wiener processes are separable and thus we are considering the sample continuous versions.

We assume that R_2 is symmetric and positive definite and that R_2 and R_2^{-1} are locally bounded. Finally, v, w, and x_0 are assumed to be completely independent, and all deterministic functions defined so far to be Borel measurable. In the same way as for x, we can write (1.4) in an intuitively more appealing form:

$$\dot{z}(t) = \int_0^t d_s H(t, s)\, x(s) + \dot{w}(t), \tag{1.5}$$

where $\dot{w}$ is Gaussian white noise with zero mean and

$$E\{\dot{w}(s)\, \dot{w}^*(t)\} = R_2(t)\, \delta(s - t).$$

Before proceeding to the formulation of the problem we should settle a point concerning integration: Integrals in this paper will usually be defined in the Lebesgue, Lebesgue–Stieltjes (LS) or Wiener–Doob–Ito[2] (q.m.) sense. It will usually be clear from the context what the appropriate concept of integration is, or else we shall point it out. However, on one occasion in Section 2, when we wish to integrate a function of bounded variation with respect to a continuous function of unbounded variation, we shall mean the Riemann–Stieltjes (RS) integral. (In this case, the LS integral does not exist.) In order to secure that the RS and LS integrals coincide whenever they both exist, we assume that the functions $s \to A(t, s)$, $s \to H(t, s)$ are continuous on the right (for every fixed t), and that the intervals of integration will be open in the left end and closed in the right end, i.e.,

$$\int_a^b = \int_{(a,b]} .$$

Now, our problem can be stated as follows: Given the observations $\{z(s); 0 \leqslant s \leqslant t\}$ we wish to determine the best estimate of $x(\tau)$ in the least-squares sense, i.e.,

$$\hat{x}(\tau \mid t) = E\{x(\tau) \mid Z_t\}, \tag{1.6}$$

where $Z_t = \sigma\{z(s); 0 \leqslant s \leqslant t\}$ is the σ-algebra generated by these observations. Since all processes involved are Gaussian (for they have been defined by linear transformations of Gaussian stochastic processes)[†], for every fixed pair (τ, t) the estimate (which is unbiased, that is $E\hat{x}(\tau \mid t) = Ex(\tau) = 0$) should be of the form

$$\hat{x}(\tau \mid t) = \int_0^t U(s; \tau, t)\, dz(s), \tag{1.7}$$

[2] Integration in quadratic mean (q.m.) with respect to a stochastic process with orthogonal increments. Compare [12, p. 425].

[†] More precisely, x and z are limits in probability of finite linear combinations of v, w and x_0 and hence jointly Gaussian. See (2.11) with $t_0 = 0$ and (1.4).

where $s \to U(s; \tau, t)$ is a square integrable $n \times m$ matrix. (The integral (1.7) should be understood in the following way:

$$\int_0^t U(s)\,dz = \int_0^t U(s) \int_0^s d_\sigma H(s, \sigma)\,x(\sigma)\,ds + \int_0^t U(s)\,dw$$

where the first integral is defined in the LS sense and the second in q.m.

We shall use the word *smoothing* whenever $\tau < t$, and the determination of $\hat{x}(t \mid t)$ will be named *filtering*. The case $\tau > t$ (*prediction*) can be transformed into a filtering problem by changing the H so that $H(t, s) \equiv 0$ for $s \geqslant t - h$, where h is the difference between the previous τ and t. Therefore we shall restrict ourselves to the case $\tau \leqslant t$.

Finally, we should point out that we can easily modify our model to include delays to act in the system from the very beginning. We may, for example, start our observations at a time $t_0 > 0$. Then $\hat{x}(\tau \mid t)$ should be found in the class:

$$\int_{t_0}^t U(s; \tau, t)\,dz(s), \tag{1.8}$$

whereas $x(t_0)$ depends on $x(t)$ for $0 < t \leqslant t_0$ (cf. Corollary 3.1.).

2. Mathematical Preliminaries

Theorem 2.1 (Existence and uniqueness). *The system of stochastic differential Eqs.* (1.1) *has a solution* $x(t, \omega)$ *with a.s. continuous* (*but not absolutely continuous*) *sample functions* $t \to x(t, \omega)$, *almost all of which are uniquely determined.*

Proof. (This is a slight modification of a standard proof in the theory of nonstochastic functional differential equations (cf. [9]):)

Let t_1 and t_2 be numbers $0 \leqslant t_1 < t_2 \leqslant T$ such that

$$\int_{t_1}^{t_2} m(t)\,dt = \alpha < 1,$$

and define a mapping K of the complete space C' (norm:

$$\| \xi \| = \sup_{0 \leqslant t \leqslant t_2} | \xi(t)|$$

where $|\cdot|$ is Euclidean vector norm) of continuous functions $[0, t_2] \to R^n$ coinciding with $\varphi(t)$ on $[0, t_1]$:

$$(K\xi)(t) = \begin{cases} \varphi(t) & \text{for } 0 \leqslant t \leqslant t_1 \\ x_0(\omega) + \int_0^t \int_0^s d_\tau A(s, \tau)\, \xi(\tau)\, ds + v(t, \omega) & \text{for } t_1 < t \leqslant t_2 . \end{cases}$$

If ω is such that $t \to v(t, \omega)$ is continuous (this is a.s. the case), K is a mapping of C' into itself. In fact, K is a contraction, for

$$\begin{aligned} | K\xi_1 - K\xi_2 | &= \left| \int_{t_1}^t \int_{t_1}^s d_\tau A(s, \tau)\, [\xi_1(\tau) - \xi_2(\tau)]\, ds \right| \\ &\leqslant \int_{t_1}^t \int_{t_1}^s | d_\tau A(s, \tau)|\, ds \sup_{t_1 \leqslant \tau \leqslant t_2} | \xi_1(\tau) - \xi_2(\tau)| \\ &\leqslant \int_{t_1}^t m(s)\, ds \, \| \xi_1 - \xi_2 \| = \alpha \| \xi_1 - \xi_2 \| \qquad \text{for } t_1 \leqslant t \leqslant t_2 , \end{aligned}$$

and

$$| K\xi_1 - K\xi_2 | = 0 \qquad \text{for } 0 \leqslant t \leqslant t_1 .$$

$$\therefore \| K\xi_1 - K\xi_2 \| \leqslant \alpha \| \xi_1 - \xi_2 \| .$$

Therefore, according to the principle of contraction mappings, there is a unique $\xi \in C'$ such that $\xi = A\xi$.

Now start with $t_1 = 0$ and C' the space of all continuous functions $[0, t_2] \to R^n$ for which $\xi(0) = x_0(\omega)$. Our procedure then defines for almost all ω a continuous function $\xi(t) = x(t, \omega)$ on $[0, t_2]$ (but not absolutely continuous, for $v(t, \omega)$ has unbounded variation). Then proceed with t_1 equal to the previous t_2, t_2 such that $\int_{t_1}^{t_2} m(t)\, dt < 1$ and C' the space of continuous functions $[0, t_2] \to R^n$ for which $\xi(t) \equiv x(t, \omega)$ on $[0, t_1]$. We continue in this manner to define $x(t, \omega)$ on subintervals of $[0, T]$ until $t_2 = T$. Our equation therefore has a unique continuous solution $x(t, \omega)$ a.s., which concludes the proof.

LEMMA 2.1. *Let A be as defined in Section 1 and let $f : [0, T] \to R^n$ be a function of bounded variation. Then the Volterra system of integral equations*

$$y(t) + \int_t^T A^*(s, t)\, y(s)\, ds = f(t) \tag{2.1}$$

has a unique solution y that is of bounded variation on $[0, T]$.

Proof. Define a mapping A of the space $L_\infty[0, T]$ into itself:

$$(Ay)(t) = f(t) - \int_t^T A^*(s, t)\, y(s)\, ds.$$

Then, since $|A^*(s, t)| \leqslant m(s)$, we have

$$\begin{aligned}
&|(A^k y_1)(t) - (A^k y_2)(t)| \\
&\quad \leqslant \int_t^T m(s)\, |(A^{k-1} y_1)(s) - (A^{k-1} y_2)(s)|\, ds \\
&\quad \leqslant \int_t^T m(s_1)\, ds_1 \int_{s_1}^T m(s_2)\, ds_2 \cdots \int_{s_{k-1}}^T m(s_k)\, ds_k \,\| y_1 - y_2 \|_\infty \\
&\quad = \frac{1}{k!} \left[\int_t^T m(s)\, ds\right]^k \| y_1 - y_2 \|_\infty . \\
&\quad \therefore \| A^k y_1 - A^k y_2 \|_\infty \leqslant \frac{1}{k!} \left[\int_0^T m(s)\, ds\right]^k \| y_1 - y_2 \|_\infty ,
\end{aligned}$$

and, therefore, A^k is a contraction mapping for sufficiently large k. Then $Ay = y$ has a unique solution in $L_\infty[0, T]$, for A is a continuous mapping. (This becomes obvious by putting $k = 1$ above.) Now, it is easily seen from the definition that this solution is of bounded variation on $[0, T]$. In fact, if $0 < t_0 < t_1 < t_2 < \cdots < t_n = T$, then

$$\begin{aligned}
\sum_{i=1}^n | y(t_i) - y(t_{i-1})| &\leqslant \int_0^T \sum_1^n | A^*(s, t_i) - A^*(s, t_{i-1})|\, | y(s)|\, ds \\
&\quad + \sum_1^n \int_{t_{i-1}}^{t_i} | A^*(s, t_{i-1})|\, | y(s)|\, ds + \sum_1^n | f(t_i) - f(t_{i-1})| \\
&\leqslant 2 \int_0^T m(s)\, | y(s)|\, ds + \operatorname*{var}_{0 \leqslant t \leqslant T} f(t) < \infty
\end{aligned}$$

independently of the subdivision, and therefore we have proved the lemma.

Now, for each $s \geqslant 0$ let $t \to X(t, s)$ be the unique absolutely continuous $n \times n$-matrix function defined on $[s, \infty)$ with initial value $X(s, s) = I$ and satisfying

$$\frac{\partial X(t, s)}{\partial t} = \int_s^t d_\tau A(t, \tau)\, X(\tau, s) \qquad \text{a.e.} \tag{2.2}$$

Then the solution of (2.1) is given by:

THEOREM 2.2. *Let y be the solution*[3] *of* (2.1). *Then for $s \leqslant t$:*

$$y(s) = X^*(t, s)\, y(t) + \int_t^T \left[\int_s^t X^*(\sigma, s)\, d_\sigma A^*(\tau, \sigma)\right] y(\tau)\, d\tau - \int_s^t X^*(\sigma, s)\, df(\sigma) \tag{2.3}$$

where X is the transfer matrix defined above.

Proof (cf. Ref. [10]). Integration by parts gives

$$y^*(t)\, X(t, s) - y^*(s)\, X(s, s) = \int_s^t y^*(\tau) \frac{\partial X(\tau, s)}{\partial \tau}\, d\tau + \int_s^t dy^*(\tau)\, X(\tau, s). \tag{2.4}$$

Now, using (2.2) and the fact that $A(\tau, \sigma) \equiv 0$ for $\sigma \geqslant \tau$, we have

$$\begin{aligned} \int_s^t y^*(\tau) \frac{\partial X(\tau, s)}{\partial \tau}\, d\tau &+ \int_t^T \left[\int_s^t X^*(\sigma, s)\, d_\sigma A^*(\tau, \sigma)\right] y(\tau)\, d\tau \\ &= \int_s^T \left[\int_s^t X^*(\sigma, s)\, d_\sigma A^*(\tau, \sigma)\right] y(\tau)\, d\tau \\ &= \int_s^t X^*(\sigma, s)\, d_\sigma \left[\int_\sigma^T A^*(\tau, \sigma)\, y(\tau)\, d\tau\right], \end{aligned} \tag{2.5}$$

where we have used a Fubini type theorem of Cameron and Martin [13]. Combining (2.4) and (2.5) and using (2.1) and the fact that $X(s, s) = I$ we have (2.3), and therefore the theorem is true.

COROLLARY 2.2. *The function $s \to X(t, s)$ defined on $[0, t]$ is the unique matrix solution of*

$$X(t, s) + \int_s^t X(t, \tau)\, A(\tau, s)\, d\tau = I. \tag{2.6}$$

Proof. Let y_i be the solution of (2.1) with $f(t) = e_i$, where e_i is the i-th unit vector. Then, from (2.3) we have,

$$y_i^*(s) = e_i^* X(T, s) \tag{2.7}$$

(which is valid for all $T \geqslant s$), i.e., the row vectors of $s \to X(T, s)$ are in fact identical to y_i^* for every $T \geqslant s$, which concludes the proof.

[3] Here, and in the sequel, f will be continuous on the right. Then, since $t \to A(s, t)$ is continuous on the right, the same is true for y due to the Lebesgue dominated convergence theorem.

Thus $t \to X(t, s)$ given by (2.2) is absolutely continuous and $s \to X(t, s)$ given by (2.6) is of bounded variation (for $s \leqslant t$). For $s > t$, define $X(t, s) \equiv 0$.

Since the continuous sample functions $t \to x(t, \omega)$ are of unbounded variation (a.s.), the integral $\int_{t_0}^t y^*(s)\, dx(s, \omega)$ cannot be defined in the LS sense. But if y is of bounded variation (like the solution of (2.1)), this integral exists in the RS sense (cf., e.g., Ref. [14, p. 7]), and the following integration by parts formula is valid:

$$y^*(t)\, x(t, \omega) - y^*(t_0)\, x(t_0, \omega) = \int_{t_0}^t y^*(s)\, dx(s, \omega) + \int_{t_0}^t x^*(s, \omega)\, dy(s). \tag{2.8}$$

(In the sequel, the ω will be suppressed from notation.)

In the same way the stochastic q.m. integral $\int_{t_0}^t y^*(s)\, dv$ (which can be defined for all square integrable y and whose sample functions are a.s. continuous), can also be defined as an RS integral for almost all ω whenever y is of bounded variation. The two stochastic processes defined by these integrals are stochastically equivalent and thus equal since they are sample continuous. In fact, q.m. convergence and a.s. convergence both imply convergence in probability, which determines the limit of the Riemann sum a.s.

LEMMA 2.2. *Let x and y be the solutions of* (1.1) *and* (2.1), *respectively. Then for $t_0 \geqslant 0$,*

$$y^*(T)\, x(T) = y^*(t_0)\, x(t_0) + \int_0^{t_0} x^*(\tau)\, d_\tau \left\{ \int_{t_0}^T A^*(s, \tau)\, y(s)\, ds \right\} + \int_{t_0}^T x^*(s)\, df(s) + \int_{t_0}^T y^*(s)\, dv(s). \qquad \text{(a.s.)}$$

Proof. (This proof is similar to that of Theorem 2.2, but we now have to be a little more careful due to the fact that x is not of bounded variation).

From (2.8) and (1.1) we have

$$\begin{aligned} y^*(T)\, x(T) &- y^*(t_0)\, x(t_0) \\ &= \int_{t_0}^T y^*(s)\, d_s \left\{ \int_0^s \int_0^\sigma d_\tau A(\sigma, \tau)\, x(\tau)\, d\sigma + v(s) \right\} + \int_{t_0}^T x^*(s)\, dy(s) \\ &= \int_{t_0}^T y^*(s)\, dv(s) + \int_{t_0}^T x^*(s)\, dy(s) + \int_{t_0}^T y^*(s) \int_0^s d_\tau A(s, \tau)\, x(\tau)\, ds. \end{aligned} \tag{2.9}$$

Now, since $A(s, \tau) \equiv 0$ for $\tau \geqslant s$ the last integral is

$$\int_{t_0}^T y^*(s) \int_0^T d_\tau A(s, \tau)\, x(\tau)\, ds.$$

We can regard the Stieltjes integral as an LS integral and use the Fubini type theorem of Cameron and Martin [13] to receive (all the conditions of the theorem are fulfilled):

$$\int_0^T x^*(\tau)\, d_\tau \left\{\int_{t_0}^T A^*(s, \tau)\, y(s)\, ds\right\}$$

$$= \int_0^{t_0} x^*(\tau)\, d_\tau \left\{\int_{t_0}^T A^*(s, \tau)\, y(s)\, ds\right\} + \int_{t_0}^T x^*(\tau)\, d_\tau \left\{\int_\tau^T A^*(s, \tau)\, y(s)\, ds\right\},$$

where we have used the fact that $A(s, \tau) \equiv 0$ for $\tau \geqslant s$. Then Eq. (2.9) gives

$$y^*(T)\, x(T) = y^*(t_0)\, x(t_0) + \int_0^{t_0} x^*(\tau)\, d_\tau \left\{\int_{t_0}^T A^*(s, \tau)\, y(s)\, ds\right\}$$

$$+ \int_{t_0}^T x^*(\tau)\, d_\tau \left\{y(\tau) + \int_\tau^T A^*(s, \tau)\, y(s)\, ds\right\} + \int_{t_0}^T y^*(s)\, dv(s). \tag{2.10}$$

Now, from (2.1) and (2.10) we obtain the result of the lemma.

THEOREM 2.3. *The solution of* (1.1) *can be expressed in the following way for* $t \geqslant t_0$:

$$x(t) = X(t, t_0)\, x(t_0) + \int_0^{t_0} d_\tau \left\{\int_{t_0}^t X(t, s)\, A(s, \tau)\, ds\right\} x(\tau) + \int_{t_0}^t X(t, s)\, dv(s), \tag{2.11}$$

where X *is the matrix function defined by* (2.2) *or* (2.6).

Proof. In Lemma 2.2, put $T = t$ and $y = y_i$ as defined in the proof of Corollary 2.2. Then (2.11) is immediately obtained from (2.7) and Lemma 2.2.

3. THE DUAL PROBLEM OF CONTROL

Consider the following class of control problems $P(T, \tau, b)$:
Determine an L_2 function $u : [0, T] \to R^m$ to minimize

$$y^*(0)\, R_0 y(0) + \int_0^T [y^*(t)\, R_1(t)\, y(t) + u^*(t)\, R_2(t)\, u(t)]\, dt, \tag{3.1}$$

when

$$y(t) + \int_t^T A^*(s, t)\, y(s)\, ds = \theta(\tau - t)\, b + \int_t^T H^*(s, t)\, u(s)\, ds, \tag{3.2}$$

where A, H, R_0, R_1 and R_2 are defined in Section 1 and θ is the step function

$$\theta(t) = \begin{cases} 1 & \text{for } t > 0, \\ 0 & \text{for } t \leqslant 0. \end{cases} \tag{3.3}$$

This problem of *control*, for which there exists a unique L_2 solution (cf. Theorem 3.2), is equivalent to the problem of *estimation* posed in Section 1 in the following sense:

THEOREM 3.1. *Let $u_0 : [0, t] \to R^m$ be the optimal solution to the control problem $P(t, \tau, b)$, $0 \leqslant \tau \leqslant t$. Then,*

$$b^*\hat{x}(\tau \mid t) = \int_0^t u_0^*(s)\, dz(s).$$

(Of course, u_0 is also a function of t, τ and b, but since these are considered fixed in the dual control problem, there should be no misunderstanding if they are suppressed from notation.)

Proof. Since $H(s, t) \equiv 0$ for $s \leqslant t$, the right member of (3.2), which we shall call $f(t)$, can be written

$$f(t) = \theta(\tau - t)\, b + \int_0^T H^*(s, t)\, u(s)\, ds.$$

The function f is of bounded variation. In fact,

$$\begin{aligned} \sum_{i=1}^n | f(t_i) - f(t_{i-1})| &\leqslant | b | + \int_0^T \sum_{i=1}^n | H^*(s, t_i) - H^*(s, t_{i-1})| \, | u(s)| \, ds \\ &\leqslant | b | + \int_0^T k(s) \, | u(s)| \, ds < \infty \end{aligned}$$

for any subdivision

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = T$$

(k and $| u |$ are both L_2 functions). With this choice of f, the state vector of our control problem y is a solution to Eq. (2.1), and from Lemma 2.2 we have ($t_0 = 0$)

$$y^*(T)\, x(T) = y^*(0)\, x(0) + \int_0^T x^*(t)\, df(t) + \int_0^T y^*(t)\, dv(t). \tag{3.4}$$

Now, again using the Fubini type theorem of Cameron and Martin [13] and the fact that $H(s, t) \equiv 0$ for $s \leqslant t$, we have $(0 \leqslant \tau \leqslant T)$

$$\begin{aligned}\int_0^T x^*(t)\, df(t) &= \int_0^T b^*x(t)\, d\theta(\tau - t) + \int_0^T x^*(t)\, d_t \left\{ \int_0^T H^*(s, t)\, u(s)\, ds \right\} \\ &= -\, b^*x(\tau) + \int_0^T u^*(s) \int_0^s d_tH(s, t)\, x(t)\, ds \qquad (3.5) \\ &= -\, b^*x(\tau) + \int_0^T u^*(s)\, dz(s) - \int_0^T u^*(s)\, dw(s)\end{aligned}$$

where we have used (1.4). Since it is clear from (3.2) that $y(T) = 0$, (3.4) and (3.5) give

$$b^*x(\tau) - \int_0^T u^*(s)\, dz(s) = y^*(0)\, x(0) + \int_0^T y^*(t)\, dv - \int_0^T u^*(t)\, dw.$$

Now, as $x(0)$, v, and w are independent, we obtain

$$\begin{aligned}&E\left[b^*x(\tau) - \int_0^T u^*(s)\, dz(s)\right]^2 \\ &\quad = y^*(0)\, R_0 y(0) + \int_0^T [y^*(t)\, R_1(t)\, y(t) + u^*(t)\, R_2(t)\, u(t)]\, dt,\end{aligned} \qquad (3.6)$$

which is equal to the objective function (3.1). In fact,

$$E[y^*(0)\, x(0)]^2 = y^*(0)\, E\{x(0)\, x^*(0)\}\, y(0) = y^*(0)\, R_0 y(0)$$

$$E\left[\int_0^T y^*(s)\, dv(s)\right]^2 = \int_0^T y^*(s)\, R_1(s)\, y(s)\, ds,$$

$$E\left[\int_0^T u^*(s)\, dw(s)\right]^2 = \int_0^T u^*(s)\, R_2(s)\, u(s)\, ds,$$

and all mixed products are zero (due to independence). Thus minimizing (3.6) = (3.1) determines the least squares estimate of $b^*x(\tau)$, which is precisely $b^*\hat{x}(\tau \mid T)$, to be $\int_0^T u_0^*(s)\, dz(s)$. This concludes the proof of the theorem.

Remark. In the case of *filtering* $(\tau = T)$ it does not make any difference if we redefine $y(T)$ to be equal to $y(T-) = b$. So in this case, (3.2) can be changed for

$$y(t) + \int_t^T A^*(s, t)\, y(s)\, ds = b + \int_t^T H^*(s, t)\, u(s)\, ds.$$

COROLLARY 3.1. *If observations are only available starting from $t_0 > 0$, i.e., if*

$$Z_t = \sigma\{z(s); t_0 \leqslant s \leqslant t\},$$

then the dual control problem is modified by putting $u(t) \equiv 0$ for $0 \leqslant t < t_0$ in (3.1) *and* (3.2).

Proof. Put

$$f(t) = \theta(\tau - t)\, b + \int_{t_0}^{T} H^*(s, t)\, u(s)\, ds$$

in the proof of Theorem 3.1.

THEOREM 3.2. *There exists a unique L_2 solution ($L_2([0, T], R^n)$) for the problem $P(T, \tau, b)$.*

Proof. For every $u \in L_2$, the solution of (3.2) can be expressed by means of (2.3):

$$\begin{aligned} y(t) &= -\int_t^T X^*(\sigma, t)\, d_\sigma \left\{\theta(\tau - \sigma)\, b + \int_\sigma^T H^*(s, \sigma)\, u(s)\, ds\right\} \\ &= X^*(\tau, t)\, b\theta(\tau - t) - \int_t^T \left[\int_t^s d_\sigma H(s, \sigma)\, X(\sigma, t)\right]^* u(s)\, ds, \end{aligned} \tag{3.7}$$

where we have used the Fubini type theorem of Cameron and Martin in the same way as in the proof of Theorem 3.1.

Therefore, the problem is of essentially the same type as the one treated in Ref. [18, p. 222]. Observing that R_1, R_2, and R_2^{-1} are bounded, the proof presented there (using the parallelogram law) applies to our problem [provided that the L_2-space to which y belongs is modified to take care of the first term of (3.1)].

So far our results concern a very wide class of problems. The problem posed in Section 1 allows for time-dependent delays, and the matrices A and H may have singular parts. Therefore, we shall specialize our problem somewhat. To this end we define A and H to be

$$A(t, s) = -\sum_{i=1}^{n} A_i(t)\, \theta(t - h_i - s) - \int_s^t A_0(t, \tau)\, d\tau, \tag{3.8}$$

$$H(t, s) = -\sum_{i=1}^{n} H_i(t)\, \theta(t - h_i - s) - \int_s^t H_0(t, \tau)\, d\tau, \tag{3.9}$$

where $0 = h_1 < h_2 < h_3 < \cdots < h_n$, θ is the step function defined by (3.3), $A_0(t, \tau) \equiv H_0(t, \tau) \equiv 0$ for $\tau \geqslant t$, $A_1, A_2 \cdots A_n$ and A_0 are integrable, and $H_1, H_2 \cdots H_n$ and H_0 square integrable (A_0 and H_0 as functions of two variables). Then $s \to A(t, s)$ and $s \to H(t, s)$ are continuous on the right, $A(t, s) \equiv H(t, s) \equiv 0$ for $s \geqslant t$, and all other conditions of Section 1 are fulfilled as well.

Now, inserting (3.8) into (1.1) we obtain the following system equation[4]:

$$\begin{aligned} dx(t) &= \left[\sum_{i=1}^{n} A_i(t)\, x(t - h_i) + \int_0^t A_0(t, s)\, x(s)\, ds\right] dt + dv(t), \\ x(0) &= x_0, \\ x(t) &= 0 \qquad \text{for } t < 0, \end{aligned} \tag{3.10}$$

and in the same way, by inserting (3.9) into (1.4), we have the observation process

$$\begin{aligned} dz(t) &= \left[\sum_{i=1}^{n} H_i(t)\, x(t - h_i) + \int_0^t H_0(t, s)\, x(s)\, ds\right] dt + dw(t), \\ z(0) &= 0. \end{aligned} \tag{3.11}$$

Note that some A_i and H_i may be identically zero, so there is no restriction in assuming the same set of delays h_i in (3.10) and (3.11). (In fact, if there are two different sets of h_i in the equations, take instead the union of these sets.)

For this class of problems, the dual vector function y turns out to be absolutely continuous except at $t = \tau$. In fact, with A defined by (3.8) we have

$$\begin{aligned} &\int_t^T A^*(s, t)\, y(s)\, ds \\ &\qquad = -\sum_i \int_t^T \theta(s - h_i - t)\, A_i^*(s)\, y(s)\, ds - \int_t^T \int_t^s A_0^*(s, \tau)\, d\tau y(s)\, ds \\ &\qquad = -\sum_i \int_{\min(t+h_i, T)} A_i^*(s)\, y(s)\, ds - \int_t^T \int_\tau^T A_0^*(s, \tau)\, y(s)\, ds\, d\tau, \end{aligned}$$

[4] Really, due to the definition of the interval of integration, the right member of (1.1) is not affected by x_0, which is the case in (3.10) for $t = h_i$ ($i = 1, 2, \ldots, n$). We have allowed this change since it does not affect x.

where we have used Fubini's theorem (A_0 is integrable). In the same way, we calculate the second integral in (3.2) to obtain

$$y(t) - \sum_i \int_{\min(t+h_i,T)}^{T} A_i^*(s)\, y(s)\, ds - \int_t^T \int_\tau^T A_0^*(s, \tau)\, y(s)\, ds\, d\tau$$
$$= \theta(\tau - t)\, b - \sum_i \int_{\min(t+h_i,T)}^{T} H_i^*(s)\, u(s)\, ds - \int_t^T \int_\tau^T H_0^*(s, \tau)\, u(s)\, ds\, d\tau.$$

Therefore, the dual vector function $y(t)$ is absolutely continuous on $[0, \tau)$ and $(\tau, T]$, and we have a differential equation for it:

$$\dot{y}(t) = -\sum_{i=1}^{n} A_i^*(t + h_i)\, y(t + h_i) - \int_t^T A_0^*(s, t)\, y(s)\, ds$$
$$+ \sum_{i=1}^{n} H_i^*(t + h_i)\, u(t + h_i) + \int_t^T H_0^*(s, t)\, u(s)\, ds \quad \text{(a.e.)}$$
$$\text{for } 0 \leqslant t \leqslant T, \tag{3.12}$$
$$y(\tau -) - y(\tau) = b,$$
$$y(t) = 0 \qquad \text{for } t \geqslant T,$$
$$u(t) = 0 \qquad \text{for } t \geqslant T.$$

Then the *dual problem of control* (corresponding to (3.10) and (3.11)) is: Determine u to minimize (3.1) subject to (3.12). If u_0 is the optimal control, we have ($\tau \leqslant T$)

$$b^* \hat{x}(\tau \mid T) = \int_0^T u_0^*(s)\, dz(s).$$

By reversing time we see that (3.12) is actually a delay differential equation. We even have delays in the control. (This is a complication which only recently has been studied in control theory. Compare Ref. [15], where also other references are given.)

4. Some Examples

1. Consider the filtering problem corresponding to the following x and z processes:

$$dx(t) = \left[A_1(t)\, x(t) + A_2(t)\, x(t-h) + \int_{t-h}^{t} A_0(t, s)\, x(s)\, ds\right] dt + dv(t)$$
$$\text{for } t \geqslant 0,$$
$$x(t) = 0 \qquad \text{for } t \leqslant 0,$$
$$dz(t) = H(t)\, x(t)\, dt + dw(t),$$
$$z(0) = 0.$$

Now, defining $A_0(t, s)$ to be identically zero for $s \leqslant t - h$, our problem belongs to the class defined by (3.10) and (3.11). Then the dual control problem is

$$\min \int_0^T [y^*(t) R_1(t) y(t) + u^*(t) R_2(t) u(t)] \, dt,$$

$$\begin{aligned} \dot{y}(t) &= - A_1^*(t) y(t) - A_2^*(t+h) y(t+h) \\ &\quad - \int_t^{t+h} A_0^*(s, t) y(s) \, ds + H^*(t) u(t) \qquad \text{for } t \leqslant T, \end{aligned}$$

when

$$\begin{aligned} y(T) &= b, \\ y(t) &= 0 \qquad \text{for } t > T. \end{aligned}$$

A feedback solution of this problem[5] by Kushner and Barnea [16] is given by

$$u_0(t) = K^*(t) y_0(t) + \int_t^{t+h} L^*(s, t) y_0(s) \, ds,$$

where we refer to [16] for a definition of K and L. (Note that we have reversed time.) Then,

$$\begin{aligned} \dot{y}_0(t) &= - [A_1(t) - K(t) H(t)]^* y_0(t) - A_2^*(t+h) y_0(t+h) \\ &\quad - \int_t^{t+h} [A_0(s, t) - L(s, t) H(t)]^* y_0(s) \, ds, \\ y_0(T) &= b, \\ y_0(t) &= 0 \qquad \text{for } t > T. \end{aligned}$$

If $\Phi(t, s)$ is the matrix solution of (2.6) with ($L(t, s) \equiv 0$ for $s \leqslant t - h$):

$$\begin{aligned} A(t, s) &= - [A_1(t) - K(t) H(t)] \theta(t - s) - A_2(t) \theta(t - h - s) \\ &\quad - \int_s^t [A_0(t, s) - L(t, s) H(s)] \, ds, \end{aligned}$$

then, from (2.3) we obtain $y_0(t) = \Phi^*(T, t) b$ for $t \leqslant T$, and therefore:

$$u_0(t) = U^*(T, t) b,$$

[5] We assume that all conditions on A_1, A_2, A_0, H, R_1, and R_2 (continuity, etc.) imposed in [16] are valid.

where

$$U(T, t) = \Phi(T, t) K(t) + \int_t^{\min(t+h,T)} \Phi(T, s) L(s, t) \, ds,$$

and, since

$$b^*\hat{x}(T \mid T) = \int_0^T u_0^*(s) \, dz(s),$$

we have

$$\hat{x}(t \mid t) = \int_0^t U(t, s) \, dz(s).$$

2. We shall solve the smoothing problem corresponding to

$$\begin{aligned} dx(t) &= A(t) x(t) \, dt + dv(t), \\ x(0) &= x_0, \\ dz(t) &= H(t) x(t) \, dt + dw(t), \\ z(0) &= 0, \end{aligned}$$

by a method which differs somewhat from that of [2].

First determine a feedback solution of the imbedded dual control problem

$$\min \left\{ y^*(0) R_0 y(0) + \int_0^\tau (y^* R_1 y + u^* R_2 u) \, dt \right\},$$

when

$$\dot{y} = - A^* y + H^* u, \qquad t > \tau.$$

It is well-known (cf., for example, Ref. [17]) that

$$u_0 = K^*(t) y_0(t),$$

where $K(t) = R(t) H^*(t) R_2^{-1}(t)$ and R is the solution of the matrix Riccati equation

$$\begin{aligned} \frac{dR}{dt} &= R_1 + AR + RA - RH^* R_2^{-1} HR, \\ R(0) &= R_0 . \end{aligned}$$

Then if $\Phi(t, s)$ is the matrix solution of

$$\begin{aligned} \frac{\partial \Phi(t, s)}{\partial t} &= (A - KH) \Phi(t, s), \\ \Phi(s, s) &= I, \end{aligned}$$

y_0 is given by $y_0(t) = \Phi^*(\tau, t)\, y_0(\tau -)$ and

$$u_0(t) = K^*(t)\, \Phi^*(\tau, t)\, y_0(\tau -) \qquad \text{for } t < \tau.$$

Furthermore, the minimum of the cost functional is

$$y_0^*(\tau -)\, R(\tau)\, y_0(\tau -)$$

(cf., Ref. [17]). Also,

$$y_0^*(\tau -)\, \hat{x}(\tau \mid \tau) = \int_0^\tau u_0^*(t)\, dz(t),$$

i.e.,

$$\hat{x}(\tau \mid \tau) = \int_0^\tau \Phi(\tau, t)\, K(t)\, dz(t). \tag{4.1}$$

Now, since $y(\tau -) = y(\tau) + b$, for $t > \tau$, we have (using the previous solution in the sense of dynamic programming) the dual control problem

$$\min \left\{ (y(\tau) + b)^*\, R(\tau)\, (y(\tau) + b) + \int_\tau^T (y^* R_1 y + u^* R_2 u)\, dt \right\},$$

when

$$\dot{y} = - A^* y + H^* u, \qquad y(T) = 0.$$

This problem has the following solution (cf., Ref. [17]):

$$u_0(t) = K^*(t)\, y_0(t) + R_2^{-1}(t)\, H(t)\, \Phi^*(\tau, t)\, R(\tau)\, b \qquad \text{for } t > \tau,$$

where we easily find y_0 to be

$$y_0(t) = - \int_t^T \Phi^*(s, t)\, H^*(s)\, R_2^{-1}(s)\, H(s)\, \Phi(\tau, s)\, ds R(\tau)\, b. \tag{4.2}$$

Then by changing the order of integration (permitted due to a Fubini type theorem for stochastic integrals (cf., Doob [12]):

$$\begin{aligned}
&\int_\tau^T u_0^*(t)\, dz(t) \\
&\quad = b^* R(\tau) \int_\tau^T \Phi^*(\tau, s)\, H^*(s)\, R_2^{-1}(s)\, \Big[dz(s) - H(s) \int_\tau^s \Phi(s, t)\, K(t)\, dz(t)\, ds\Big] \\
&\quad = b^* R(\tau) \int_\tau^T \Phi^*(\tau, s)\, H^*(s)\, R_2^{-1}(s)\, [dz(s) - H(s)\, \hat{x}(s \mid s)\, ds] - y_0^*(\tau)\, \hat{x}(\tau \mid \tau),
\end{aligned}$$

where we have used (4.1), and, to obtain the last term, (4.2) and the fact that

$$\Phi(s, t) = \Phi(s, \tau)\,\Phi(\tau, t).$$

Since

$$\int_0^\tau u_0^*(t)\,dz(t) = y_0^*(\tau -)\,\hat{x}(\tau \mid \tau) = b^*\hat{x}(\tau \mid \tau) + y_0^*(\tau)\,\hat{x}(\tau \mid \tau),$$

the optimal estimate is

$$\hat{x}(\tau \mid T) = \hat{x}(\tau \mid \tau) + \int_\tau^T R(\tau)\,\Phi^*(\tau, s)\,H^*(s)\,R_2^{-1}(s)\,[dz(s) - H(s)\,\hat{x}(s \mid s)\,ds].$$

This is a well-known result, and we have presented it for the sole purpose to demonstrate how the difficulty created by the jump condition can be overcome by modifying the cost functional for $t > \tau$.

5. A Remark on Estimation with Sampled Observations

In many practical situations we have access to the observation process at discrete times only. That is, at time t the following observations are available:

$$z(t_1), z(t_2), z(t_3), \ldots z(t_{n(t)}),$$

where

$$0 < t_1 < t_2 < t_3 < \cdots t_{n(t)} \leqslant t < t_{n(t)+1} < \cdots.$$

Then, for a fixed T the estimate $b^*\hat{x}(\tau \mid T)$ belongs to the class

$$\sum_{i=1}^{n(T)} c_i^* z(t_i) = \int_0^T u^*(t)\,dz(t), \tag{5.1}$$

where c_i is an m-vector (which depends on T and τ), and (θ is defined by (3.3)):

$$u(t) = \sum_{i=1}^{n(T)} c_i \theta(t_i - t). \tag{5.2}$$

All results obtained in the previous sections remain valid for this problem, except that we here confine our research for an optimal u to the class (5.2). That is, our problem is to find a vector sequence $c_1, c_2 \cdots c_{n(T)}$ forming an $n(T) \times m$-vector c, such that (3.1) is minimized when the constraints (3.2) and (5.2) are fulfilled.

This problem is equivalent to minimizing a quadratic function in c_i, $i = 1, 2 \cdots n(T)$:

$$c^*Qc + 2b^*P^*c + \alpha, \tag{5.3}$$

where Q is a positive definite and symmetric matrix, P is another matrix, and α is a real number. In fact, for control functions of type (5.2) we have from (3.7)

$$\begin{aligned} y(t) &= X^*(\tau, t)\, b\theta(\tau - t) + \sum_{i=1}^{n(T)} \int_t^{\max(t_i, t)} ds \left[\int_0^s d_\sigma H(s, \sigma)\, X(\sigma, t) \right]^* c_i \\ &= X^*(\tau, t)\, b\theta(\tau - t) + M(t)\, c. \end{aligned} \tag{5.4}$$

Inserting (5.4) into (3.1) and also observing the fact that

$$\int_0^T u^* R_2 u \, dt = c^* R c$$

for a suitably chosen positive definite and symmetric matrix R, we obtain an expression of type (5.3) and our assertion is therefore true. So essentially our problem is now solved and the optimal c is given by $c_0 = -Q^{-1}Pb$, i.e.,

$$\hat{x}(\tau \mid T) = -P^* Q^{-1} \bar{z}, \tag{5.5}$$

where $\bar{z}$ is the $n(T) \times m$-vector formed by the observations $z(t_i)$.

Note, however, that P and Q depend on T (and on τ); so in order to construct a recursive estimator, we shall have to look into this problem a little more thoroughly. But in this paper we shall not pursue this matter any further.

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