Sensitivity shaping in feedback control and analytic interpolation theory*

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Dedicated to Alain Bensoussan on his 60th birthday

1 Introduction

The problem of stabilizing a linear control system leads to several important considerations. In particular, there are certain basic requirements which need to satisfied in any practical feedback system [6, 7, 11, 18]. First, the closed-loop system needs to be *internally stable*, i.e., the transfer function between any two points in the loop should be stable. Internal stability guarantees that all signals in the system remain bounded when a bounded signal is injected at any location. Secondly, the absolute value of the closed-loop transfer function needs to be bounded in the right half-plane. Prescribing a fixed uniform bound, the design of an internally stable feedback system leads to a Nevanlinna-Pick interpolation problem [12, 13, 16].

In many situations, we require in addition that the closed-loop transfer function has bounded degree, often chosen to be equal to the number of interpolation conditions. In general, this ensures not only low degree of the controller but also that the feedback system behaves like a low-order system, a common specification in many applications. In general, there are infinitely many transfer functions satisfying these these conditions, and one would like to select one which best satisfied some additional specifications. However, classical theory does not provide procedures for determining an arbitrary such solution, but only a particular one, known as the *central solution*.

Recently, however, a new theory has been developed [1, 2, 5, 9, 10], which provides a complete parametrization of all solutions and a procedure for determining each of them. This is a modification of a theory previously developed for the Caratheodory extension problem with degree constraint [3, 4, 8].

In this paper, we shall apply this theory to the design problem described above. In Section 2 we set up and motivate the problem, in Section 3 we review pertinent fact about the theory of Nevanlinna-Pick interpolation with degree constraint, and in Section 4 we discuss various desighn strategies and give some examples. The sensitivity shaping problem considered here is a special case of the model matching problem [6, 7, 11], and the theory can also be appled to this problem.

Only the single-input-single-output case is considered here. For such problems good results can be obtained by manual loop shaping. It is therefore desirable to extend

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these results to the multi-input-multi-output case, since in this case the advantages of our procedure become much more convincing.

2 Sensitivity shaping in feedback control

Consider a standard linear control system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

with a scalar input u and a scalar output y. In Figure 1 we depict this system as an input-output system with the transfer function

$$P(s) = C(sI - A)^{-1}B + D.$$

We shall refer to it as the "plant." The plant is said to be *stable* if all the poles of P lie in the open left half-plane. Then, since P is proper, it belongs to $\mathcal{RH}^\infty$, the space of all rational functions which are bounded and analytic in the closed right half-plane. If, in addition, P has all its zeros in the open left half-plane it is said to be *minimum-phase*.

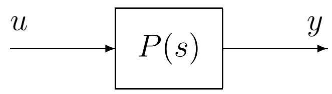

Figure 1: The plant

By contrast, suppose that $P(s)$ has n_p poles and n_z zeros in the closed right half-plane (including point at infinity; let us denote them $p_1, p_2, \cdots, p_{n_p}$ and $z_1, z_2, \cdots, z_{n_z}$, respectively. For simplicity, we assume that these zeros and poles are simple. If $n_p \neq 0$, the plant is unstable, and we need to stabilize it by feedback. Consider, therefore, the standard feedback system depicted in Figure 2. Here $C(s)$ is a controller to be designed so that the closed-loop system fulfills certain design specifications, r is a reference signal, d a disturbance, and n measurement noise.

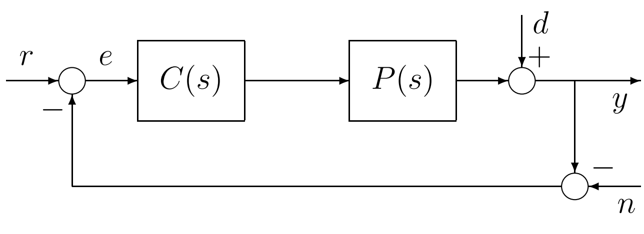

Figure 2: The closed-loop system

It is straight-forward to see that the closed-loop transfer function from the reference signal r to the error signal e is given by

$$S = \frac{1}{1 + PC}. \tag{1}$$

The function S is called the *sensitivity function*, and it is also the transfer function from the disturbance d to the output y.

The first object is to stabilize the system in such a way that the transfer functions between any two arbitrary points in the closed-loop system is stable. This stronger concept of stablity is called *internal stability.* For internal stability, you require not only that the closed-loop system transfer function (from r to y) is stable, but that there is no cancelation of right-half-plane (unstable) pole and zeros in the product PC [11, p.13]. Internal stability is needed in all practical feedback systems, since it guarantees that all signals in the system remain bounded when a bounded signal, such as r, d or n in Figure 2, is injected at any location in the loop.

It is well-known [11, Theorem 7.2.2] that internal stability is achieved if and only if the sensitivity function is stable and satisfies the interpolation conditions

$$S(z_k) = 1, \quad k = 1, 2, \cdots, n_z, \quad \text{and} \quad S(p_k) = 0, \quad k = 1, 2, \cdots, n_p. \tag{2}$$

Secondly, for robustness it is also desirable to put a specified uniform bound on the absolute value of the sensitivity function. For example, inserting a reference signal r in the feedback system of Figure 2, we have $e = Sr$, and hence

$$\|e\|_2 \leq \|S\|_\infty \|r\|_2.$$

To bound the error norm, we need to bound $\|S\|_\infty := \sup_{\omega\in(-\infty,\infty)} |S(i\omega)|$. Similarly, we note that the output y produced by a disturbance d satisfies

$$\|y\|_2 \leq \|S\|_\infty \|d\|_2,$$

so for disturbance attenuation, we also need to bound $\|S\|_\infty$. Therefore, we require that

$$\|S\|_\infty < \gamma \tag{3}$$

for some prescribed $\gamma > 0$.

With this bound, the $\mathcal{RH}^\infty$-function S/γ maps the right half-plane into the open unit disc. Hence the problem of finding an S satisfying both (2) and (3) is equivalent to the *Nevanlinna-Pick interpolation problem* [16]. It has a solution if and only if a certain matrix depending on the interpolation data, the Pick matrix (8) defined in Section 3, is positive semidefinite. If the Pick matrix is positive definite, and from now on we shall assume this, there are infinitely many analytic S solutions, so we need to specify what kind of solution we prefer.

A third requirement is that of low complexity: We would like the degree of closed-loop transfer function

$$T = \frac{PC}{1 + PC} \tag{4}$$

to be as small as possible. Except that there may be compeling reasons for T itself to have low degree, as mentioned in the introduction, an important reason is that a low degree of T guarantees a low degree of the compensator C. This is seen by the following observation, which is proved in [**?**]. Since $S = 1 - T$, $\deg T = \deg S$.

Proposition 2.1 *Suppose P is strictly proper and that S satisfies the interpolation conditions (2). Then the controller*

$$C = \frac{1 - S}{PS} \tag{5}$$

satisfies the degree bound

$$\deg C \leq \deg P - n_z - n_p + \deg S, \tag{6}$$

where n_z and n_p are the number of unstable zeros and poles of P, respectively.

As we shall see in the next section, we need to have $\deg S \geq n_z + n_p - 1$ to guarantee the existence of an interpolant. But, even if $\deg S = n_z + n_p - 1$, there may be infinitely many analytic interpolants, so we may choose one that is most suitable with respect to the particular application in mind.

The lowest bound (3), namely

$$\gamma_{\text{opt}} = \inf\{\|S\|_\infty \mid S \text{ satisfied (2)}\}.$$

is attained for an S such that $|S(i\omega)| = \gamma_{\text{opt}}$ for all $\omega \in \mathbb{R}$. However, if we want to achieve lower sensitivity in some selected frequency band, we must allow a higher upper bound

$$\gamma > \gamma_{\text{opt}}. \tag{7}$$

In fact, decreasing the sensitivity at low frequences will ofte increase the sensitivity in other parts of the spectrum. In particular, if P has relative degree at least two, Bode's sensitivity integral yields

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \log |S(i\omega)| d\omega = \sum_{k=1}^{m} \text{Re}\{\rho_k\},$$

where $\rho_1, \rho_2, \cdots, \rho_n$ are the unstable poles of PC. This is called the *water-bed effect* (see, e.g., [18]).

In this regard, the most common design specification is to require the absolute value of the senitivity function to be small for low frequencies. In fact, typical reference signals and disturbances have low frequencies, so having low sensitivity for these frequencies reduces the corresponding contributions to the error e and output y. Design specifications, as well as interpolation conditions, can also be expressed in terms of the closed-loop transfer function T, which is the transfer functions not only from the reference signal r to the output y, but also from the measurment noise n to the output y. Therefore, it is also called the *complementary sensitivity function*. Typically, we want low complementary sensitivity at high frequencies, since the measurement noise generally consists of high frequencies. The difficulty of modeling at high frequencies may also require low complementary sensitivity at these frequencies. In fact, to increase the robust stability margin against the multiplicative unstructured uncertainty, which is large at high frequencies, the complementary sensitivity should be low over this frequency band, due to the small gain theorem [18].

There are situations where such design specification cannot be met and where we need to modify the interpolation problem. Typical examples are when either n_z or n_p is zero. In this case we have an interpolation problem of the type

$$S(s_k) = w, \quad k = 0, 1, 2, \cdots, n,$$

where $s_0, s_1, \cdots, s_n$ are some arbitrary interpolation points. With the degree constraint $\deg S \leq n$, S takes the form

$$S(z) = \frac{b(z)}{a(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n},$$

so any interpolant must satisfy $b(s_k) = S(s_k)a(s_k)$, $k = 0, 1, \cdots, n$, or, equivalently, the linear system of equations

$$\begin{bmatrix} s_0^n & s_0^{n-1} & \cdots & 1 \\ s_1^n & s_1^{n-1} & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ s_n^n & s_n^{n-1} & \cdots & 1 \end{bmatrix} \begin{bmatrix} b_0 - w \\ b_1 - wa_1 \\ \vdots \\ b_n - wa_n \end{bmatrix} = 0.$$

Assuming that s_k, $k = 0, 1, \cdots, n$ are distinct, the Vandermonde matrix in the left-hand side is nonsingular, which shows that $S \equiv w$ is the only possible solution.

For this reason we shall sometimes introduce additional interpolation conditions. How this should be done is discussed in Section 4.

3 Nevanlinna-Pick interpolation with degree constraint

The basic analytic interpolation problem (also allowing for extra interpolation conditions) described in the previous section can be summerized as follows. Given interpolation points $s_0, s_1, \cdots, s_n$ in the right half-plane, interpolation values $v_0, v_1, \cdots, v_n$ in the open unit disc and abound γ, find a function S with the properties

(i) S is analytic in RHP and $|S(s)|/\gamma < 1$ in RHP

(ii) $S(s_k)/\gamma = v_k, \quad k = 0, 1, \cdots, n$

(iii) $\deg S \le n$.

By Pick's Theorem, this problem has a solution if and only if the Pick matrix

$$\left[\frac{1 - v_k \bar{v}_\ell}{s_k + \bar{s}_\ell}\right]_{k,\ell=0}^{n} \tag{8}$$

is positive definite [16]. For simplicity, we assume that the interpolation points are distinct and that the sets of interpolation points and interpolation values are self-conjugate. Hence we are only interested in real S. Without loss of generality, we can also assume that $s_0 = 1$ and that v_0 is real. In fact, a simple conformal mapping moves s_0 to arbitrary point.

In this formulation of the Nevanlinna-Pick interpolation problem with degree constraint, S is analytic in the right half-plane and maps it into the open unit disc. We would like to consider instead functions which are *strictly positive real*, i.e., functions which are analytic and have positive real part in the complement of the open unit disc. By transforming the domain and codomain via

$$z = \frac{1+s}{1-s} \quad \text{and} \quad w = \frac{1+v}{1-v},$$

respectively, we transform S to the strictly positive real function

$$f(z) = \frac{\gamma + S\left(\frac{z-1}{z+1}\right)}{\gamma - S\left(\frac{z-1}{z+1}\right)}. \tag{9}$$

Then, in this setting, the basic interpolation problem becomes: Given interpolation points $z_0, z_1, \cdots, z_n$ in the complement of the closed unit disc and interpolalation values $w_0, w_1, \cdots, w_n$ in the open right half-plane, find a function f with the properties

(i)′ f is strictly positive real

(ii)′ $f(z_k) = w_k, \quad k = 0, 1, \cdots, n$

(iii)′ $\deg f \le n$.

We assume that the corresponding Pick matrix

$$\left[\frac{w_k + \bar{w}_\ell}{1 - z_k^{-1}\bar{z}_\ell^{-1}}\right]_{k,\ell=0}^{n} \tag{10}$$

is positive definite, so that there are infintely many solutions. In this setting, the normalization $s_0 = 1$ becomes $z_0 = \infty$.

A strictly positive real funcion f of degree less than or equal to n can always be written as

$$f(z) = \frac{b(z)}{a(z)}, \tag{11}$$

where the polynomials

$$\begin{aligned} a(z) &= a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \quad a_0 > 0 \\ b(z) &= b_0 z^n + b_1 z^{n-1} + \cdots + b_n, \quad b_0 > 0 \end{aligned}$$

are stable, i.e., they have all roots in the open unit disc. They both have degree n but may have common factors. Then,

$$f(z) + f(z^{-1}) = \frac{a(z)b(z^{-1}) + a(z^{-1})b(z)}{a(z)a(z^{-1})},$$

which must be positive on the unit circle. Therefore, there exists a unique stable polynomial $\rho(z)$ of degre n such that

$$a(z)b(z^{-1}) + a(z^{-1})b(z) = \rho(z)\rho(z^{-1}). \tag{12}$$

The roots of $\rho(z)$, all located in the open unit disc, are called the *spectral zeros* of f.

It was shown in [1] that, for each choice of spectral zeros, i.e., for each stable polynomial

$$\rho(z) = z^n + \rho_1 z^{n-1} + \cdots + \rho_n,$$

there is one and only one pair (a, b) such that (11) satisfies (i)$'$,(ii)$'$ and (iii)$'$ and that this pair can be determined by solving a convex optimization problem.

To formulate this optimization problem, we need some notation. Let $\mathcal{Q}$ be the space of all functions

$$Q(e^{i\theta}) = \left|\frac{\pi(e^{i\theta})}{\tau(e^{i\theta})}\right|^2$$

on the unit circle, where

$$\tau(z) = \prod_{k=0}^{n}(z - z_k^{-1})$$

is stable, and $\pi(z)$ is an arbitrary polynomial of degree less than or equal to n. Of course, all $Q \in \mathcal{Q}$ are nonnegative on the unit circle. We denote by $\mathcal{Q}_+$ the subset of all $Q \in \mathcal{Q}$ which are positive on the unit circle. Next, let $w(z)$ be any real rational function which is analytic on and outside the unit circle and satisfies the interpolation conditions

$$w(z_k) = w_k, \quad k = 0, 1, \ldots, n.$$

Note that w need not be positive real, so it is easy to find such a function. In fact, choose any stable polynomial $\alpha(z)$ of degree n and determine the polynomial $\beta(z)$, also of degree n, satisfying the Vandermonde system $\beta(z_k) = w_k\alpha(z_k), k = 0, 1, \cdots, n$. Since the points $z_0, z_1, \cdots, z_n$ are distinct, there is a unique such $\beta(z)$, and the corresponding w is given by $w = \beta/\alpha$.

Then, for each stable $\rho(z)$ and any $Q \in \mathcal{Q}$, define the functional

$$\mathbb{J}_\rho(Q) = \frac{1}{\pi}\int_{-\pi}^{\pi} Q(e^{i\theta})\mathrm{Re}\{w(e^{i\theta})\}d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\{Q(e^{i\theta})\}\left|\frac{\rho(e^{i\theta})}{\tau(e^{i\theta})}\right|^2 d\theta. \tag{13}$$

It can be shown that the value of $\mathbb{J}_\rho$ does not depend on the particular choice of w but only on the values $w_0, w_1, \cdots, w_n$ of w at the interpolation points. The function $\mathbb{J}_\rho$ is strictly convex. The following (nontrivial) result was proven in [1].

Theorem 3.1 *The function $\mathbb{J}_\rho$ has a unique minimum $\hat{Q}$ in $\mathcal{Q}$, which is an interior point. That is, $\hat{Q} \in \mathcal{Q}_+$.*

The minimum-phase spectral factor of $\hat{Q}$ turns out to be precisely a/τ. More precisely, a in (11) is the unique stable $a(z)$ with $a_0 > 0$ such that

$$a(z)a(z^{-1}) = \tau(z)\tau(z^{-1})\hat{Q}(z). \tag{14}$$

Then, given $a(z)$ and $\rho(z)$, $b(z)$ can be uniquely determined from (12). In fact, the linear operator

$$S(a)v = a(z)v(z^{-1}) + a(z^{-1})v(z)$$

from the vectorspace V_n of polynomials of degre less than or equal to n to the space $\mathcal{D}_n$ of symmetric pseudo-polynomials of degree at most n is nonsigular, since $a(z)$ has no reciprocal roots; see, e.g., [**?**]. We qoute from [1]:

Theorem 3.2 *For each stable $\rho(z)$, there is one and only one pair (a, b) such that (11) satisfies (i)′, (ii)′, (iii)′ and (12), and this pair of polynomials are obtained by solving first (14) for a, where $\hat{Q}$ is the minimizing function in Theorem 3.1, and the (12) for b.*

The convex optimization problem of Theorem 3.1 can be solved using Newton's method; see [**?**, 1] for details.

Finally, returning to the original problem of finding a sensitivity function S satisfying (i), (ii) and (iii), we first observe that

$$\mathrm{Re}(f) = \frac{\gamma^2 - |S|^2}{|\gamma - S|^2}.$$

Therefore, in this context, the spectral zeros, to be chosen, are precisely the zeros of

$$\Gamma(s) = \gamma^2 - S(s)S(-s), \tag{15}$$

and in terms of the interpolant f, determined by the convex optimization algorithm,

$$S(s) = \gamma\frac{f\left(\frac{1+s}{1-s}\right) - 1}{f\left(\frac{1+s}{1-s}\right) + 1}. \tag{16}$$

4 A new procedure for sensitivity shaping

To ensure internal stability, the sensitivity function must always satisfy the interpolation conditions (2). Hoverver, as pointed out in Section 2, sometimes we need to introduce extra interpolation conditions

$$S(\lambda_k) = \alpha_k, \quad k = 0, 1, \cdots, n_e, \tag{17}$$

so the integer n in the interpolation problem of Section 3 is

$$n = n_z + n_p + n_e.$$

Of course, like the other interpolation conditions, the extra ones need to be chosen in complex conjugate pairs, if they are not real.

The design parameters, to be selected, are therefore

1. n spectral zeros,
2. $(\lambda_k, \alpha_k), \quad k = 0, 1, \cdots, n_e$.

The question is how to choose them to achieve the prescribed design specifications.

First, placing a spectral zero close to the imaginary axis at the frequency ω_1, raises the modulus of the sensitivity to a level close to the upper bound γ at that frequency, as depicted in Figure 3. In fact, if λ is a spectal zero close to $i\omega_1$,

$$\gamma^2 - S(i\omega_1)S(-i\omega_1) \approx \gamma^2 - S(\lambda)S(-\lambda) = 0,$$

and hence $|S(i\omega_1)| \approx \gamma$. Then, by the water-bed effect, the sensitivity will often be lowered in other parts of the spectrum.

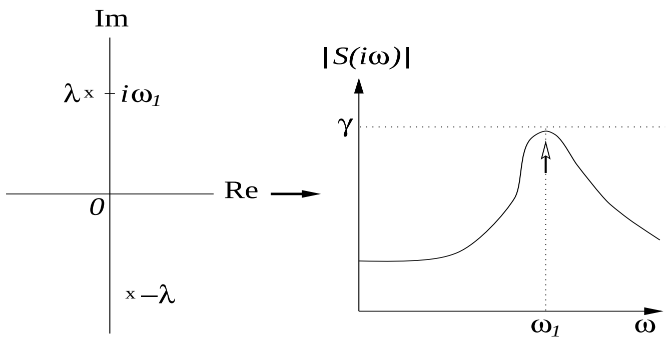

Figure 3: The influence of a spectral zero λ near the imaginary axis on the sensitivity frequency response

Introducing an extra interpolation condition $S(\lambda) = \alpha$ with α real and λ close to the imaginary axis at the frequency ω_2, fixes the modulus of the sensitivity at a level close to α at that frequency, as depicted in Figure 4. To see this, observe that

$$|S(i\omega_2)| \approx |S(\lambda)| = \alpha.$$

By an approriate choice of (λ, α) we can thus prescribe the senitivity at a selected frequency.

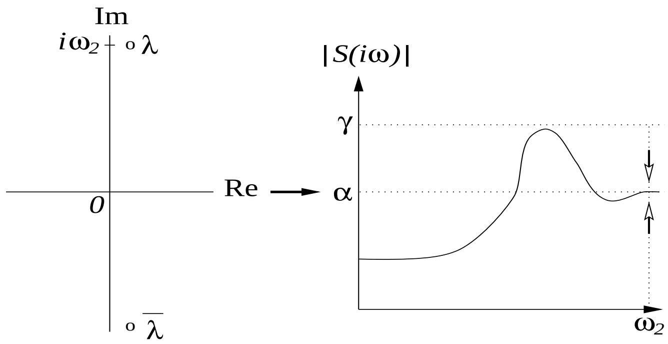

Figure 4: The influence of an extra interpolation constraint $S(\lambda) = \alpha$ on the sensitivity frequency response

Example 4.1 Consider a plant

$$P(s) = \frac{(s-1)(s-2)}{(s+1)(s^2+s+1)},$$

which is stable but has nonminimum-phase zeros at $s = 1$ and $s = 2$. This example is discussed in [7, p. 77], where the performance specification

$$|S(i\omega)| \leq 0.1, \quad \text{for } \omega < 0.01$$

is imposed. We also impose the bound $\gamma = 1.3$ over the whole spectrum. Transforming this to the discrete-time setting, we obtain the unstable zeros $\infty, -1, -3$. To move one of these zeros off theunit circle, we make the transformation $z \to z/(1+\epsilon)$, where we choose $\epsilon = 0.005$. Then the interpolation points become $z_0 = \infty, z_1 = -1.005, z_2 = -3.015$, and the corresponding interpolation conditions are

$$f(z_0) = f(z_1) = f(z_2) = \frac{\gamma+1}{\gamma-1} = \frac{2.3}{0.3}.$$

As explained at the end of Section 2, a senitivity fuction of at most degree two satisfying these interpolation conditions would have to be constant, which would not allow us to satisfy the specifications. Therefore, we add another interpolation condition. Choosing a real $\lambda > 0$ close to the origin, and choosing $\alpha = 0$ reduces the sensitivity for low frequencies. Taking $\lambda = 0.001/1.001$ yields the extra interpolation condition

$$f(1.002) = 1.$$

Then $n = 3$, and choosing the spectral zeros $0, 0.9e^{\pm 1.7453i}$ yields the sensitivity function

$$S(s) = \frac{s^3 + 2.6532s^2 + 6.3989s + 0.0096}{s^3 + 3.0042s^2 + 5.3459s + 0.7115},$$

depicted with solid line in Figure 5, and the controller

$$C(s) = \frac{1 - S(s)}{P(s)S(s)} = \frac{0.3510s^3 + 0.7020s^2 + 0.7020s + 0.3510}{s^3 + 2.6532s^2 + 6.3989s + 0.0096}.$$

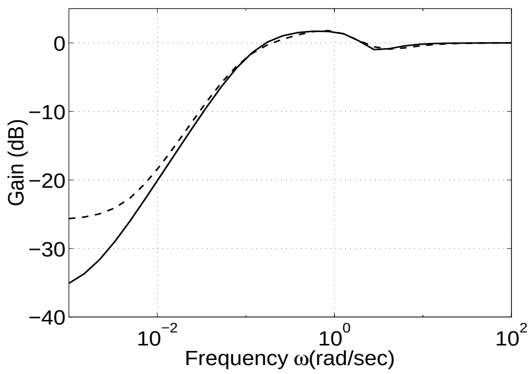

Figure 5: The frequency response of S

Both these functions have degree three, and, as can be seen in Figure 5, the specifications are satisfied with broad margin.

The corresponding sensitivity function, determined in [7], using conventional H^∞ methods, is depicted with a dashed line in Figure 5. It barely satisfies the specifications, despite the fact that the degree of the controller is four.

Example 4.2 As a second example, we consider a mixed sensitivity shapinging problem for the discrete-time plant

$$P(z) = \frac{1}{z - 1.05},$$

having an unstable pole at 1.05 and a nonminimum-phase zero at infinity. For internal stability, we need the interpolation conditions

$$S(\infty) = 1, \qquad S(1.05) = 0.$$

The specifications are

$$\begin{aligned}
&|S(e^{j\theta})| < 2(\approx 6.02\text{dB}), && \theta \in [0, \pi]\,(\text{rad/sec}),\\
&|S(e^{j\theta})| < 0.1(= -20\text{dB}), && \theta \in [0, 0.3]\,(\text{rad/sec}),\\
&|T(e^{j\theta})| < 0.5(\approx -6.02\text{dB}), && \theta \in [2.5, \pi]\,(\text{rad/sec}),
\end{aligned}$$

and hence we choose $\gamma = 2$.

However, there turns out to be no sensitivity function of degree two which satisfies these specifications, we need to add an extra interpolation conditions. To this end, we choose $S(-1.01) = 1$ to obtain small $|T|$ for high frequencies and $S(1.01e^{\pm 0.3i}9 = 0$ to obtain small $|S|$ for low frequencies.This yields the sensitivity function S depicted with solid line in Figure 6. The complementary sensitivity function T is depicted with dashed line in the same figure. Here we have $\deg S = \deg T = 4$ and $\deg C = 3$.

Solving the same problem by the conventional method of mixed sensitivity minimization with weighting functions, i.e.

$$\min_C \left\| \begin{array}{c} W_1 S \\ W_2 T \end{array} \right\| - \infty \tag{18}$$

subject to internal stability for the nominal system, we were unable to satisfy the specifications even with a sensitivity function of degree seven and a corresponding controller of degree six. (See [?] for further details.)

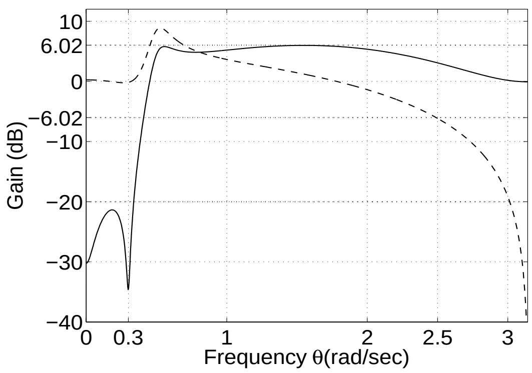

Figure 6: The gains of S (solid line) and of T (dashed line)

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