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What are moment problems and why are they useful in systems and control?

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What is the talk about

- A classical problem – the moment problem – with a decidedly non-classical twist motivated by applications to systems and control.
- What is new are certain rationality constraints imposed by applications that alter the mathematical problem and make it nonlinear.
- A global-analysis approach that studies the class of solutions as a whole.
- A powerful paradigm for smoothly parametrizing, comparing, and shaping solutions to specifications.

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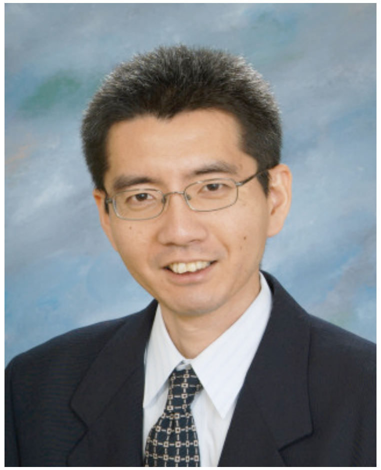
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The moment problem

Given $c_0, c_1, \ldots, c_n$,
find $d\mu$ such that

$$\int_a^b \alpha_k(t) d\mu = c_k, \quad k = 0, 1, \ldots, n$$

- Power moment problem: $\alpha_k(t) = t^k$
- Trigonometric moment problem: $\alpha_k(t) = e^{ikt}, \quad [a, b] = [-\pi, \pi]$
- Nevanlinna-Pick interpolation: $\alpha_k(t) = \frac{e^{it} + z_k}{e^{it} - z_k}, \quad [a, b] = [-\pi, \pi]$

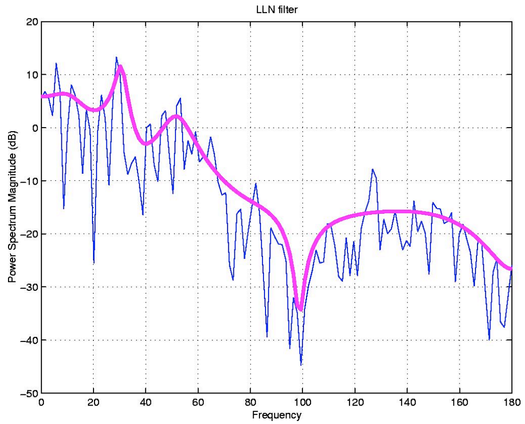
Chebyshev

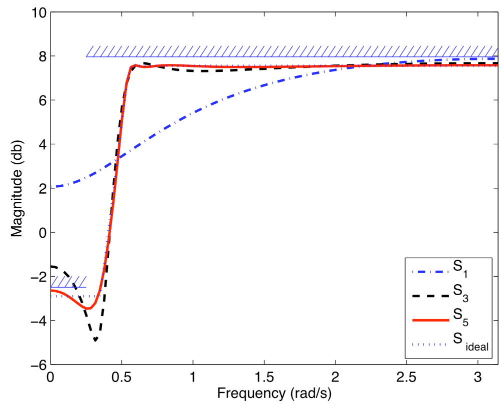
Markov

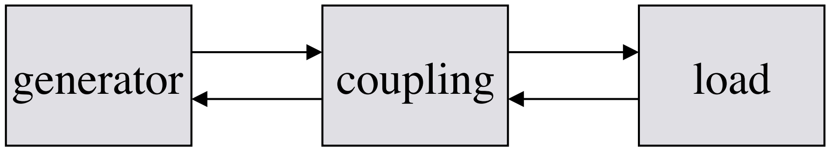
Lyapunov

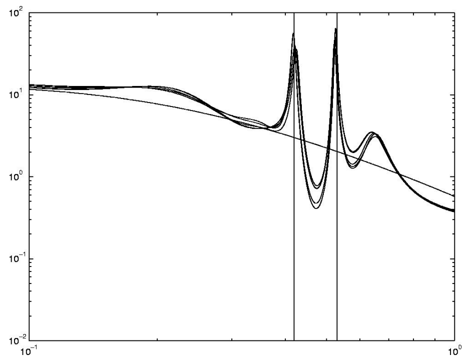
Where do we find moment problems in systems and control?

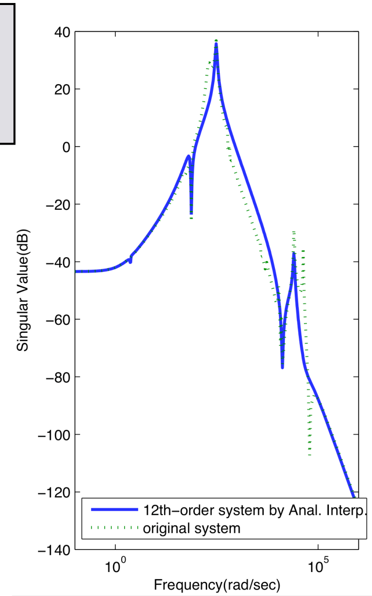
- spectral estimation
- speech synthesis
- system identification
- image processing
- optimal control
- robust control
- model reduction
- model matching problems
- simultaneous stabilization
- optimal power transfer











Moment problems don’t always look like moment problems

Let us look at a few that don’t, and return to them throughout the lecture

Ex. 1: Covariance extension

- $c_k = E\{y(t+k)y(t)\}$, where y stationary stochastic process

PROBLEM: Given
$c_0, c_1, \dots, c_n$ such that

$$T_n = \begin{bmatrix} c_0 & c_1 & \cdots & c_n \\ \bar{c}_1 & c_0 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{c}_n & \bar{c}_{n-1} & \cdots & c_0 \end{bmatrix} > 0$$

Find infinite extension
$c_{n+1}, c_{n+2}, \dots$ such that

$$T_\infty = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots \\ \bar{c}_1 & c_0 & c_1 & \cdots \\ \bar{c}_2 & \bar{c}_1 & c_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} > 0$$

spectral estimation, speech synthesis, system identification

Ex 2: Circulant covariance extension

- $c_k = E\{y(t+k)y(t)\}$, where y periodic stochastic process on $[0, 2N]$

$$c_{2N-k} = c_k, \quad k = 0, 1, \ldots, N$$

reciprocal process
(Jamison, Krener, Levy, Frezza, Ferrante, Picci, Pavon, Carli)

PROBLEM: Given $n < N$ and $c_0, c_1, \ldots, c_n$ such that $T_n > 0$, find extension $c_{n+1}, c_{n+2}, \ldots, c_N$ such that

$$T_{2N-1} = \begin{bmatrix} c_0 & c_1 & \cdots & c_1 \\ \bar{c}_1 & c_0 & \cdots & c_2 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{c}_1 & \bar{c}_2 & \cdots & c_0 \end{bmatrix} > 0$$

circulant matix

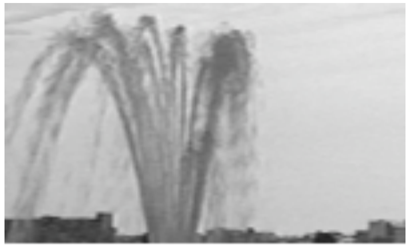
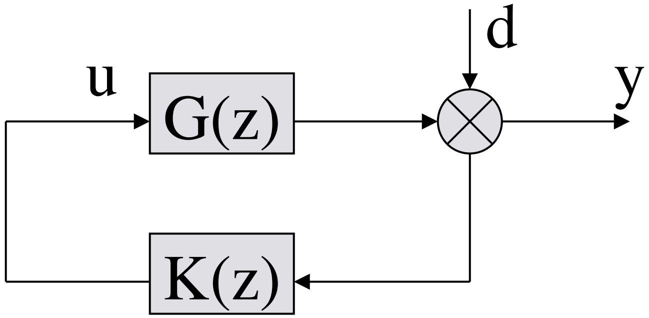


Image processing
(in vector case)

(Carli, Ferrante, Pavon, Picci)

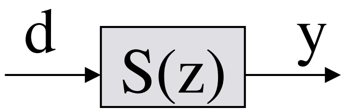
Ex. 3: Robust control



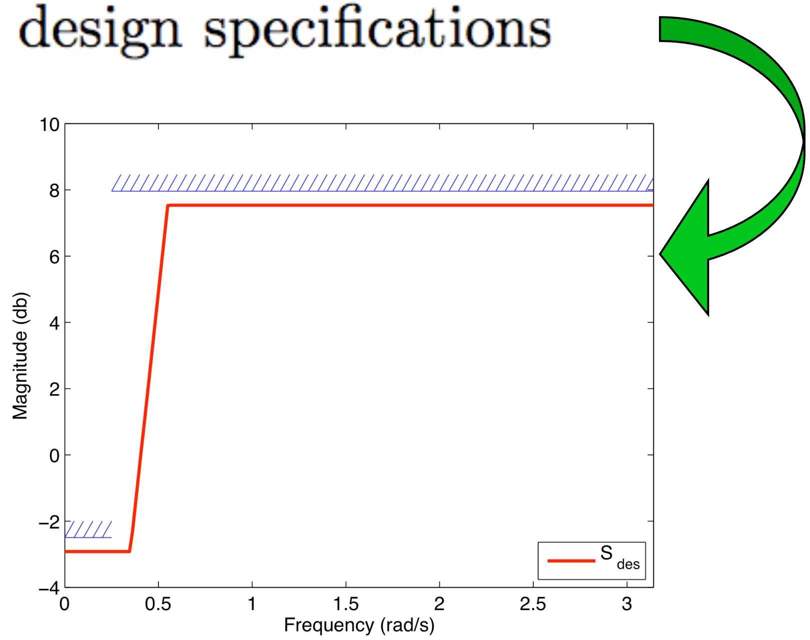
PROBLEM: Find an internally stabilizing controller K such that

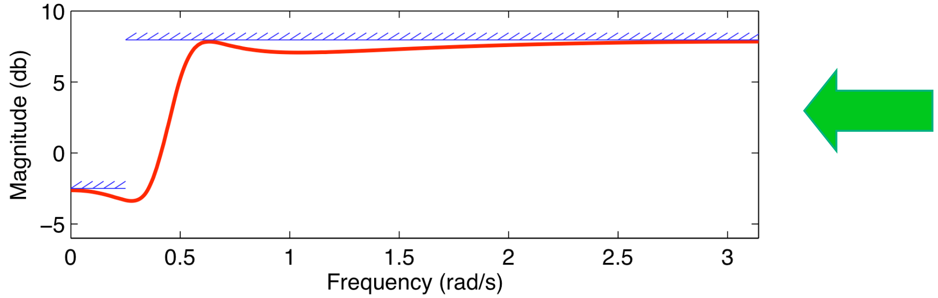
$$S = (I - GK)^{-1}$$

has low degree and satisfies the design specifications



sensitivity function





What do we want to achieve?

- A basic paradigm for smooth parameterization of the whole class of solutions in a systems-theoretical language
- Methods for determining solutions by convex optimization

NB. This is a systematic design methodology that is still in progress

Moment problem in the style of Krein

$\mathfrak{P}$ finite-dimensional subspace of $C[a,b]$

$(\alpha_0, \alpha_1, \ldots, \alpha_n)$ basis in $\mathfrak{P}$

Given $c := (c_0, c_1, \ldots, c_n) \in \mathbb{C}^{n+1}$,

find positive measure $d\mu$ such that

$$\int_a^b \alpha_k(t) d\mu = c_k, \quad k = 0, 1, \ldots, n$$

$$\mathfrak{P}_+ := \{p \in \mathfrak{P} \mid P(t) := \mathrm{Re}(p) \geq 0 \quad \forall t \in [a,b]\}$$

positive cone

closed convex

Suppose $\mathfrak{P}_+$ has a nonempty interior $\mathring{\mathfrak{P}}_+$

$$\mathfrak{M} : \mathcal{M}_+ \to \mathbb{C}^{n+1}, \quad d\mu \mapsto c = \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix}$$

space of positive measures

$$c_k = \int_a^b \alpha_k(t) d\mu$$

$$p \in \mathfrak{P}, \quad p(t) = \textstyle\sum_0^n p_k \alpha_k(t)$$

$$P(t) = \mathrm{Re}\{p(t)\}$$

$$\langle c, p \rangle := \mathrm{Re}\left\{ \sum_{k=0}^{n} c_k p_k \right\} = \int_a^b P(t) d\mu \geq 0 \quad \forall p \in \mathfrak{P}_+$$

$\Rightarrow$ $c \in \mathfrak{C}_+$ positive sequence $\Rightarrow$ $\mathfrak{M}(\mathcal{M}_+) \subset \mathfrak{C}_+$

$$\mathfrak{C}_+ := \{ c \in \mathbb{C}^{n+1} \mid \langle c, p \rangle \geq 0 \quad \forall p \in \mathfrak{P}_+ \} = (\mathfrak{P}_+)^{\mathsf{T}}$$

dual cone

closed convex

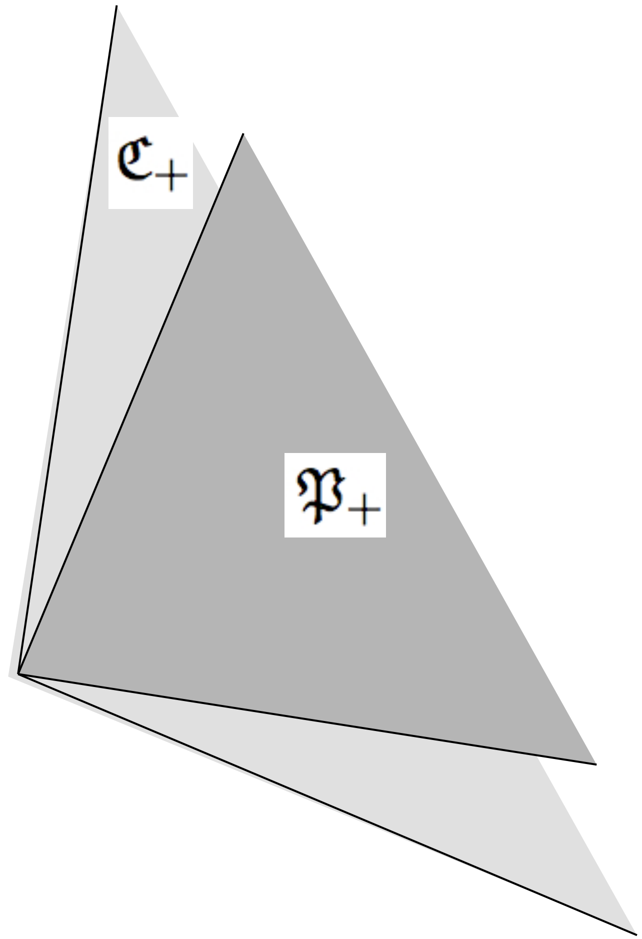
THEOREM(Krein-Nudelman) $\mathfrak{M}(\mathcal{M}_+) = \mathfrak{C}_+$

The moment problem is solvable if and only if $c \in \mathfrak{C}_+$

Dual cones

$$\mathfrak{P}_+ = \{p = \sum_{k=0}^{n} p_k \alpha_k \in \mathfrak{P} \mid$$
$$P(t) = \mathrm{Re}\{p(t)\} \geq 0, \quad a \leq t \leq b\}$$

positive cone



$$\mathfrak{C}_+ = \{c \in \mathbb{C}^{n+1} \mid \langle c, p \rangle \geq 0 \quad \forall p \in \mathfrak{P}_+\}$$
$$= (\mathfrak{P}_+)^\mathsf{T}$$

positive sequences

$$\langle c, p \rangle := \mathrm{Re}\left\{ \sum_{k=0}^{n} c_k p_k \right\}$$

Trigonometric moment problem

$\mathfrak{P} = \text{span}\{1, e^{it}, \ldots, e^{int}\}$ $\qquad$ $\boxed{\alpha_k(t) = e^{ikt}}$

$[a, b] = [-\pi, \pi]$

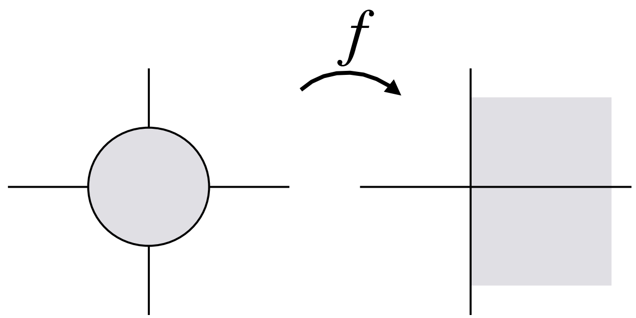
$$c \in \mathfrak{C}_+ \quad \Longleftrightarrow \quad T_n = \begin{bmatrix} c_0 & c_1 & \cdots & c_n \\ \bar{c}_1 & c_0 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{c}_n & \bar{c}_{n-1} & \cdots & c_0 \end{bmatrix} \geq 0$$

Equivalent formulation:

Given $c := (c_0, c_1, \ldots, c_n) \in \mathbb{C}^{n+1}$, find an infinite extension $c_{n+1}, c_{n+2}, \ldots$ such that

$$f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

is positive real (Carathéodory function)



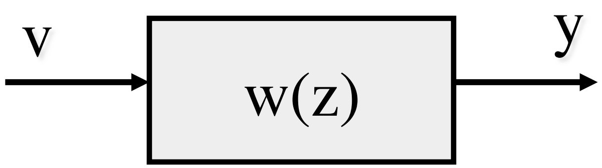
analytic in $\mathbb{D}$

$\text{Re}\{f(z)\} \geq 0$ in $\mathbb{D}$

Spectral estimation as a trigonometric moment problem

white noise

$E\{v(s)v(t)\} = \delta_{st}$



stationary process
with spectral density

$\Phi(e^{i\theta}) = |w(e^{i\theta})|^2$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta})d\theta = E\{y(t+k)y(t)\}$$

$$= \lim_{N\to\infty} \frac{1}{N+1}\sum_{t=0}^{N-k} y_{t+k}y_t$$

where $y_0, y_1, y_2, \ldots, y_N$

observed data

For small k we can use the ergodic estimate

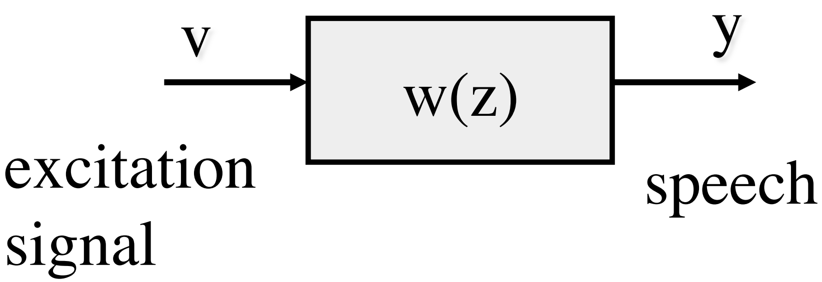
$$c_k = \frac{1}{N+1}\sum_{t=0}^{N-k} y_{t+k}y_t$$

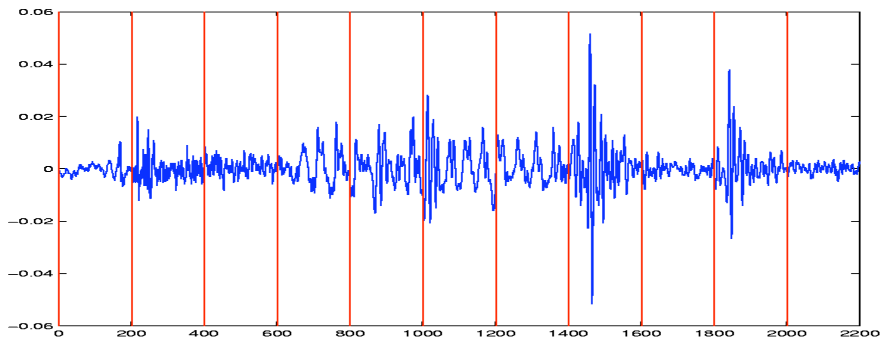
Hence we can only estimate

$c_0, c_1, c_2, \ldots, c_n, \quad n << N$

$$\int_{-\pi}^{\pi} e^{ik\theta}\Phi(e^{i\theta})\frac{d\theta}{2\pi} = c_k, \; k = 0, 1, \ldots, n$$

Ex: Modeling speech





on each (30 ms) subinterval

w(z) constant, y stationary

observation: $y_0, y_1, \ldots, y_N$

$N \approx 250$

$$\int_{-\pi}^{\pi} e^{ik\theta} d\mu = c_k := \frac{1}{N+1} \sum_{t=0}^{N-k} y_{t+k} y_t, \quad k = 0, 1, \ldots, n \quad n = 10$$

$\mathfrak{P}$ consists of trigonometric polynomials

$$d\mu = \left| w(e^{i\theta}) \right|^2 \frac{d\theta}{2\pi} = \frac{P(\theta)}{Q(\theta)} d\theta$$

rational positive measure

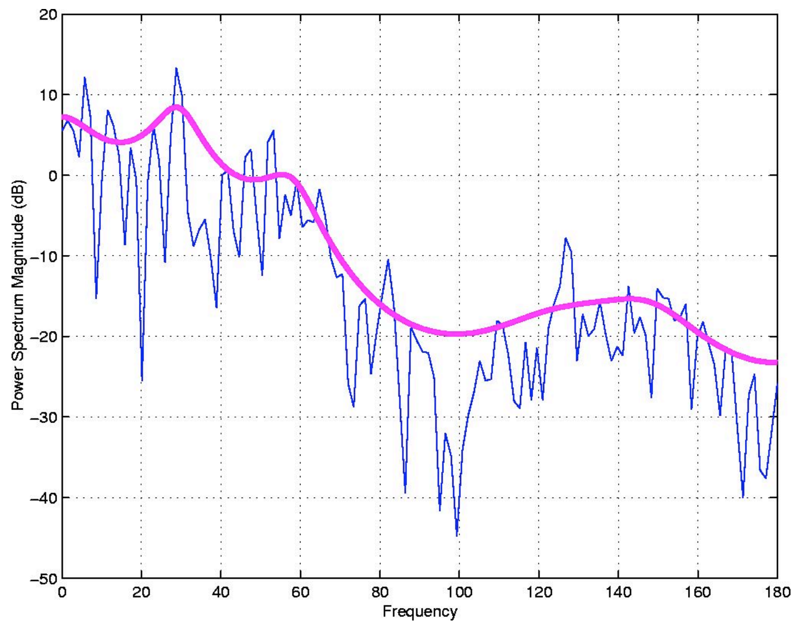
$P(\theta) = \mathrm{Re}\{p(\theta)\}$

$Q(\theta) = \mathrm{Re}\{q(\theta)\}$

where $p, q \in \mathring{\mathfrak{P}}_+$

Cellular telephone:

$$d\mu = \frac{\rho_n}{|\varphi_n(e^{it})|^2} dt$$



FFT in blue envelope in purple

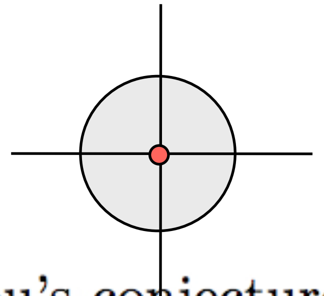
$\varphi_n(z)$ n:th Szegö polynomial orthogonal on the unit circle

$P(t) = \rho_n \qquad Q(t) = |\varphi_n(e^{it})|^2$

$\sigma(z)$ stable polynomial spectral factor: $|\sigma(e^{it})|^2 = P(t)$

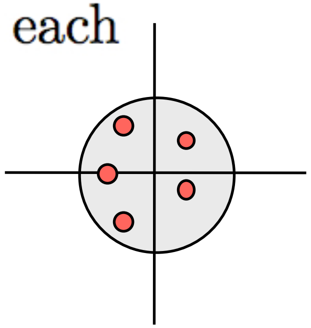
$\sigma(z) = \sqrt{\rho_n} z^n$

spectral zeros



Is there a solution $d\mu$ for each choice of spectral zeros?

YES (Georgiou 1983)

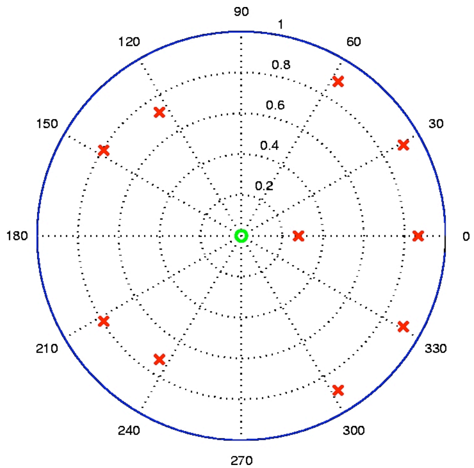


Unique? (Georgiou's conjecture)

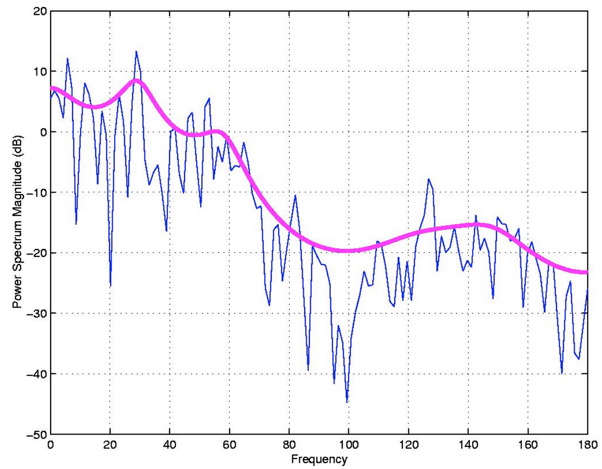
Well-posed?

YES (Byrnes, Lindquist Gusev, Matveev 1993)

$$w(z) = \frac{\sqrt{\rho_n} z^n}{\varphi_n(e^{it})}$$

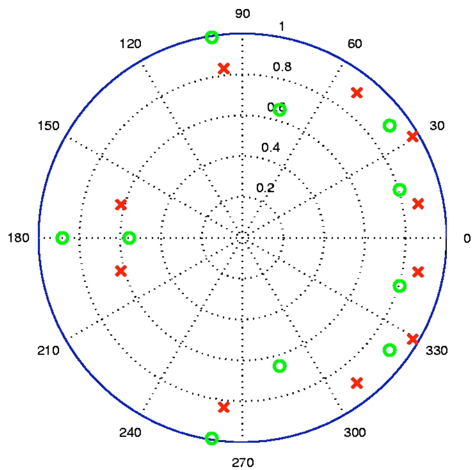


zeros/poles

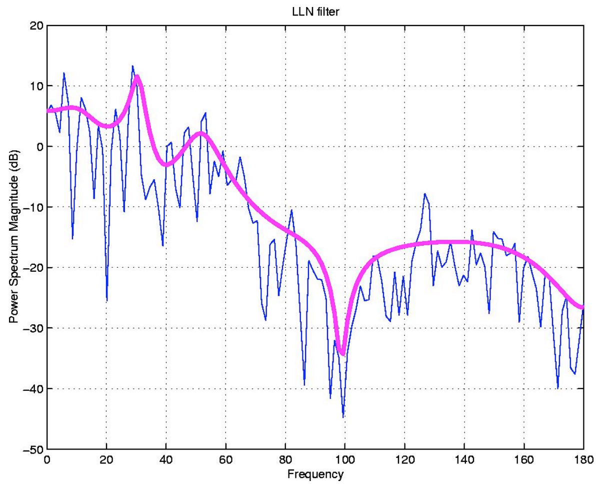


envelope

A $w(z)$ with other spectral zeros, but with the same degree



zeros/poles



envelope

The moment problem for rational measures

Byrnes - L

DEF. $p \in \mathfrak{P}, \quad p = \sum_{k=0}^{n} p_k \alpha_k$ polynomial in $\mathfrak{P}$

$P = \mathrm{Re}(p)$

P/Q, where $p, q \in \mathfrak{P}$ real rational function for $\mathfrak{P}$

$$\mathcal{R}_+ = \left\{ d\mu \mid d\mu = \frac{P(t)}{Q(t)} dt,\ p, q \in \mathring{\mathfrak{P}}_+ \right\} \subset \mathcal{M}_+$$

rational positive measure

$$\int_a^b u_k d\mu = c_k, \quad k = 0, 1, \ldots, n \quad (\dagger)$$

Find $d\mu \in \mathcal{M}_+$ satisfying $(\dagger)$ linear problem

Find $d\mu \in \mathcal{R}_+$ satisfying $(\dagger)$ nonlinear problem

$\mathring{\mathfrak{C}}_+$ interior of $\mathfrak{C}_+$

$c \in \mathring{\mathfrak{C}}_+$ strictly positive sequence

From now on we assume that all $p \in \mathfrak{P}$ are Lipschitz continuous.

THEOREM. $\mathfrak{M}(\mathcal{R}_+) = \mathring{\mathfrak{C}}_+$. In other words, the moment problem for rational measures is solvable if and only if c is strictly positive.

For each $p \in \mathring{\mathfrak{P}}_+$, define

$$\mathcal{P}_+(p) = \{d\mu \in \mathcal{R}_+ \mid p \in \mathring{\mathfrak{P}}_+ \text{ fixed}\}$$

THEOREM. For each $p \in \mathring{\mathfrak{P}}_+$, $\mathfrak{M}(\mathcal{P}_+(p)) = \mathring{\mathfrak{C}}_+$. In other words, the moment problem for rational measures with fixed $p \in \mathring{\mathfrak{P}}_+$ is solvable if and only if c is strictly positive.

We want to show that there is a unique solution for each $p \in \mathring{\mathfrak{P}}_+$.

A Dirichlet principle

For fixed $p \in \mathring{\mathfrak{P}}_+$, consider the moment equations

$$f_k^p(q) := c_k - \int_a^b \alpha_k \frac{P}{Q} dt = 0, \quad k = 0, 1, \dots, n,$$

where $f^p : \mathring{\mathfrak{P}}_+ \to \mathring{\mathfrak{C}}_+$.

Dirichlet Principle: $f_k^p(q) = 0, \; k = 0, 1, \dots, n$ are the critical point equations for some smooth function $\mathbb{J}_p : \mathring{\mathfrak{P}}_+ \to \mathbb{R}$, which has a unique minimum and no other critical points.

Define a 1-form on $\mathring{\mathfrak{P}}_+$: $\omega = \mathrm{Re}\left\{ \sum_{k=0}^{n} f_k^p(q) dq_k \right\}$

$$\omega = \mathrm{Re} \sum_{k=0}^{n} c_k dq_k - \int_a^b \frac{P}{Q} dQ dt$$

$\mathring{\mathfrak{P}}_+$ open and convex

$$d\omega = \int_a^b \frac{P}{Q^2} dQ \wedge dQ dt = 0$$ $\Rightarrow$ ω closed $\Rightarrow$ ω exact

By the Poincaré Lemma, we can integrate along any curve:

$$\mathbb{J}_p(q_1) := \int_{q_0}^{q_1} \left(\mathrm{Re} \sum_{k=0}^{n} c_k dq_k - \int_a^b \frac{P}{Q} dQ dt \right)$$ $\Rightarrow$

$$\boxed{\mathbb{J}_p(q) = \langle c, q \rangle - \int_a^b P \log Q \, dt}$$

(modulo a constant of integration)

- This is a strictly convex functional

$$\mathbb{J}_p(q) = \langle c, q \rangle - \int_a^b P \log Q \, dt$$

strictly convex function

$\mathbb{J}_p : \mathring{\mathfrak{P}}_+ \to \mathbb{R}$

Moment equations:

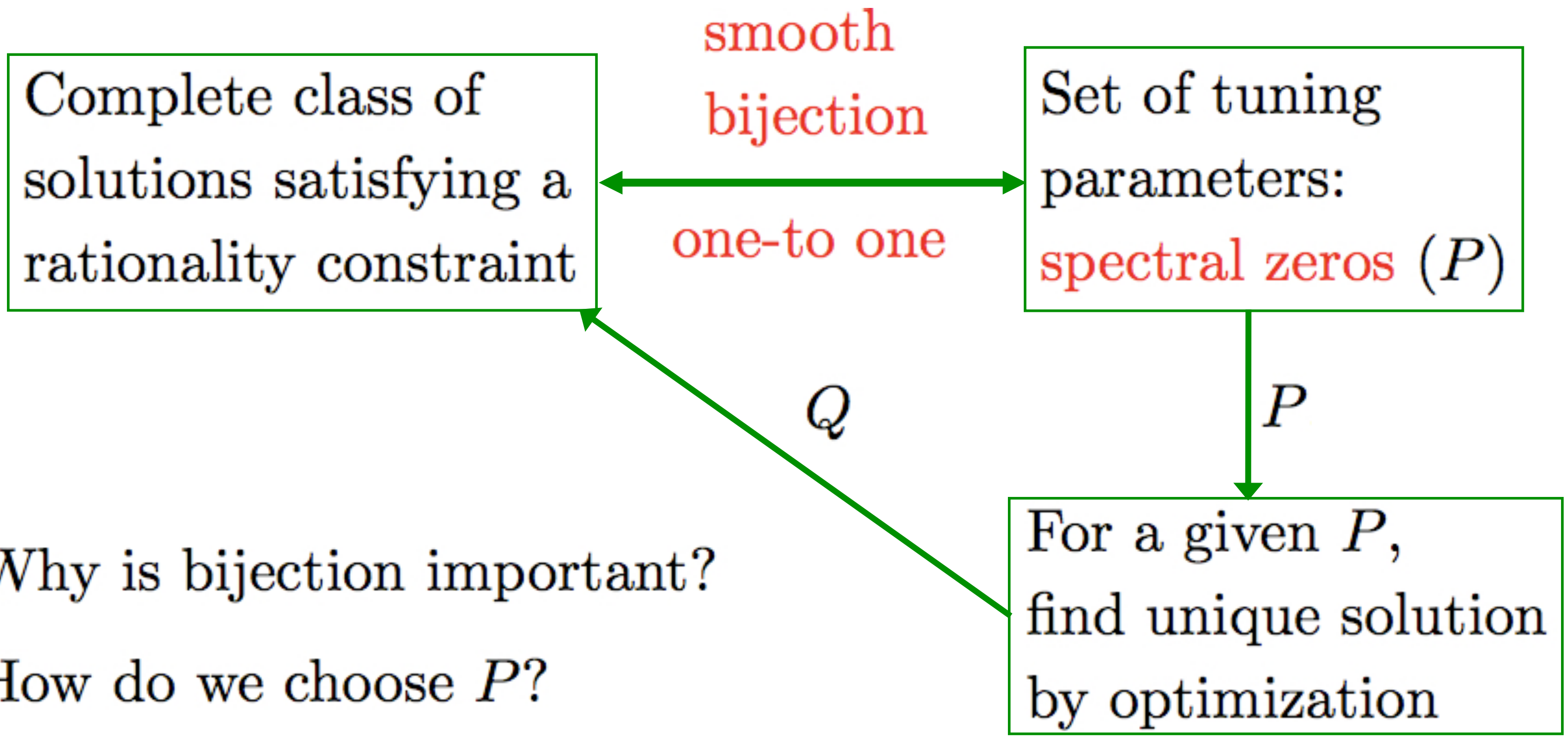
$$\frac{\partial \mathbb{J}_p}{\partial q_k} = c_k - \int_a^b u_k \frac{P}{Q} dt = 0, \quad k = 0, 1, \ldots, n \quad (\dagger)$$

We have already shown that the moment equations $(\dagger)$ have a solution $\hat{q} \in \mathring{\mathfrak{P}}_+$ for all $(c, p) \in \mathring{\mathfrak{C}}_+ \times \mathring{\mathfrak{P}}_+$. Since $\mathbb{J}_p$ is strictly convex, $\hat{q}$ is a unique minimum. Hence $(\dagger)$ has a unique solution.

THEOREM. Let $(c, p) \in \mathfrak{C}_+ \times \mathfrak{P}_+$, and set $P := Re\{p\}$. Then the functional $\mathbb{J}_p$ has a unique minimizer $\hat{q} \in \mathfrak{P}_+$: the unique solution of the moment equations $(\dagger)$. If $p \in \mathring{\mathfrak{P}}_+$, then $\hat{q} \in \mathring{\mathfrak{P}}_+$.

A global analysis approach

Object: Finding a solution that best satisfies additional design specifications (without increasing the complexity)



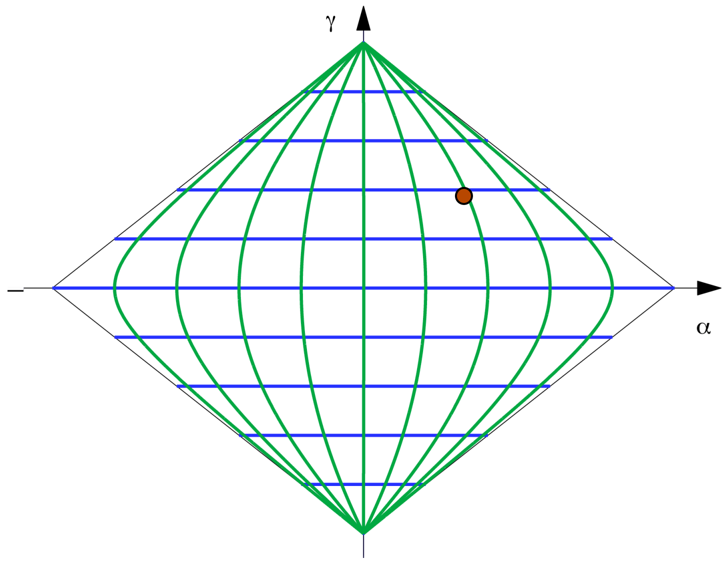
- Why is bijection important?
- How do we choose P?

EXAMPLE. $\mathfrak{P} = \text{span}\{1, e^{it}, \ldots, e^{int}\}$

The solutions $d\mu \in \mathcal{R}_+$ form a manifold of dimension $2n$.

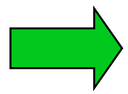
A foliation with one leaf for each choice of $p \in \mathring{\mathfrak{P}}_+$ (Kalman filtering)

A foliation with one leaf for each choice of $c \in \mathring{\mathfrak{C}}_+$ (normalized)



THEOREM. The two foliations intersect transversely so that each leaf in one meets each leaf in the other in exactly one point.

$$\min_{q \in \mathring{\mathfrak{P}}_+} \mathbb{J}(q)$$



unique solution $d\mu = \frac{P}{Q} dt$

Primal problem

spectral zeros

$$(\mathbf{P}) \quad \mathbb{I}_p(\Phi) = \int_a^b P \log \Phi dt \to \max$$

subject to $\int_a^b u_k \Phi dt = c_k, \quad k = 0, 1, \dots, n$

Lagrange multipliers

THEOREM. (P) has a unique solution

$$\Phi = \frac{P}{Q}, \quad Q := \mathrm{Re}\{\hat{q}\},$$

where $\hat{q} \in \mathfrak{P}_+$ is the unique minimizer of $\mathbb{J}_p$.

Dual problem:

$$\min_{q \in \mathfrak{P}_+} \mathbb{J}_p(q)$$

maximum entropy solution for $P = 1$

Alternative cost function:

$$\mathbb{D}(P \| \Phi) = \int_a^b P \log \frac{P}{\Phi} dt \to \min$$

Kullback-Leibler divergence

Circulant covariance extension

$$c_{2N-k} = c_k, \quad k \le N$$

$$\mathfrak{P}_+(N) = \{p \in \mathfrak{P} \mid P(e^{ik\pi/N}) > 0, \ k = 0, 1, \dots, 2N\} \supset \mathfrak{P}_+$$

$$\mathfrak{C}_+(N) = \mathfrak{P}_+(N)^{\mathsf{T}} \subset \mathfrak{C}_+$$

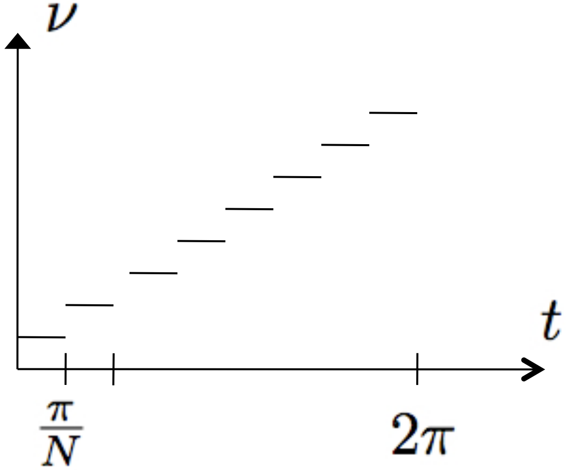
$\mathfrak{C}_+(N) \to \mathfrak{C}_+$ as $N \to \infty$

$$\mathcal{R}(N) = \{d\mu \in \mathcal{M}_+ \mid d\mu = \frac{P}{Q} d\nu, \ p, q \in \mathfrak{P}_+(N)\}$$

sum of Dirac measures

PROBLEM. Given $c_0, c_1, \dots, c_n$ $(n < N)$ find $d\mu \in \mathcal{R}(N)$ such that

$$\int_{-\pi}^{\pi} e^{ikt} d\mu = c_k, \ k = 0, 1, \dots, n \quad (\dagger)$$



For each $(c, p) \in \mathfrak{C}_+(N) \times \mathfrak{P}_+(N)$, there is a unique $q \in \mathfrak{P}_+(N)$ such that $(\dagger)$ holds. It is the unique minimizer of

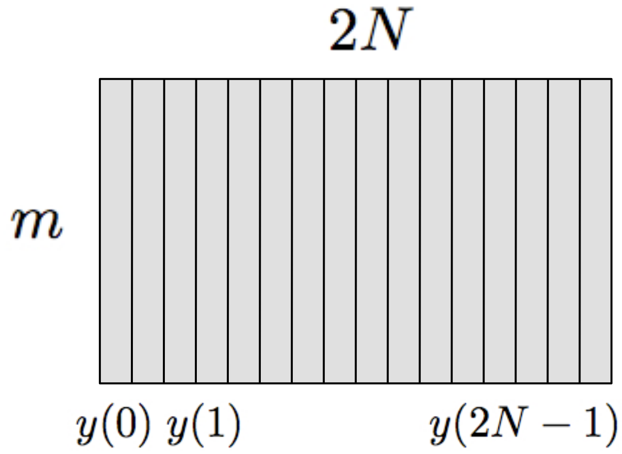
$$\mathbb{J}_p(q) = \langle c, q \rangle - \int_{-\pi}^{\pi} P \log Q \, d\nu,$$

Image processing

$y(t)$ reciprocal m-vector process

$c_k = E\{y(t+k)y(t)^{\mathsf{T}}\} \quad m \times m$

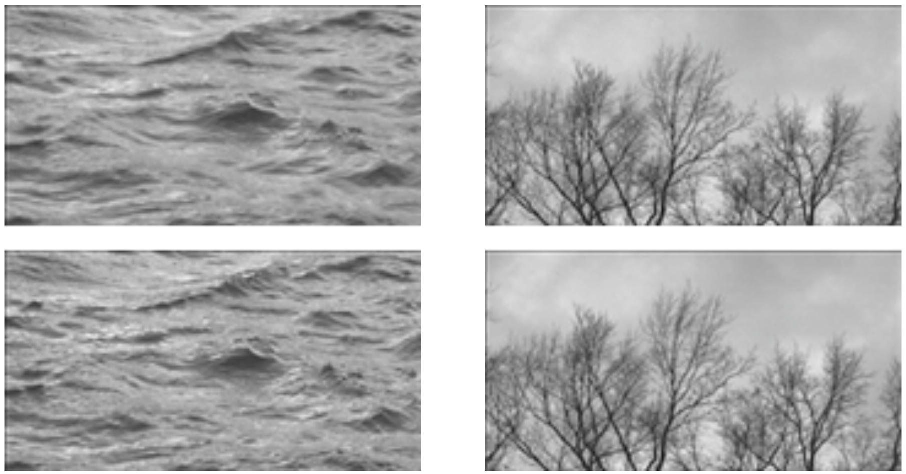
$c_{2N-k} = {c_k}^{\mathsf{T}}, \quad k \leq N$



For scalar P there is a matrix version of

$$\mathbb{J}_p(q) = \langle c, q \rangle - \int_{-\pi}^{\pi} P \log Q \, d\nu,$$

original below



Francesca Carli, Augusto Ferrante, Michele Pavon, and Giorgio Picci

reconstructions with $P = 1$ (maximum entropy) and $n = 1$ $(m = 125,\ 2N - 1 = 175)$

Nevanlinna-Pick interpolation

$$\alpha_k(t) = \frac{e^{it} + z_k}{e^{it} - z_k}, \quad k = 0, 1, \ldots, n \qquad z_0, z_1, \ldots, z_n \in \mathbb{D}$$

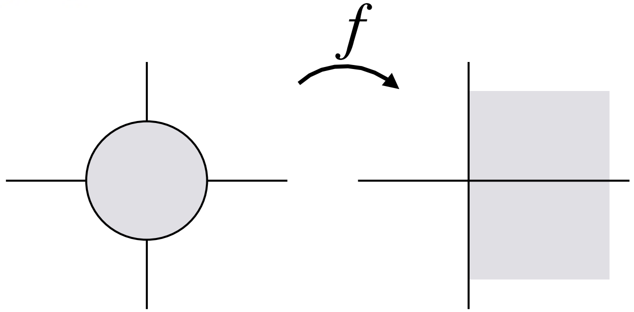
(distinct)

$$c \in \mathfrak{C}_+ \iff P_n = \left[\frac{c_j + \bar{c}_k}{1 - z_j \bar{z}_k}\right]_{j,k=0}^{n} \geq 0$$

Pick matrix

Given $z_0, z_1, \ldots, z_n \in \mathbb{D}$ (distinct), find a Carathéodory function f such that

$$f(z_k) = c_k, \quad k = 0, 1, \ldots, n$$



analytic in $\mathbb{D}$

$\mathrm{Re}\{f(z)\} \geq 0$ in $\mathbb{D}$

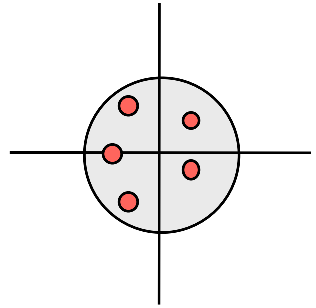
$\Updownarrow$

$$\int_{-\pi}^{\pi} \alpha_k(t) \mathrm{Re}\{f(e^{it})\} dt = c_k, \quad k = 0, 1, \ldots, n$$

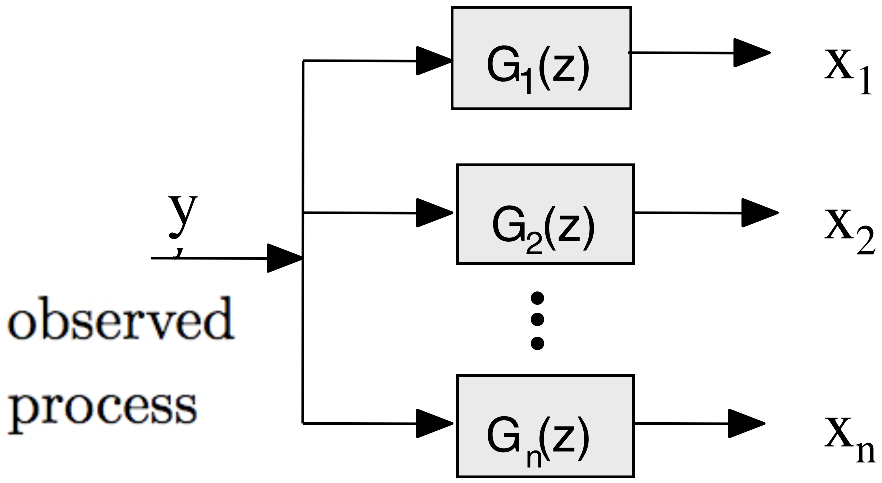
$$d\mu = \mathrm{Re}\{f(e^{it}\}dt \in \mathcal{R}_+ \iff \deg f \leq n$$

A tunable high resolution spectral estimator (THREE)

Zoom into a selected spectral band by moving interpolation points from the origin closer to the unit circle.



Byrnes-Georgiou-L



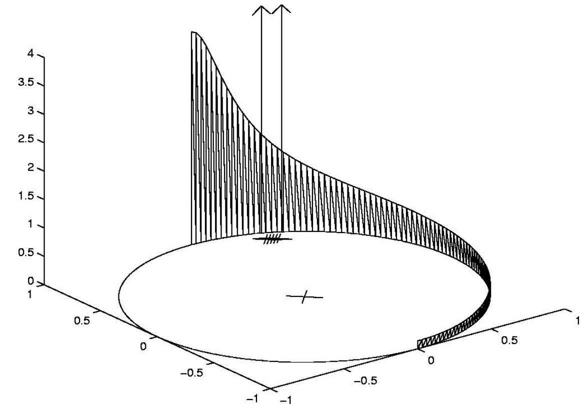
$$G_k(z) = \frac{1}{1 - \bar{z}_k z}$$

$$f(z_k) = w_k := \tfrac{1}{2}(1 - z_k^2) E\{x_k^2\}$$

$$E\{x(t)x(t)^{\mathsf{T}}\} = \left[\frac{w_k + \bar{w}_\ell}{1 - z_k \bar{z}_\ell}\right]_{k,\ell=0}^{n}$$

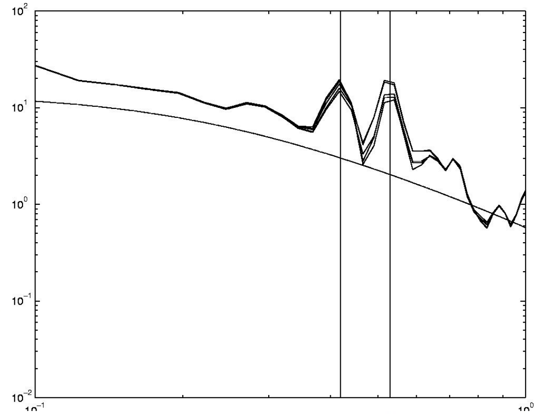
Two sets of tuning parameters:
- filter bank poles
- spectral zeros (P)

Estimation of spectral lines in colored noise

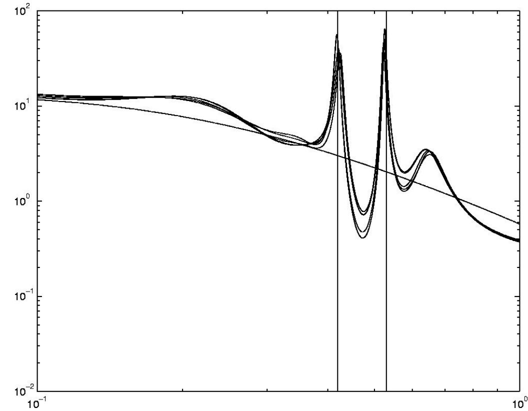


separation between
spectral lines = 0.11

five runs superimposed

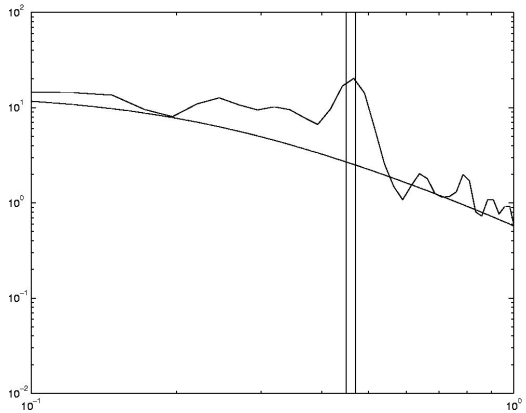


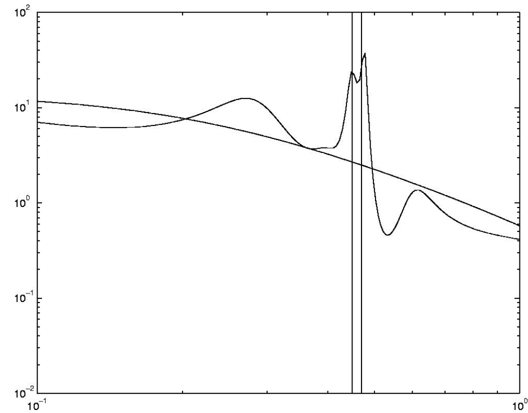
Periodogram (FFT)



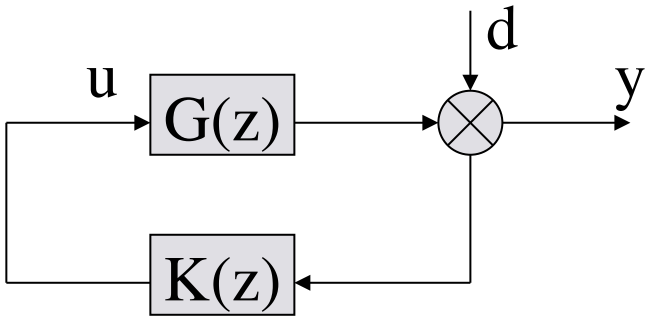
THREE (default setting)

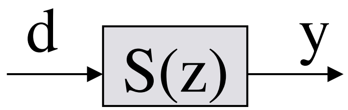
separation between
spectral lines = 0.02





Loop shaping in robust control





$S = (1 - GK)^{-1}$

Sensitivity function

- Internal stability requires

  S analytic in $\mathbb{D}^c := \{z \mid |z| > 1\}$

  $S(z_k) = 0$ at all unstable poles of G

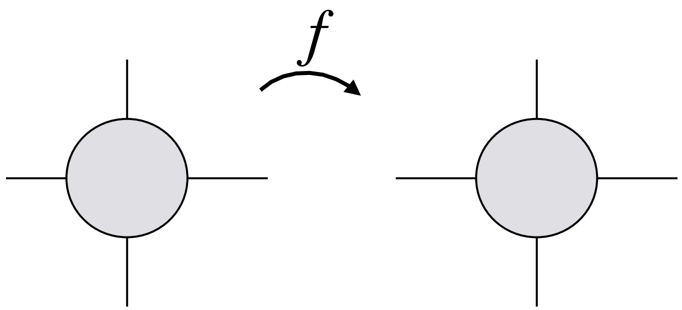
  $S(z_j) = 1$ at all zeros of G in $\mathbb{D}^c$

- Disturbance attenuation requires

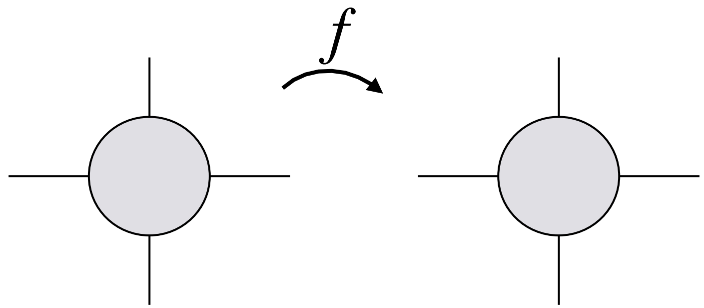
  $\|S\|_\infty \leq \gamma$

- We want $\deg S$ to be small

There is a minimum bound γ_{opt} but we choose $\gamma > \gamma_{\text{opt}}$ and define $f(z) := \frac{1}{\gamma} S(z^{-1})$



Nevanlinna-Pick interpolation for Schur functions



$$f(z_k) = w_k, \quad k = 0, 1, \ldots, n$$

class of Schur functions $\mathcal{S}$

The interpolants of degree at most n are parameterized by the $p \in \mathfrak{P}_+$ in a $1-1$ fashion

(P)
$$\max_{f \in \mathcal{S}} \mathbb{K}_p(f)$$
$$\text{subject to } f(z_k) = c_k, \quad k = 0, 1, \ldots, n$$

has unique solution $\hat{f}$

$$F : p \mapsto \hat{f}$$

where
$$\mathbb{K}_p(f) = \int_{-\pi}^{\pi} P \log\left(1 - |f(e^{it})|^2\right) dt$$

$p \in \mathfrak{P}_+$ ⇒ $\deg \hat{f} \leq n$

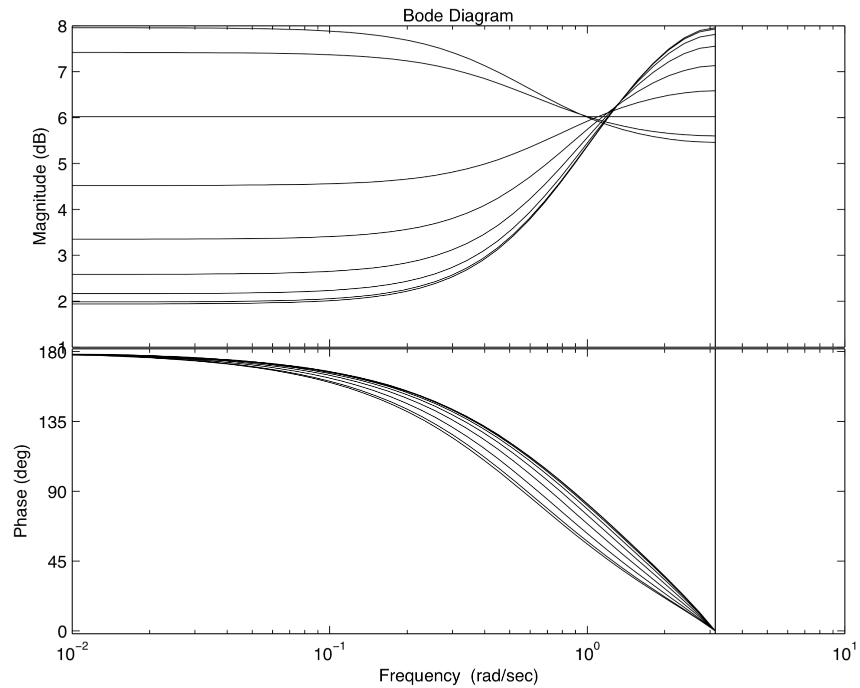
Example $G(z) = \dfrac{1}{z-2}$

$$S(2) = 0, S(\infty) = 1$$

Find all S of degree at most $n = 1$.

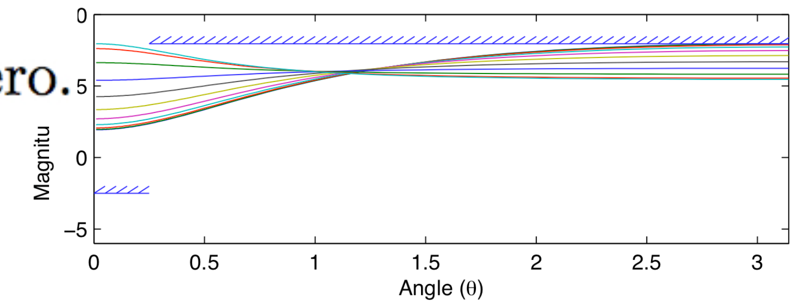
$$S(z) = \frac{z-2}{z-a}, \quad -1 < a < 1$$

$$\gamma = 2.5 > \gamma_{\text{opt}} = \min \|S\|_\infty = 2$$



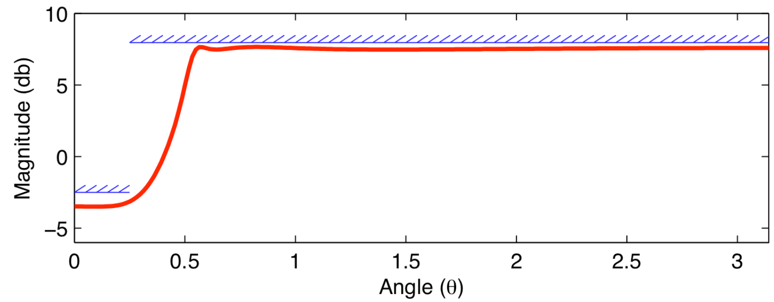
There is exactly one solution for each $p \in \mathfrak{P}_+$ represented by its (spectral) zero.

None of these solutions satisfies the design specifications



Enlarge the family of parameters $\mathfrak{P}_+$

$$\mathfrak{P}_m := \{p = p_1 p_2 \mid p_1 \in \mathfrak{P}, \; \deg p_2 \le m, \mathrm{Re}\{p\} \ge 0\}$$



Shaping by model reduction

$F:\ p \mapsto f$ unique interpolant maximizing of $\mathbb{K}_p$

Karlsson - Georgiou - L

interpolant of high or infinite degree but with $|g|$ of desired shape

difficult approximation problem

$$\begin{array}{ccc} g & \xrightarrow{\;F^{-1}\;} & \pi \in L_1 \\ \downarrow & & \downarrow \\ f & \xleftarrow{\;F\;} & p \in \mathfrak{P}_m \end{array}$$

approximation by quasi-convex optimization

interpolant of degree at most $n+m$ with a shape close to that of g

$p \in \mathfrak{P}_m \Rightarrow \deg f \leq n+m$

Example $G(z) = \dfrac{1}{z-2}$

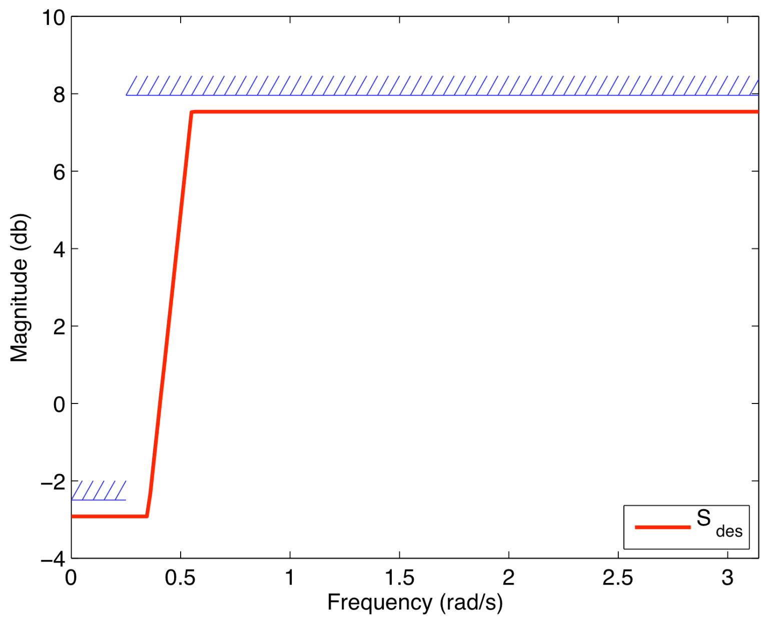
For internal stability:

$f = S/\gamma$ satisfies

$f(0) = 0.4$ and $f(0.5) = 0$

Find interpolant g with

$|g|$ as in figure (not rational)



$$\Pi = \mathrm{Re}\{\pi\} := \left(|g|^{-2} - 1\right) \in F^{-1}(g)$$

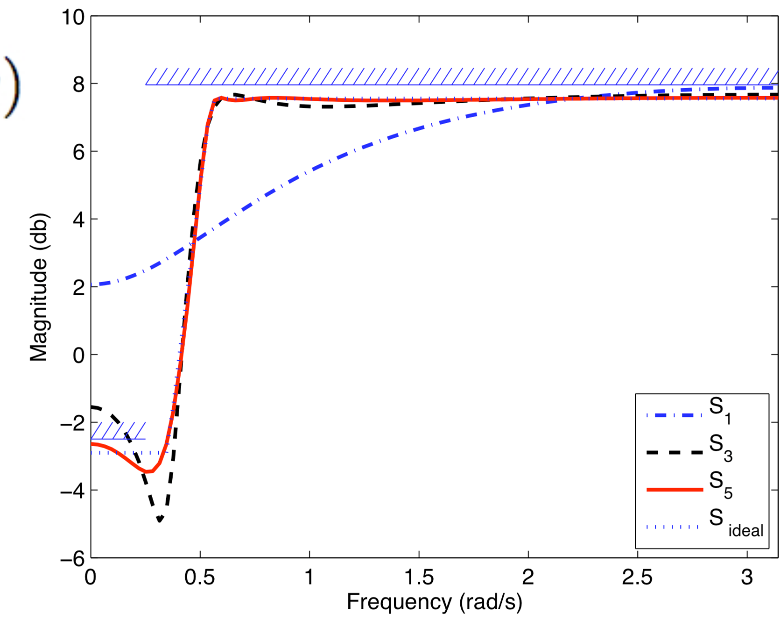
Use quasi-convex optimization

to find $p \in \mathfrak{P}_m$ (m=0, 2 and 4)

such that P is close to Π

Sensitivity functions

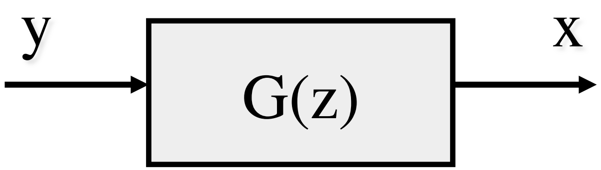
of degrees 1, 3 and 5



A generalization of THREE

stationary process

spectral density Φ



$\mathrm{E}\{x(t)x(t)^{\mathsf{T}}\} = \Sigma$

$$G(z) = (I - zA)^{-1}B$$

(P) $\min_{\Phi \in C_+} \mathbb{D}(P\|\Phi)$ subject to $\int_{-\pi}^{\pi} G\Phi G^* d\theta = \Sigma$

$$\text{where } \mathbb{D}(P\|\Phi) = \int_{-\pi}^{\pi} P \log \frac{P}{\Phi} dt$$

Kullback-Leibler divergence

(D) $\min_{\Lambda \in \mathfrak{L}_+} \mathbb{J}_p(\Lambda)$ where $\mathbb{J}_p(\Lambda) = \mathrm{tr}(\Sigma\Lambda) - \int_{-\pi}^{\pi} P \log G^*\Lambda G \, d\theta$

$$\mathfrak{L}_+ = \{\Lambda \in \text{range } \Gamma \mid Q := G^*\Lambda G > 0\} \quad \text{where } \Gamma : \ \Phi \mapsto \Sigma$$

Multi-variable case

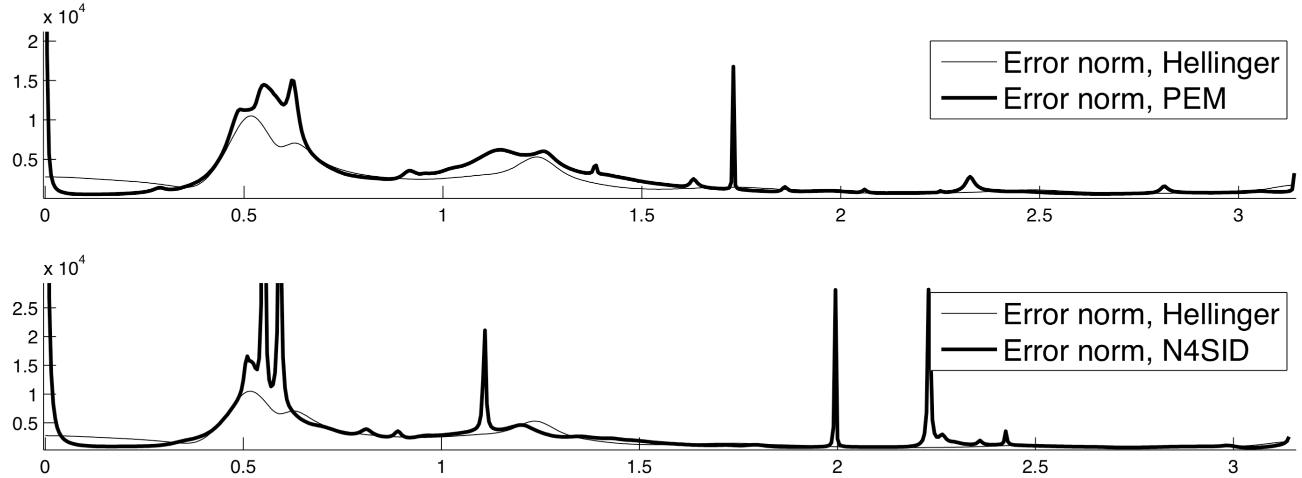
$$\mathbb{D}(P\|\Phi) = \int_{-\pi}^{\pi} \operatorname{tr}\left(P(\log P - \log \Phi)\right) d\theta$$

works well in the multi-variable case for scalar P

von Neumann's generalization of Kullback-Leibler divergence

Ferrante, Pavon and Ramponi have suggested replacing $\mathbb{D}(P\|\Phi)$ by the Hellinger distance:

$$d_H(\Phi, P) = \inf\left\{\|W_P - W_\Phi\|_2 \mid W_P W_P^* = P,\ W_\Phi W_\Phi^* = \Phi\right\}$$



Some other problems

- Prediction-error approximation

  Given spectral density Φ, find approximant $\hat{\Phi}$ in the model class $\hat{\Phi} = Q^{-1}$, where $q \in \mathfrak{P}_+$.

  Blomqvist - Wahlberg

$$\underbrace{\int_{-\pi}^{\pi} (\Phi Q - \log Q)\, d\theta}_{} \rightarrow \min$$

$$\mathbb{J}_1(q) = \langle c, q \rangle - \int_{-\pi}^{\pi} \log Q d\theta$$

where $c_k = \int e^{ik\theta} \Phi d\theta$

prefiltering for nontrivial P

- Covariance and cepstral matching

  Given $c_0, c_1, \dots, c_n$ and $\sigma_1, \dots \sigma_n$, find $p, q \in \mathfrak{P}_+$ such that

  Byrnes - Enqvist - L, Enqvist, Georgiou - L, Avventi - Enqvist

$$\begin{cases} \int_{-\pi}^{\pi} e^{ik\theta} \frac{P}{Q} d\theta = c_k \\ \int_{-\pi}^{\pi} e^{ik\theta} \log \frac{P}{Q} d\theta = \sigma_k \end{cases}$$

- Operator Theory   Byrnes - Georgiou - Lindquist - Megretski

Conclusions

An enhanced theory for generalized moment problems that incorporates rationality constraints prescribed by applications.

- Complete parameterizations of solutions with smooth tuning strategies.
- A global analysis approach that studies the class of solutions as a whole.
- Convex optimization for determining solutions.