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Spectral Estimation of Periodic and Skew Periodic Random Signals and Approximation of Spectral Densities

Axel Ringh[1] and Anders Lindquist[2,1]

1. Royal Institute of Technology, Stockholm Sweden
E-mail: aringh@kth.se

2. Shanghai Jiao Tong University, Shanghai, P. R. China
E-mail: alq@kth.se

Abstract: This paper discusses extensions of the theory of rational covariance extension to periodic and skew-periodic processes. It is also shown how these methods can be used to construct fast algorithms for approximate spectral estimation of (non-periodic) processes.

Key Words: Moment problem, Covariance extension, Periodic processes, Skew-periodic processes, Maximum entropy, Speech processing

1 Introduction

The covariance extension problem or, as it is also called, the (truncated) trigonometric moment problem is ubiquitous in signal processing and control. A case in point is the mobile telephone, in which such a problem is solved every 30 ms. The covariance extension problem with complexity (rationality) constraints required in applications have been studied in a long series of papers; see, e.g., [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], and references therein.

This theory was generalized in [14] to the circulant rational covariance extension problem, modeling periodic random processes, and it was shown that these methods also provide a fast procedure for solving the original rational covariance extension problem. Further results in this direction are given in [15] and [16].

In [16], a Master's thesis written while the first author was an exchange student at Shanghai Jiao Tong University, these methods were generalized to model also skew-periodic processes. The present paper is based on results and simulations in [16].

The outline of the paper goes as follows. In Section 2 we review basic fact from the theory of rational covariance extension, and we provide a motivation example from speech processing. Section 3 is devoted to the circulant rational covariance extension problem and how it can be used for approximation. In Section 4 we modify the theory to modeling of skew-periodic processes. Finally, in Section 5 we discuss applications to speech data.

2 Background

Let $\{y(t);\, t \in \mathbb{Z}\}$ be a (discrete-time) stationary stochastic process with spectral density Φ. Such a density has a Fourier transform

$$\Phi(e^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k e^{-ik\theta}, \tag{1}$$

This work was done at the Shanghai Jiao Tong University when the first author was an exchange student from the Royal Institute of Technology, Stockholm, Sweden. It was supported as a Minor Field Study by Swedish International Development Cooperation Agency, SIDA.

where $c_k = \mathrm{E}\{y(t+k)y(t)\}$ are the covariance lags. The covariance lags are given by the inverse Fourier transform

$$c_k = \int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta})\frac{d\theta}{2\pi}. \tag{2}$$

Now suppose that the process y is produced by passing a white noise process u through a linear system with a transfer function W as in Fig. 1. We assume that this linear system

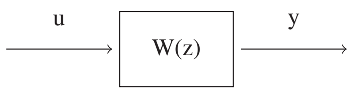

Fig. 1: Linear system

is finite-dimensional, requiring W to be rational of some degree n. If we also require that W is stable with no poles and zeros on the unit circle, the output process y will be stationary in steady state with the spectral density

$$\Phi(e^{i\theta}) = |W(e^{i\theta})|^2. \tag{3}$$

Consequently, the spectral density is also rational and takes the form

$$\Phi(e^{i\theta}) = \frac{P(e^{i\theta})}{Q(e^{i\theta})} \tag{4}$$

where P and Q belong to the class $\mathfrak{P}_+$ of positive trigonometric polynomials

$$Q(e^{i\theta}) = \sum_{k=-n}^{n} q_k e^{-ik\theta} > 0, \quad \theta \in [-\pi, \pi] \tag{5}$$

of degree at most n.

2.1 A motivating example

To motivate the approach taken in this paper we consider an example from speech processing in the context of a mobile telephone with simulations taken from [9]. The speech signal is divided into short intervals of 30 ms. In each of these windows the process could be considered to be stationary. Spectral estimation via the fast Fourier transform (FFT) for a short window in the phoneme [ng] leads to the rugged (blue) curve in Fig. 2. This spectral estimate contains

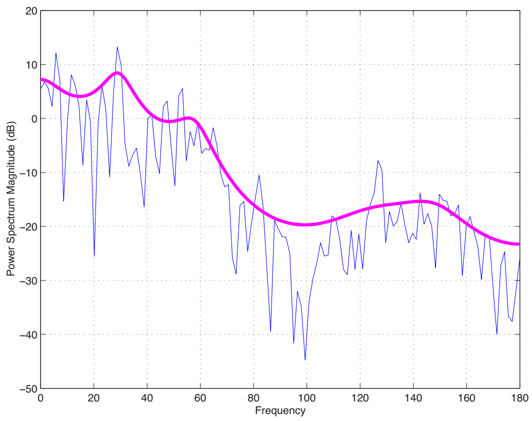

Fig. 2: Maximum entropy solution

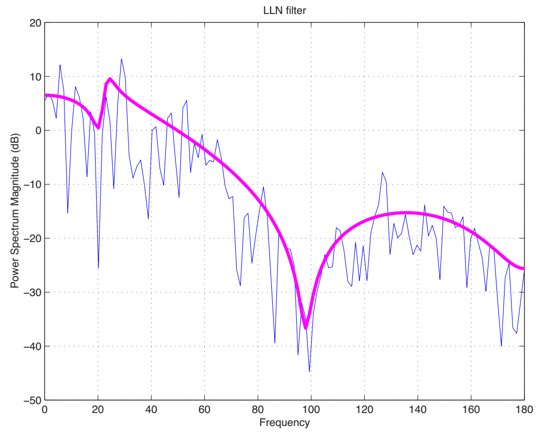

Fig. 3: Solution with nontrivial P

too much structure so we approximate it by a rational function leading to the smooth curve in Fig. 2. More precisely, for each window of observations we construct a filter as in Fig. 1, where now u is a more general excitation signal also containing pitch information. However, the analysis above still applies in this situation. Hence, given the 30 ms observation record $y_0, y_1, y_2, \dots, y_M$, where M is approximately 250 for the mobile telephone, n covariance lags are estimate by the truncated ergodic estimates

$$c_k = \frac{1}{M} \sum_{j=0}^{M-k-1} y_j y_{j+k}. \tag{6}$$

Then the moment problem

$$\int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) \frac{d\theta}{2\pi} = c_k, \quad k = 0, 1, 2, \dots, n \tag{7}$$

is solved for the spectral density Φ. This is an inverse problem with infinitely many solutions. In the mobile telephone example, where $n = 10$, Φ is chosen to be of the form (4) with P constant (of degree zero). This leads to a unique solution, the *maximum entropy* solution, which is easily computed from the so called normal equations. The corresponding spectral density is plotted as the smooth curve in Fig. 2. This is a flat spectral estimate which does pick up the valley in the spectrum. However, by choosing a nontrivial numerator polynomial P we can smoothly tune the estimate without increasing the degree to obtain the spectrum in Fig. 3. All this is explained in detail in, e.g., [6], [9]. Next, we explain how to determine this solution.

2.2 The rational covariance extension problem

Let $c_0, c_1, c_2, \dots, c_n$ be a sequence of covariance lags satisfying the Toeplitz condition

$$T_n = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_0 & c_1 & \cdots & c_{n-1} \\ c_2 & c_1 & c_0 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_0 \end{bmatrix} > 0. \tag{8}$$

Then for an arbitrary $P \in \mathfrak{P}_+$, we maximize the generalized entropy functional

$$\mathbb{I}_P(\Phi) = \int_{-\pi}^{\pi} P(e^{i\theta}) \log \Phi(e^{i\theta}) \frac{d\theta}{2\pi} \tag{9}$$

subject to the moment conditions (7). It was shown in [5], [6], [9] that this problem has the solution (4) where Q is the unique solution of the dual problem to minimize the strictly convex functional

$$\mathbb{J}_P(Q) = \int_{-\pi}^{\pi} \left(C(e^{i\theta}) Q(e^{-i\theta}) - P(e^{i\theta}) \log Q(e^{i\theta}) \right) \frac{d\theta}{2\pi}, \tag{10}$$

where

$$C(e^{i\theta}) = \sum_{-n}^{n} c_k e^{-ik\theta}. \tag{11}$$

This provides a complete parameterization of all solutions to the moment problem (7) of the rational form (4) in terms of $P \in \mathfrak{P}_+$ [3], [4], [6]. If we choose $P = 1$, the functional (9) becomes the usual entropy gain (Burg entropy), which motivates the concept maximum-entropy solution.

2.3 How to choose P

As the solutions of the rational covariance extension problem are in one-one correspondence with the $P \in \mathfrak{P}_+$, one is left with the task to find a suitable P. Without prior information, it is common to choose the maximum-entropy solution, which, as we have seen, corresponds to $P = 1$.

If the *cepstral coefficients*

$$m_k = \int_{-\pi}^{\pi} e^{ik\theta} \log \Phi(e^{i\theta}) \frac{d\theta}{2\pi}, \quad k = 1, 2, \dots, n \tag{12}$$

could be computed from the data, P could also be determined. In fact, if we normalize the covariance lags so that $c_0 = 1$, the spectral density Φ is uniquely determined by the *theoretical* coefficients $c_1, c_2, \dots, c_n, m_1, m_2, \dots, m_n$, and there is a unique pair (P, Q) with P normalized so that the constant term $p_0 = 1$ such that $\Phi = P/Q$ satisfies both (7) and (12), namely the unique minimizer of the strictly convex functional

$$\mathbb{J}(P, Q) = \int_{-\pi}^{\pi} \left(C(e^{i\theta}) Q(e^{-i\theta}) - M(e^{i\theta}) P(e^{-i\theta}) \right) \frac{d\theta}{2\pi} + \int_{-\pi}^{\pi} P(e^{i\theta}) \log \frac{P(e^{i\theta})}{Q(e^{i\theta})} \frac{d\theta}{2\pi}, \tag{13}$$

where M is the pseudo-polynomial formed from $m_1, \dots, m_n$ as in (11); see [9, Theorem 3.2].

However, if $c_0, c_1, \ldots, c_n, m_1, m_2, \ldots, m_n$ are determined from measurements, P may end up on the boundary of $\mathfrak{P}_+$, where (12) may not hold. Therefore as suggested in [10], one may add a regularization term and minimize

$$J_\lambda(P, Q) = \mathbb{J}(P, Q) - \lambda \int_{-\pi}^{\pi} \log P(e^{i\theta}) \frac{d\theta}{2\pi} \tag{14}$$

instead. As $\lambda \to \infty$, the solution tends to the maximum-entropy solution [17].

3 Circulant Rational Covariance Extension

Next we consider periodic stationary stochastic processes y of period $2N$. Such processes occur in many important applications. Then instead of moment conditions of the type (7) and (12) we consider

$$\int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) d\nu = c_k, \quad k = 0, 1, 2, \ldots, n, \tag{15}$$

and

$$\int_{-\pi}^{\pi} e^{ik\theta} \log \Phi(e^{i\theta}) d\nu = m_k, \quad k = 1, 2, \ldots, n, \tag{16}$$

where $d\nu$ is the Stieltjes measure

$$d\nu(\theta) = \sum_{j=-N+1}^{N} \delta(e^{i\theta} - \zeta_j) \frac{d\theta}{2N} \tag{17}$$

and $\zeta_j := e^{ij\pi/N}$. Thus the spectral density has its mass in discrete points on the unit circle. This has the effect that the computations become simpler as they reduce to FFT calculations. Therefore these methods can be used as an approximation procedure for the regular rational covariance extension problem described above. Indeed, as $N \to \infty$, one obtains the non-periodic case [14].

3.1 Modeling of periodic processes

A periodic process of period $2N$ has the representation

$$y(t) = \frac{1}{2N} \sum_{j=-N+1}^{N} \zeta_j^t \hat{y}(\zeta_j), \tag{18}$$

where $\hat{y}$ is given by

$$\hat{y}(\zeta_j) = \sum_{t=-N+1}^{N} y(t) \zeta_j^{-t}. \tag{19}$$

Here

$$\mathrm{E}\{\hat{y}(z_j)\hat{y}(z_k)^*\} = w_j \delta_{jk}, \tag{20}$$

where $w_0, w_1, \ldots, w_n$ are weights to be determined from data and * denotes the conjugate transpose. Since $\zeta_j^{2N} = 1$,

$$y(t + 2N) = y(t) \tag{21}$$

as required. The periodic process y has a discrete spectral density

$$\sum_{j=-N+1}^{2N-1} w_j \delta(e^{i\theta} - \zeta_j), \tag{22}$$

on the unit circle. However, we want to determine a spectral density Φ of the form $\Phi = P/Q$ such that

$$\Phi(\zeta_j) = w_j, \quad j = 0, 1, 2, \ldots, 2N - 1, \tag{23}$$

which can then be used as an approximation for the solution of the corresponding continuous covariance extension problem considered in Section 2.

The moment conditions (15) and (16) can then be written in the form

$$\frac{1}{2N} \sum_{j=-N+1}^{N} \zeta_j^k \Phi(\zeta_j) = c_k, \quad k = 0, 1, \ldots, n \tag{24}$$

$$\frac{1}{2N} \sum_{j=-N+1}^{N} \zeta_j^k \log \Phi(\zeta_j) = m_k, \quad k = 1, 2, \ldots, n \tag{25}$$

and therefore $c_{k+2N} = c_k$ and $m_{k+2N} = m_k$, as expected. Consequently, as demonstrated in detail in [14], periodic processes y can be modeled by the same methods as in Section 2 by merely exchanging the measure $d\theta/(2\pi)$ by $d\nu$ everywhere.

The functional (10) corresponds to

$$\mathbb{J}_P(Q) = \sum_{j=0}^{2N-1} \left(C(\zeta_j) Q(\bar{\zeta_j}) - P(\zeta_j) \log Q(\zeta_j) \right), \tag{26}$$

where, for simplicity of notation we have removed the term $\frac{1}{2N}$. The pseudo-polynomials P and Q now only need to be restricted to the larger cone $\mathfrak{P}_+(N) \supset \mathfrak{P}_+$ of pseudo-polynomials (5) such that $Q(\zeta_j) > 0$. On the other hand, this has the consequence that the set of admissible sequences $c := (c_0, c_1, \ldots, c_n)$ may be smaller than the set of c that satisfy the Toeplitz condition (8). However, any c satisfying (8) will be admissible for a sufficiently large N [14]. Minimizing (26) with $P = 1$ yields the maximum entropy solution.

Similarly, the functional (13) corresponds to

$$\begin{aligned} \mathbb{J}(P, Q) = \sum_{j=0}^{2N-1} \left(C(\zeta_j) Q(\bar{\zeta_j}) - M(\zeta_j) P(\bar{\zeta_j}) \right) \\ + \sum_{j=0}^{2N-1} P(\zeta_j) \log \frac{P(\zeta_j)}{Q(\zeta_j)}, \end{aligned} \tag{27}$$

to which we add a regularization term to obtain

$$\mathbf{J}_\lambda(P, Q) = \mathbb{J}(P, Q) - \lambda \sum_{j=-N+1}^{2N-1} \log P(\zeta_j), \tag{28}$$

which we then minimize over $\mathfrak{P}_+(N) \times \mathfrak{P}_+(N)$ with the extra side condition $p_0 = 1$ normalizing P.

Fig. 4 shows zeros and poles of a spectral density $\Phi = P/Q$ for which $P, Q \in \mathfrak{P}_+$ with $n = 10$. This spectrum is used to numerically compute covariance lags and cepstral coefficients. These are then in turn used to reconstruct a periodic spectrum with $N = 100$ by minimizing (28) for a suitable $\lambda = 0.001$. As a comparison, we minimize (26) with $P = 1$ to reconstruct the maximum-entropy (ME) solution. All three spectra are shown in Fig. 5, where the curve obtained from (28) is marked “Log”. We see that using the optimization problem (28) provides a better reproduction than the ME solution.

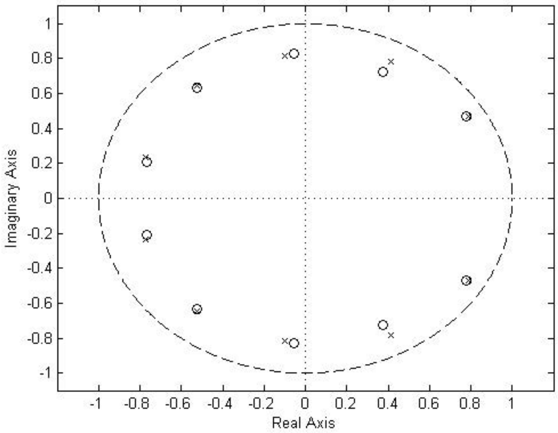

Fig. 4: Randomly generated poles and zeros

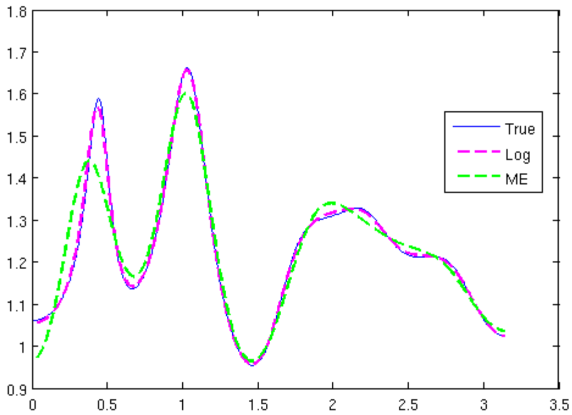

Fig. 5: Periodic spectrum

3.2 Approximation of spectral densities

Given a $c = (c_0, c_1, \dots, c_n)$ satisfying (8), we noted in Section 2 that, for each $P \in \mathfrak{P}_+$, there is a $Q \in \mathfrak{P}_+$ such that

$$\int_{-\pi}^{\pi} e^{ik\theta} \frac{P(e^{i\theta})}{Q(e^{i\theta})} \frac{d\theta}{2\pi} = c_k, \quad k = 0, 1, 2, \dots, n \tag{29}$$

Next, let Q_N be the unique minimizer of (26), where N is sufficiently large for c to be admissible. Then it was shown in [14, Theorem 7] that $Q_N \to Q$ as $N \to \infty$. Consequently, the circulant covariance extension solution provides an approximation for the non-periodic case. The proof of [14, Theorem 7] can be trivially modified to allow Φ_N to be P_N/Q_N.

Now, given a $\Phi = P/Q$ with zeros and poles as in Fig. 6 and $n = 6$, we have computed a Q_N for the ME solution and (P_N, Q_N) for the minimizer of (27) with $N = 60$ and $\lambda = 0.1$ and depicted the corresponding Φ_N in Fig. 7. As we can see, solving the full problem to minimize (27) yields a better approximation.

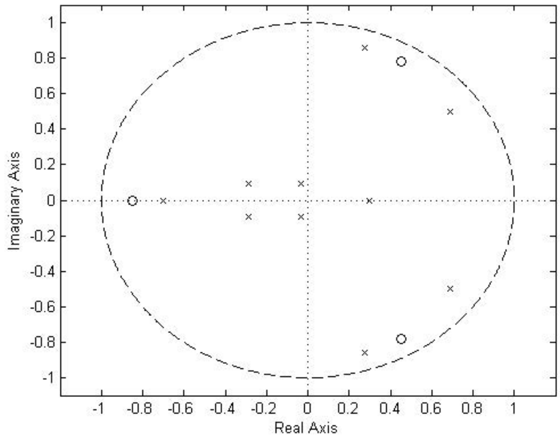

Fig. 6: Poles and Zeros of true spectrum

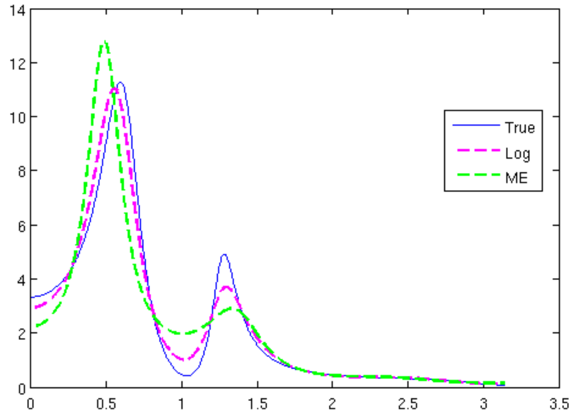

Fig. 7: Periodic approximation

4 Modeling of skew-periodic processes

A skew-periodic process of period N has a representation

$$\begin{aligned} y(t) &= \frac{1}{N} \sum_{j=1}^{N} (\zeta_{2j-1})^t \hat{y}(\zeta_{2j-1}) \\ &= \int_{-\pi}^{\pi} e^{it\theta} \hat{y}(e^{i\theta}) d\nu, \end{aligned} \tag{30}$$

where

$$\hat{y}(\zeta_{2j-1}) := \sum_{t=1}^{N} y(t) (\zeta_{2j-1})^{-t} \tag{31}$$

and

$$d\nu = \frac{1}{N} \sum_{j=1}^{N} \delta(e^{i\theta} - \zeta_{2j-1}) d\theta. \tag{32}$$

Then

$$y(t+N) = \frac{1}{N} \sum_{j=1}^{N} (\zeta_{2j-1})^t \underbrace{(\zeta_{2j-1})^N}_{-1} \hat{y}(\zeta_{2j-1}), \tag{33}$$

which yields the skew-periodic property

$$y(t+N) = -y(t). \tag{34}$$

Since $(\zeta_{2j-1})^{2N} = 1$, it is also a periodic process of period $2N$.

Since $\zeta_{2j-1}^k = \zeta_k^{2j-1}$, it is easy to show that

$$\int_{-\pi}^{\pi} e^{ik\theta} d\nu = \frac{1}{N} \sum_{j=1}^{N} \zeta_k^{2j-1} = \delta_{k0}. \tag{35}$$

Consequently, setting

$$\Phi(\zeta_{2j-1}) = \frac{1}{N} \sum_{\ell=1}^{N} c_\ell (\zeta_{2j-1})^{-\ell}, \tag{36}$$

we have

$$\begin{aligned} \int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) d\nu &= \frac{1}{N} \sum_{j=1}^{N} (\zeta_{2j-1})^k \Phi(\zeta_{2j-1}) \\ &= \sum_{\ell=1}^{N} c_\ell \frac{1}{N} \sum_{j=1}^{N} (\zeta_{2j-1})^{k-\ell} \\ &= \sum_{\ell=1}^{N} c_\ell \int_{-\pi}^{\pi} e^{i(k-\ell)\theta} d\nu = c_k, \end{aligned}$$

and therefore

$$c_k = \int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) d\nu. \tag{37}$$

In the same way, we introduce the cepstral coefficients

$$m_k = \int_{-\pi}^{\pi} e^{ik\theta} \log \Phi(e^{i\theta}) d\nu. \tag{38}$$

With the new measure (32) and $\mathfrak{P}_+$ being the larger cone of psedo-polynomials (5) which are positive in the odd points ζ_{2j-1} we can now proceed as in Section 3 to minimize (26) and (28).

Given a spectral density, depicted as "True" in Fig. 9, with zeros and poles as in Fig. 8, we generate the coefficients $c_0, c_1, \dots, c_n, m_1, \dots, m_n$ from (7) and (12). By minimizing (26) for $P = 1$ we obtain the maximum entropy (ME) solution. Finally, minimizing (28) yields the solution marked "Log" in Fig. 9, which clearly is the best reproduction of this skew-periodic spectral density.

Finally, we apply the skew-periodic algorithms also to the density of Fig. 6 for a $N = 60$ to obtain the solutions Fig. 10. Here $n = 6$ and $\lambda = 0.1$, the same as in Fig. 7. As we can see, the results are very close to those of Fig. 7.

5 Conclusions

We have shown how we can model periodic and skew-periodic processes using methods of moment problems with positive rational measures. We have also demonstrated that such methods lead to good approximations of the corresponding covariance extension problem with a continuous spectrum. To apply these methods to more realistic data,we apply them to speech data in Fig. 11. Here the blue rugged curve marked "True" is the estimate obtained by FFT, while the envelopes marked "ME" and "Estimated" are the solutions of (26) for $P = 1$ and (28), respectively, with $n = 10$, $N = 250$ and $\lambda = 0.01$. As we can see the envelope "Estimated" picks up the valleys of the spectrum much better than ME, which is used, for example, in mobile telephones. Our goal is to speed up this computation so that it can be done on line in a speech processing device.

The present methods can also be generalized handle vector-valued processes [18], [15], [16]. This opens up for applications also in image processing [19], [20].

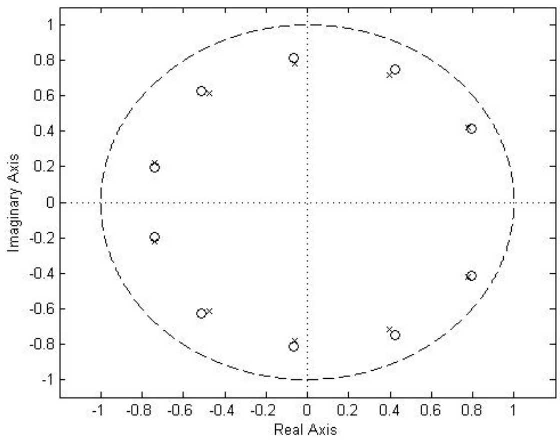

Fig. 8: Randomly generated poles and zeros

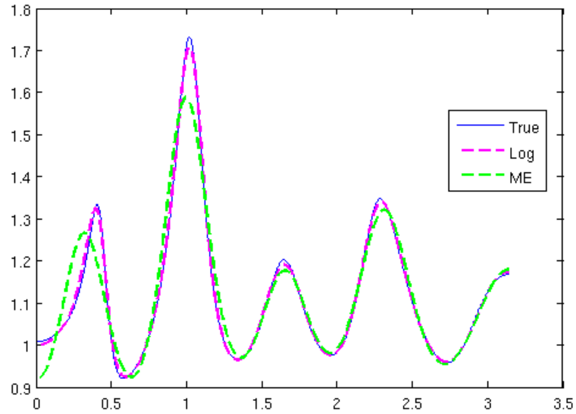

Fig. 9: Skew-periodic spectrum

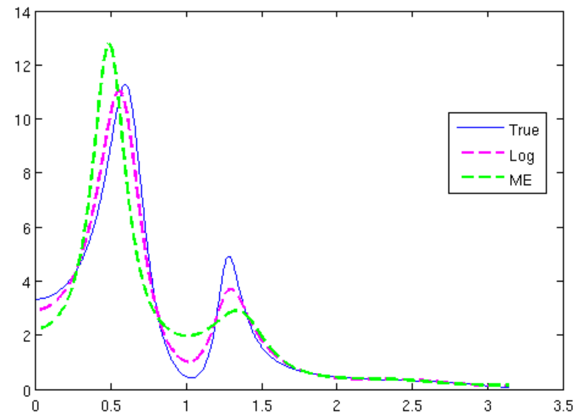

Fig. 10: Skew-periodic approximation

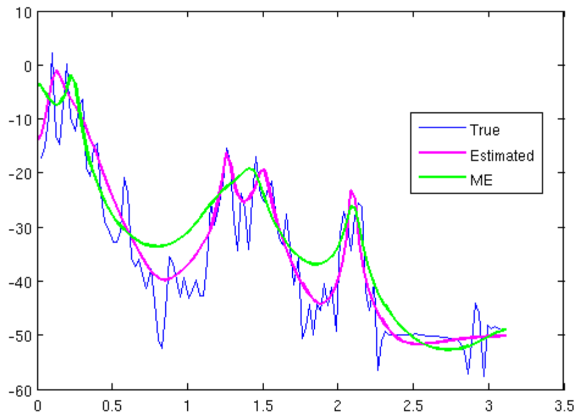

Fig. 11: Periodic approximation applied to speech data

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