Group Steering: Approaches Based on Power Moments

Guangyu Wu [a], Anders Lindquist [b]

[a] *Department of Automation, Shanghai Jiao Tong University, Shanghai, China*

[b] *Department of Automation and School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai, China*

Abstract

This paper considers the problem of steering a colossal group of agents of which the dynamics are governed by a discrete-time first-order linear system, which is a very preliminary version. The group of agents are characterized as a probability density function and an occupation measure respectively in the paper and two corresponding treatments are given. We propose to use the power moments to characterize the density function/occupation measure of the agents. A moment system representation of the original system is put forward for control and an empirical control scheme corresponding to it is proposed. By the designed control law, the moment sequence of the control at each time step is positive, which ensures the existence of the control for the moment system. We then realize the control as an analytic form of function by a convex optimization scheme of which the existence and uniqueness of the solution have been proved in our previous paper. An error analysis of the terminal density from the specified one is also provided. For the problem where the group of agents is characterized as an occupation measure, the control for each agent is determined by drawing independent and identically-distributed(i.i.d) samples from the realized analytic function. Finally we simulate both unconstrained and constrained controls of a colossal group of agents, which validate our proposed algorithms.

Key words: Density steering; power moments; multiple agents.

1 Introduction

We are interested in the problem of steering the states of a colossal group of agents, of which the dynamics are governed by a discrete-time stable first-order linear system, between an initial and a final probability densities or occupation measures without stochastic disturbance. We consider the linear dynamics of the i_{th} agent

$$x_i(k+1) = a(k)x_i(k) + u_i(k), \quad i = 1, \cdots, N \qquad (1)$$

Since the system is stable and we assume $a(k)$ to be positive, we have $a(k) \in (0, 1)$. The control input on the i_{th} agent is defined as u_i, and x_i is its state. We assume that the agents are non-interactive and the volume of the agents is ignored. It means that the agents are allowed to occupy the same state and the collisions are ignored.

It is a significant problem both theoretically and empirically. From the perspective of theory, the task is to control numerous agents. For the conventional control problems, there is always only one object to control and the feedback control law is proposed as a function of the state of the single object. However it is quite difficult for a colossal group of agents. In this problem, we don't aim to steer the state of each agent to a specified state, which is not necessary and very computationally consuming. Instead, we attempt to control the agents as a whole, and require the whole group to have specific global properties. Moreover, with the number of agents approaching to infinity, the problem is an infinite dimensional one, which is intractable only if dimension reduction, or we call approximation, is applied to this problem. The intrinsic infinite-dimensionality makes the problem an interesting and open one. As Dr. Roger Brockett asserted in [5], "Important limitations standing in the way of the wider use of optimal control can be circumvented by explicitly acknowledging that in most situations the apparatus implementing the control policy will be judged on its ability to cope with a distribution of initial states, rather than a single state". To the end of real practice, there are quite some scenarios where we need to control a group of agents. Typical applications include but are not limited to the the steering of swarms (UAV's, large collection of microsatelites, ensemble control, etc.), modeling of the flow and collective motion of agents.

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Email addresses: chinarustin@sjtu.edu.cn (Guangyu Wu), alq@kth.se (Anders Lindquist).

There have been several results on the problem of steering a group of agents. One branch of results formulate the problem as stabilizing a discrete-time Markov process evolving on a compact subset of $\mathbb{R}^d$ to an arbitrary target density [2,3]. These insightful results made significant contributions to this problem, however the results proposed in these papers are not able to steer the initial density to an arbitrary one within limited steps, either is an error of the terminal distribution from the specified one proposed. Moreover, the support of the inputs are assumed to be bounded. Another significant branch of treatments in the previous results is to assume the distribution as a Gaussian [1, 14, 15]. Then the problem reduces to steering the first two orders of the moments of the Gaussian distributions to the specified ones. We would also like to mention results for the continuous-time systems. Chen, Georgiou and Paven have proposed fundamental results using the Schrödinger Bridge strategy for Gaussian distributions [8, 9] and more general distributions [10]. The results are extended to nonlinear continuous-time system and hard state constraints in [7]. Moreover, a robust optimal density control of robotic swarms is proposed in [18].

When the density of the agents is assumed to be Gaussian, we use the first and second order moments to characterize the density, which turns the problem to a finite-dimensional one. However, by generalizing the mean and covariance to all the power moments, we will have a more conceptual view of this problem. Controlling the system state as a distribution function, if only assumed to be Lebesgue integrable, is an uncountably infinite-dimensional problem. By probability theory, we note that a distribution function can be uniquely determined by its full power moment sequence [24]. By controlling the full power moment sequence instead of the distribution of system state, the problem is reduced to a countably infinite-dimensional one. By properly truncating the first several terms of the power moment sequence for characterizing the density of the system state, the problem is now steering a truncated power moment sequence to another, which is finite-dimensional and tractable.

In this paper we provide what can be regarded as the first **computable** and **implementable** solution to the distribution steering problem of the discrete-time linear system within limited steps, where the specified initial and terminal distributions, including probability densities and occupation measures of the agents, are **arbitrary** (only assuming the existence of first several orders of power moments). The paper is a very preliminary version, which is structured as follows. In Section 2, we first treat the density steering problem. We propose a moment system representation as a counterpart of the discrete-time linear system. Then there follows a definition to group steering by moments based on the moment system. Different from the conventional control problems, the Hankel matrices of the moments of control inputs and system states need to be positive definite, which makes it difficult to treat the control problem by prevailing methods such as optimal control. We propose an empirical scheme to treat this problem. Then we use a density parametrization algorithm proposed in our previous paper [22] to realize the control inputs as analytic functions by the power moments. Since the density steering problem, where the densities are not assumed to fall within specific functions, is intrinsically infinite-dimensional, the error of the terminal density is inevitable. We then propose an error upper bound of the terminal density from the specified one. Based on the density-steering algorithm proposed in Section 2, we propose an algorithm of steering finitely many agents which are characterized as an occupation measure. The control input of each agent is obtained by drawing i.i.d. samples from the realized control functions where the sampling is enabled by the acceptance-rejection sampling strategy. At last, we give six examples to validate the two steering algorithms we propose. The densities in the examples include Gaussian, mixture of Gaussians and mixture of Laplacians.

2 Steer the group of agents as a probability density function

We first consider the problem of steering an initial density function to a terminal one. Then the steering of arbitrary occupation measures will be treated in the following section, of which there has not been a solution in the previous papers.

2.1 A moment system representation

With a colossal group of agents, i.e., N is large, a conventional approach is to approximate $x(k)$ and $u(k)$ as random variables of which the density functions are denoted as q_k and p_k. The problem of steering the group of agents is then turned to steering an initial density function to a terminal one. The density control problem is formulated as follows.
Problem 2.1 (Density control problem)**.** The dynamics of the system is

$$x(k+1) = a(k)x(k) + u(k). \qquad (2)$$

Given an initial probability density function $q_0(x)$ of random variable $x(0)$, determine the control sequence, i.e., $(u(0), \cdots, u(K-1))$ such that the terminal density function is $q_T(x)$ for $x(K)$.

However it is not always feasible for us to obtain a closed form of solution to this problem. If the distributions are not assumed to fall within specific classes, the problem is intrinsically infinite-dimensional. We note that the density function of $x(k+1)$ can be written as

$$
\begin{aligned}
q_{k+1}(t) &= \int_{\mathbb{R}} f_k(\xi, t-\xi)\, d\xi \\
&\underline{\underline{x \perp\!\!\!\perp u}} \int_{\mathbb{R}} q_k\left(\frac{\xi}{a(k)}\right) p_k(t-\xi)\, d\xi \\
&\underline{\underline{x \perp\!\!\!\perp u}} \left(q_k\left(\frac{t}{a(k)}\right) * p_k(t)\right)(t).
\end{aligned}
\tag{3}
$$

where $x \perp\!\!\!\perp u$ denotes that $x(k)$ and $u(k)$ are independent. To tackle the problem defined in Problem 2.1, we need to obtain an analytic form of solution to $q_{k+1}(t)$ in (3). However except for limited classes of functions such as Gaussian distributions and trigonometric functions, it is infeasible for most others. That is main reason that in the previous results which have similar problem setting as Problem 2.1, the examples are almost Gaussian or trigonometric densities. This severely limits the use of these results in the real applications.

There is a similar problem in non-Gaussian Bayesian filtering. In our previous results [22], we proposed a method of using the power moments to treat this intractable problem, mainly for characterizing the macroscopic property of the distributions. However, even it is theoretically feasible to characterize the distribution of the agents by the full power moment sequence, the problem is infinite dimensional. A common treatment is to truncate the first $2n$ moment terms [6, 11], which turns the problem we treat to a truncated moment problem.

By the system equation (2), the power moments of the states up to order $2n$ are written as

$$
\mathbb{E}\left[x^l(k+1)\right] = \sum_{j=0}^{l} \binom{l}{j} a^j(k) \mathbb{E}\left[x^j(k) u^{l-j}(k)\right]
$$

for $l \in \mathbb{N}_0$ ($\mathbb{N}_0$ denotes the set of all nonnegative integers), $l \leq 2n$. We note that it is difficult to treat the term $\mathbb{E}\left[x^j(k) u^{l-j}(k)\right]$, since we are confronted with a colossal group of agents, even an infinite number of them. However, we note that if $x(k)$ and $u(k)$ are independent from each other, i.e., $\mathbb{E}\left[x^j(k) u^{l-j}(k)\right] = \mathbb{E}\left[x^j(k)\right] \mathbb{E}\left[u^{i-j}(k)\right]$, the dynamics of the moments can be written in a linear matrix equation

$$
\mathcal{X}(k+1) = \mathcal{A}(\mathcal{U}(k))\mathcal{X}(k) + \mathcal{U}(k) \tag{4}
$$

where the new state vector is composed of the power moment terms up to order $2n$,

$$
\mathcal{X}(k) = \begin{bmatrix} \mathbb{E}[x(k)] & \mathbb{E}[x^2(k)] & \cdots & \mathbb{E}[x^{2n}(k)] \end{bmatrix}^T. \tag{5}
$$

The new input vector is written as

$$
\mathcal{U}(k) = \begin{bmatrix} \mathbb{E}[u(k)] & \mathbb{E}[u^2(k)] & \cdots & \mathbb{E}[u^{2n}(k)] \end{bmatrix}^T. \tag{6}
$$

Here we have defined

$$
\mathbb{E}\left[x^l(k)\right] = \int_{\mathcal{S}} x^l q_k(x) dx \tag{7}
$$

and

$$
\mathbb{E}\left[x^j(k) u^{l-j}(k)\right] = \int_{\mathcal{S}} x^j q_k(x) dx \int_{\mathcal{C}} u^{l-j} p_k(u) du.
$$

Similarly we have

$$
\mathbb{E}\left[u^l(k)\right] = \int_{\mathcal{C}} u^l p_k(u) du \tag{8}
$$

The sets $\mathcal{S}$ and $\mathcal{C}$ are the support of $q_k(x)$ and $p_k(u)$ correspondingly. For control problems without state constraints, $\mathcal{S}$ can be chosen as the real line $\mathbb{R}$. Or it can be chosen as a compact subset of $\mathbb{R}$ according to its bounds. Similarly, $\mathcal{C}$ can be chosen as $\mathbb{R}$ if the control input is unconstrained, or a compact subset of $\mathbb{R}$ if constrained.

The matrix $\mathcal{A}(\mathcal{U}(k))$ of the new system can then be written as (9).

By using the truncated power moments to characterize the dynamics of system (1) where $x(k)$ and $u(k)$ are probability densities, we shall reformulate the control problem as steering the power moments of the probability densities. System (4) is called the moment system corresponding to system (1).

By the proposed moment system, the original problem in Problem 2.1 can be reduced to distribution steering by power moments, which is formulated as follows.

Problem 2.2 (Density steering problem by power moments)**.** The dynamics of the moment system is

$$
\mathcal{X}(k+1) = \mathcal{A}(\mathcal{U}(k))\mathcal{X}(k) + \mathcal{U}(k)
$$

where $\mathcal{X}(k), \mathcal{U}(k)$ are obtained by (7),(8). Given an **arbitrary** initial density $q_0(x)$, determine the control sequence $(u(0), \cdots, u(K-1))$ such that the first $2n$ order power moments of the terminal density are identical to those of an **arbitrarily** specified one, i.e.,

$$
\int_{\mathbb{R}} x^l q_T(x) dx = \int_{\mathbb{R}} x^l q_K(x) dx \tag{10}
$$

for $l = 1, \cdots, 2n$, where q_T is the specified terminal density function.

$$\mathcal{A}(\mathcal{U}(k)) = \begin{bmatrix} a(k) & 0 & 0 & \cdots & 0 \\ 2a(k)\mathbb{E}[u(k)] & a^2(k) & 0 & \cdots & 0 \\ 3a(k)\mathbb{E}[u^2(k)] & 3a^2(k)\mathbb{E}[u(k)] & a^3(k) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ \binom{2n}{1}a(k)\mathbb{E}[u^{2n-1}(k)] & \binom{2n}{2}a^2(k)\mathbb{E}[u^{2n-2}(k)] & \binom{2n}{3}a^3(k)\mathbb{E}[u^{2n-3}(k)] & & a^{2n}(k) \end{bmatrix}. \quad (9)$$

Remark. The advantage of our proposed problem is twofold. First, by this formulation, the problem is now finite dimensional and there exists a closed form of solution to this problem. Second, it doesn't require the initial and terminal density functions of the agents to fall within specific classes, which makes the algorithm applicable to a wide range of real applications.

However with the moment system to control, there remains to design control laws which satisfies

$$\mathbb{E}\left[x^j(k)u^{l-j}(k)\right] = \mathbb{E}\left[x^j(k)\right]\mathbb{E}\left[u^{i-j}(k)\right]. \quad (11)$$

With limited number of agents, it might be possible to propose feedback control laws for each agent, i.e., the control input is a function of the states of all the agents. However with the increase of the number of agents, it is no longer tractable. In our problem, N is infinity (for the problem of steering occupation measures, N is still a large number). A feasible control law needs to satisfy that the control vector is independent of the current state vector. In the next part of section, we propose an algorithm for steering power moments to desired ones.

2.2 An empirical control scheme for the moment system

In the previous section, controlling the group of agents has been reduced to controlling the moment system corresponding to it. Then the task is now to figure out an algorithm to determine a sequence of $(\mathcal{U}(0), \cdots, \mathcal{U}(K-1))$. However, there are two main differences from the conventional control problems. First, the system matrix of the moment system is a function of the control vector in this problem. Second, the sequence of the elements of the control vector $\mathcal{U}(k)$ needs to satisfy the condition that the corresponding Hankel matrix

$$[\mathcal{U}(k)]_H = \begin{bmatrix} \mathbb{E}[u^0(k)] & \mathbb{E}[u^1(k)] & \cdots & \mathbb{E}[u^n(k)] \\ \mathbb{E}[u^1(k)] & \mathbb{E}[u^2(k)] & & \mathbb{E}[u^{n+1}(k)] \\ \vdots & \vdots & \ddots & \\ \mathbb{E}[u^n(k)] & \mathbb{E}[u^{n+1}(k)] & & \mathbb{E}[u^{2n}(k)] \end{bmatrix}$$

is positive definite. Here $[\mathcal{U}]_H$ denotes the Hankel matrix of the vector $\mathcal{U}$. We define such subspace of $\mathbb{R}^{2n}$ as $\mathbb{V}^{2n}_{++} := \{\mathcal{U} \in \mathbb{R}^{2n} \mid [\mathcal{U}]_H \succ 0\}$.

In the previous results, optimal control strategy is always used in the distribution steering problems. However, it is not quite feasible in this problem. The reason is that we always have to ensure that $\mathcal{X}(k), \mathcal{U}(k) \in \mathbb{V}^{2n}_{++}$. To the best of our knowledge, there has not been a result feasible of treating the optimal control problem constraining the states and control inputs to fall within $\mathbb{V}^{2n}_{++}$, i.e., the corresponding Hankel matrices to be positive definite.

Now we formulate the problem we are to treat in this part of section. Let $\mathcal{U}$ be the feasible set of control sequences $\boldsymbol{\mathcal{U}} := (\mathcal{U}(0), \cdots, \mathcal{U}(K-1))$, which satisfies

$$\sum_{k=0}^{K-1} \mathbb{E}\left[\mathcal{U}^T(k)\mathcal{U}(k)\right] < \infty$$

and effects the terminal system state $x(K)$ to be distributed satisfying (10). Then the family $\mathcal{U}$ represents admissible control inputs which achieve the desired moment transfer.

Denote the error of the moments from the specified ones as

$$e(k) = \mathcal{X}_T - \mathcal{X}(k) \quad (12)$$

It is not always feasible to handle both the constraints on $\mathcal{U}(k)$ and $\mathcal{X}(k)$ to fall within the set $\mathbb{V}^{2n}_{++}$ simultaneously. However, we note that a sub-optimal solution to the control problem can be achieved by first to determine the trajectory of the state then to obtain the control inputs corresponding to this trajectory. We first determine the trajectory of the state.

Lemma 2.3. *Given*

$$e(k_0) = \mathcal{X}_T - \mathcal{X}(k_0) \in \mathbb{V}^{2n}_{++},$$

we have

$$\mathcal{X}(k) = \mathcal{X}(k_0) + \sum_{i=k_0}^{k-1} \omega_i e(k_0) \in \mathbb{V}^{2n}_{++} \quad (13)$$

for $k = k_0+1, \cdots, K$ where $\omega_i \in \mathbb{R}_+$ for $i = k_0, \cdots, K-1$ and $\sum_{i=k_0}^{K-1} \omega_i = 1$. Here the elements of $\mathcal{X}_T$ are the power moments corresponding to the specified terminal density function $q_T(x)$.

Proof. The proof is straightforward. Since $\mathcal{X}(k_0), e(k_0) \in \mathbb{V}_{++}^{2n}$, we have $[\mathcal{X}(k_0)]_H \succ 0$, $[e(k_0)]_H \succ 0$. We note that the sum of positive definite matrices is still positive definite. Since $\omega_i > 0$, we have $\omega_i e(k_0) \in \mathbb{V}_{++}^{2n}$. Then $\mathcal{X}(k) \in \mathbb{V}_{++}^{2n}$. □

Now it remains to prove that there exists a time step k_0 at which $\mathcal{X}_T - \mathcal{X}(k_0) \in \mathbb{V}_{++}^{2n}$.

Proposition 2.4. *There exists a time step k_0 which satisfies $\mathcal{X}_T - \mathcal{X}(k_0) \in \mathbb{V}_{++}^{2n}$, assuming that $\mathcal{X}(k), 0 \leq k \leq k_0$ are uncontrolled moment states, i.e., $u(k) = 0, 0 \leq k \leq k_0$.*

Proof. We write the Hankel matrix form of $\mathcal{X}_T - \mathcal{X}(k)$ as (14). Since $u(k) = 0, 0 \leq k \leq k_0$, we obtain (15).

Now it remains to prove that $\exists k_0$, $[\mathcal{X}_T - \mathcal{X}(k_0)]_H \succ 0$ with $u(k) = 0, 0 \leq k \leq k_0$. By definition of the positive definiteness, it is equivalent to prove that each leading principal minor, the determinant of leading principal submatrix, is positive.

Denote the i_{th}-order leading principal submatrix of $[\mathcal{X}_T - \mathcal{X}(k)]_H$ as $H_i(k)$, and the corresponding minor as $\det(H_i(k))$. We note that each $\det(H_i(k))$ is a polynomial of $\prod_{i=0}^{k-1} a(i)$, of which the degree is even. Therefore, if there exists no real zero for all the $\det(H_i(k))$, all $k_0 \in \mathbb{N}_0$ satisfies $[\mathcal{X}_T - \mathcal{X}(k_0)]_H \succ 0$. Now we consider the case that $\det(H_i(k))$ has at least a real zero in $(0, 1)$. We note that $\det(H_i(k_0)) > 0$ with $k_0 \to +\infty$. Let $\breve{k}_i$ be the smallest integer greater than the largest zero of the polynomial $\det(H_i(k))$. By the continuity of $\det(H_i(k))$, we have that $\det(H_i(k)) > 0, k \in (\breve{k}_i, +\infty)$.

Let $\breve{k} = \max_i(\breve{k}_i)$. With $k_0 > \breve{k}$, we have $H_i(k_0) \succ 0$, for $i = 1, \cdots, n$. Therefore we have $[\mathcal{X}_T - \mathcal{X}(k_0)]_H \succ 0$, which ensures the positiveness of all $H_i(k_0)$ and completes the proof. □

By Proposition 2.4, it is feasible for us to choose a time step k_0 which satisfies $\mathcal{X}_T - \mathcal{X}(k_0) \in \mathbb{V}_{++}^{2n}$. We assume that the system is uncontrolled before k_0, i.e. $u(k) = 0, k \leq k_0$. From step k_0, we impose controls on the system. Lemma 2.3 has proved the positiveness of $\mathcal{X}(k), k = k_0, \cdots, K$. Therefore it remains to determine the parameters $\omega_k, k = k_0, \cdots, K-1$ and the corresponding control inputs $\mathcal{U}(k)$.

It is a non-trivial problem. We give an empirical scheme to treat it. To obtain a relatively smooth transition of states, it is desired that the ω_i's are close to each other. It is usually feasible for us to choose

$$\omega_{k_0} = \cdots = \omega_{K-1} = \frac{1}{K - k_0}$$

After that the parameters $\omega_i' s$ are determined, the control inputs of the moment system $\mathcal{U}(i)$ for $i = k_0, \cdots, K-1$ can then be calculated by solving the equation (4), provided with $\mathcal{X}(k), k = k_0 + 1, \cdots, K$ calculated by (13).

However sometimes the control inputs $\mathcal{U}(k) \notin \mathbb{V}_{++}^{2n}$ by choosing the ω_i's to be all equal. It usually happens when the specified initial/terminal density has several modes (peaks). If so, we can choose a larger ω_0/ω_{K-1}.

In conclusion, we have proposed to use the moment system to control the colossal group of agents characterized as a probability density function. And an empirical control law has been proposed which ensures the existence of $u(k)$. However, it is only feasible for us to obtain the power moments of the control inputs $u(k)$. We need to obtain the $u(k)$ given its power moments, which we call realization of the control inputs.

2.3 Realization of the control inputs

In this part of section, we are to realize the probability density of $u(k)$ given the power moments of the designed controls $\mathcal{U}(k)$ for the moment system.

For the sake of simplicity, we omit k if there is no ambiguity in the following part of this section. The problem now comes to proposing an estimation algorithm which is feasible of estimating the probability density of which the power moments are as specified.

Typically we consider two situations for the controller design, one is unconstrained control, i.e., u is supported on the whole real line. And the other is constrained control, where we consider u to be supported on a compact interval on $\mathbb{R}$. These two situations refer to two types of moment problems we are to treat, namely the Hamburger moment problem for the first situation and the Hausdorff moment problem for the second one.

A convex optimization scheme for density estimation by Kullback-Leibler distance has been proposed in [22] considering the Hamburger moment problem. We adopt this strategy in this paper for treating the unconstrained control realization. The treatment is as follows.

Let $\mathcal{P}$ be the space of probability density functions on the real line with support there, and let $\mathcal{P}_{2n}$ be the subset of all $p \in \mathcal{P}$ which have at least $2n$ finite moments (in addition to $\mathbb{E}[u^0(k)]$, which of course is 1). The Kullback-Leibler distance is then defined as

$$\mathbb{KL}(r\|p) = \int_{\mathbb{R}} r(u) \log \frac{r(u)}{p(u)} du \tag{16}$$

where r is an arbitrary probability density in $\mathcal{P}$. And we

$$
\begin{aligned}
&[\mathcal{X}_T - \mathcal{X}(k)]_H \\
&= \begin{bmatrix} 1 & \mathbb{E}[x_T] - \mathbb{E}[x(k)] & \cdots & \mathbb{E}[x_T^n] - \mathbb{E}[x^n(k)] \\ \mathbb{E}[x_T] - \mathbb{E}[x(k)] & \mathbb{E}[x_T^2] - \mathbb{E}[x^2(k)] & \cdots & \mathbb{E}[x_T^{n+1}] - \mathbb{E}[x^{n+1}(k)] \\ \vdots & \vdots & \ddots & \\ \mathbb{E}[x_T^n] - \mathbb{E}[x^n k] & \mathbb{E}[x_T^{n+1}] - \mathbb{E}[x^{n+1}(k)] & & \mathbb{E}[x_T^{2n}] - \mathbb{E}[x^{2n}(k)] \end{bmatrix}
\end{aligned} \tag{14}
$$

$$
[\mathcal{X}_T - \mathcal{X}(k_0)]_H = \begin{bmatrix} 1 & \mathbb{E}[x_T] - \prod_{i=0}^{k-1} a(i)\mathbb{E}[x(0)] & \cdots & \mathbb{E}[x_T^n] - \left(\prod_{i=0}^{k-1} a(i)\right)^n \mathbb{E}[x^n(0)] \\ \mathbb{E}[x_T] - \prod_{i=0}^{k-1} a(i)\mathbb{E}[x(0)] & \mathbb{E}[x_T^2] - \left(\prod_{i=0}^{k-1} a(i)\right)^2 \mathbb{E}[x^2(0)] & \cdots & \mathbb{E}[x_T^{n+1}] - \left(\prod_{i=0}^{k-1} a(i)\right)^{n+1} \mathbb{E}[x^{n+1}(0)] \\ \vdots & \vdots & \ddots & \\ \mathbb{E}[x_T^n] - \left(\prod_{i=0}^{k-1} a(i)\right)^n \mathbb{E}[x^n(0)] & \mathbb{E}[x_T^{n+1}] - \left(\prod_{i=0}^{k-1} a(i)\right)^{n+1} \mathbb{E}[x^{n+1}(0)] & & \mathbb{E}[x_T^{2n}] - \left(\prod_{i=0}^{k-1} a(i)\right)^{2n} \mathbb{E}[x^{2n}(0)] \end{bmatrix} \tag{15}
$$

denote the linear integral operator Γ as

$$
\Gamma : p(u) \mapsto \Sigma = \int_{\mathbb{R}} G(u)p(u)G^T(u)du
$$

where $p(u)$ is defined on the space $\mathcal{P}_{2n}$. According to the type of the control problem, $\mathcal{C}$ is either $\mathbb{R}$ (unconstrained control), or a compact interval on $\mathbb{R}$ (constrained control). Here

$$
G(u) = \begin{bmatrix} 1 & u & \cdots & u^{n-1} & u^n \end{bmatrix}^T
$$

and

$$
\Sigma = \begin{bmatrix} 1 & \mathbb{E}[u] & \cdots & \mathbb{E}[u^n] \\ \mathbb{E}[u] & \mathbb{E}[u^2] & \cdots & \mathbb{E}[u^{n+1}] \\ \vdots & \vdots & \ddots & \\ \mathbb{E}[u^n] & \mathbb{E}[u^{n+1}] & & \mathbb{E}[u^{2n}] \end{bmatrix}
$$

where $\mathbb{E}[u^i], i = 1, \cdots, 2n$ are the elements of the designed control $\mathcal{U}$. Moreover, since $\mathcal{P}_{2n}$ is convex, then so is $\text{range}(\Gamma) = \Gamma\mathcal{P}_{2n}$.

We let

$$
\mathcal{L}_+ := \left\{ \Lambda \in \text{range}(\Gamma) \mid G(u)^T \Lambda G(u) > 0, x \in \mathbb{R} \right\}.
$$

Given any $r \in \mathcal{P}$ and any $\Sigma \succ 0$, there is a unique $\hat{p} \in \mathcal{P}_{2n}$ that minimizes (16) subject to $\Gamma(\hat{p}) = \Sigma$, namely

$$
\hat{p} = \frac{r}{G^T \hat{\Lambda} G} \tag{17}
$$

where $\hat{\Lambda}$ is the unique solution to the problem of minimizing

$$
\mathbb{J}_r(\Lambda) := \text{tr}(\Lambda\Sigma) - \int_{\mathbb{R}} r(u) \log \left[G(u)^T \Lambda G(u) \right] du \tag{18}
$$

over all $\Lambda \in \mathcal{L}_+$. Here $\text{tr}(M)$ denotes the trace of the matrix M.

Then the density estimation is formulated as a convex optimization problem. Unlike other methods of moments, the power moments of our proposed density estimate are exactly identical to those specified, which makes it a satisfactory approach for realization of the control inputs. Since the prior density $r(u)$ and the density estimate $\hat{p}(u)$ are both supported on $\mathbb{R}$, $r(u)$ can be chosen as a Gaussian distribution (or a Cauchy distribution if $\hat{p}(u)$ is assumed to be heavy-tailed).

We note that for the Hausdorff moment problem, where the control u is supported on a compact subset of $\mathbb{R}$, the proofs and results for the Hamburger moment problem are also valid by substituting the domain $\mathbb{R}$ by a compact interval on the real line. By doing this, we will obtain a density estimate $\hat{p}$ in the form of (17). The prior $r(u)$ can then be chosen as a truncated Gaussian or a truncated Laplacian.

We conclude the algorithm for density steering of the colossal group of agents in this section as in Algorithm 1.

2.4 Error analysis of the terminal density function

Since we used the truncated power moments of the initial and terminal density functions for steering, there may exist an error between the terminal density and the desired one. In this part, we will propose an upper bound

Algorithm 1 Density steering of a colossal group of agents.

Input: The maximal time step K; the parameter of the system $a(k)$ for $k = 0, \cdots, K-1$; the initial system density $q_0(x)$; the specified terminal density $q_T(x)$.

Output: The controls $u(k)$, $k = 0, \cdots, K-1$.

1: $k \Leftarrow 0$
2: **while** $k < K$ and $e(k) \notin \mathbb{V}_{++}^{2n}$ **do**
3: Calculate $\mathcal{X}(k)$ by (4) if $k > 0$ or by (5) if $k = 0$
4: Calculate $e(k)$ by (12)
5: **if** $e(k) \in \mathbb{V}_{++}^{2n}$ **then**
6: Calculate the states of the moment system $\mathcal{X}(i)$ for $i = k+1, \cdots, K-1$ by (13) with $\omega_k = \cdots = \omega_{K-1}$
7: Calculate the controls of the moment system $\mathcal{U}(i)$ for $i = k, \cdots, K-1$ by (4)
8: **if** $\exists i, \mathcal{U}(i) \notin \mathbb{V}_{++}^{2n}$ **then**
9: Back to Step 6, adjust $\omega_k, \cdots, \omega_{K-1}$
10: **end if**
11: Optimize the cost function (18) and obtain the analytic estimates of the densities $\hat{p}_i(u)$ for $i = k, \cdots, K-1$
12: **else**
13: $u(k) = 0$
14: **end if**
15: Calculate the power moments of the system state $x(k+1)$, i.e., $\mathcal{X}(k+1)$
16: $k \Leftarrow k+1$
17: **end while**

of error of the terminal density, as to characterize the maximal difference between the terminal density by our proposed algorithm and the specified one.

In [22], we proposed an error upper bound for the Hamburger moment problem in the sense of total variation distance, which is a measure widely used in the moment problem [19, 20].

The total variation distance between the terminal density $q_K(x)$ and the desired terminal density $q_T(x)$ is defined as follows:

$$V(q_K, q_T) = \sup_x \left| \int_{(-\infty, x]} (q_K - q_T) dx \right| = \sup_x |F_{q_K} - F_{q_T}|$$

where F_{q_K} and F_{q_T} are the two distribution functions of q_K and q_T.

Shannon-entropy is used to calculate the upper bound of the total variation distance in [20]. The Shannon-entropy [17] is defined as

$$H[q] = -\int_{\mathcal{C}} q(x) \log q(x) dx.$$

We first introduce the Shannon-entropy maximizing distribution $F_{\breve{q}_K}$, of which the moments are the power moments of q_K. It has the following density function [12],

$$\breve{q}_K(x) = \exp\left(-\sum_{i=0}^{2n} \lambda_i x^i\right)$$

where $\lambda_0, \cdots, \lambda_{2n}$ are determined by the following constraints,

$$\int_{\mathcal{C}} x^k \exp\left(-\sum_{i=0}^{2n} \lambda_i x^i\right) dx = \int_{\mathcal{C}} x^k q_T(x) dx$$

for $k = 0, 1, \cdots, 2n$. By referring to [20], the KL distance between the true density and the Shannon-entropy maximizing density can be written as

$$\begin{aligned} KL\left(q_T \| \breve{q}_K\right) &= \int_{\mathcal{C}} q_T(x) \log \frac{q_T(x)}{\breve{q}_K(x)} dx \\ &= -H\left[q_T\right] + \sum_{i=0}^{2n} \lambda_i \int_{\mathcal{C}} x^k q_T(x) dx \\ &= H\left[\breve{q}_K\right] - H\left[q_T\right]. \end{aligned}$$

Similarly, we can obtain $KL\left(q_T \| q_K\right) = H\left[q_K\right] - H\left[q_T\right]$.

By [13, 20], we obtain

$$\begin{aligned} V\left(\breve{q}_K, q_K\right) &\leq 3\left[-1 + \left\{1 + \frac{4}{9} KL\left(q_K \| \breve{q}_K\right)\right\}^{1/2}\right]^{1/2} \\ &= 3\left[-1 + \left\{1 + \frac{4}{9}\left(H\left[\breve{q}_K\right] - H\left[q_K\right]\right)\right\}^{1/2}\right]^{1/2} \end{aligned}$$

and

$$V\left(\breve{q}_K, q_T\right) \leq 3\left[-1 + \left\{1 + \frac{4}{9}\left(H\left[\breve{q}_K\right] - H\left[q_T\right]\right)\right\}^{1/2}\right]^{1/2}$$

Then the error upper bound of the terminal density can be written as

$$\begin{aligned} & V\left(q_K, q_T\right) \\ = & \sup_x \left|F_{q_K}(x) - F_{q_T}(x)\right| \\ \leq & \sup_x \left(\left|F_{q_K}(x) - F_{\breve{q}_K}(x)\right| + \left|F_{\breve{q}_K}(x) - F_{q_T}(x)\right|\right) \\ \leq & \sup_x \left|F_{q_K}(x) - F_{\breve{q}_K}(x)\right| + \sup_x \left|F_{\breve{q}_K}(x) - F_{q_T}(x)\right| \\ \leq & 3\left[-1 + \left\{1 + \frac{4}{9}\left(H\left[\breve{q}_K\right] - H\left[q_K\right]\right)\right\}^{1/2}\right]^{1/2} \\ & + 3\left[-1 + \left\{1 + \frac{4}{9}\left(H\left[\breve{q}_K\right] - H\left[q_T\right]\right)\right\}^{1/2}\right]^{1/2} \end{aligned}$$

3 Steer the group as an occupation measure

In the previous section, we proposed an algorithm steering a colossal group of agents characterized as a probability density function to a terminal one. However, the characterization of agents as a density function is an approximation when the number of agents tends to infinity. For applying our proposed algorithm to real applications, we need to put forward control inputs for all finite number of agents. Hence in this section, we use the occupation measure to characterize the group of agents and propose a control scheme to steer an initial occupation measure to a terminal one.

We first define the occupation measure of the agents at time step k as in [23]

$$dq_k(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - x_i(k)\right)dx,$$

then the state of the group of agents can be written as

$$x(k) = \frac{1}{N}\sum_{i=1}^{N} x_i(k)\delta\left(x(k) - x_i(k)\right) \tag{19}$$

Since the controls are applied to the agents, the control on the group of agents is defined as

$$u(k) = \frac{1}{N}\sum_{i=1}^{N} u_i(k)\delta\left(x(k) - x_i(k)\right). \tag{20}$$

The power moments of the occupation measures can then be written as

$$\mathbb{E}\left[x^l(k)\right] = \frac{1}{N}\sum_{i=1}^{N} x_i^l(k), \tag{21}$$

and

$$\mathbb{E}\left[u^l(k)\right] = \frac{1}{N}\sum_{i=1}^{N} u_i^l(k). \tag{22}$$

Now we define the problem we will treat in this section.
Problem 3.1 (Occupation measure steering problem by power moments)**.** The dynamics of the moment system is

$$\mathcal{X}(k+1) = \mathcal{A}(\mathcal{U}(k))\mathcal{X}(k) + \mathcal{U}(k).$$

where $\mathcal{X}(k), \mathcal{U}(k)$ are defined as (21),(22). Given an **arbitrary** initial state $x(0)$, determine the control sequence $(u_i(0), \cdots, u_i(K-1))$ for each agent i such that the power moments of the terminal occupation measure are identical to those of an **arbitrarily** specified state, i.e.,

$$\mathbb{E}\left[x_T^l\right] = \int_{\mathbb{R}} x^l dq_T(x) = \frac{1}{N}\sum_{i=1}^{N} x^l(K) \tag{23}$$

for $l = 1, \cdots, 2n$, where q_T is the specified terminal occupation measure.
Remark. To our best knowledge, the previous results mainly focused on steering the targets by characterizing them as analytic density functions. However, there has not been a solution for a large but finite number of agents, which we will treat in this section. By our definition of the problem, the outputs are the control sequences for each agent. The control inputs can then be directly applied to each agent of the colossal group.

The main difference of the occupation measure steering problem from the density steering one lies in determining the control inputs for each agent. We naturally consider designing feedback control laws for the agents. It might be feasible with limited number of agents, however it is quite expensive and problematic with quite a large number of agents as in our problem. Moreover, as to implement the feedback control, we need to obtain the states of each agent and calculate the control inputs based on all of them, which requires us to install numerous sensors on the agents and we have to treat the issues of the communications between the agents. However, by our proposed algorithm using moments, it is not necessary to collect all states of the agents and to transmit them back to the center to calculate the control inputs at all time steps.

Since N is large, we consider first estimating the occupation measure of $u(k)$ as a continuous function $\hat{p}(u)$. Then we draw N i.i.d samples from it and assign them to u_i, i.e., $u_i \sim \hat{p}(u), i \in \mathbb{N}_0, i \leq N$. By the strong law of large numbers, we note that

$$\frac{1}{N}\sum_{i=1}^{N} u_i^l(k) \xrightarrow{a.s.} \int_{\mathbb{R}} u^l \hat{p}_k(u)du, \text{ with } N \to +\infty \tag{24}$$

which means that the power moments of $u(k)$ converge almost surely to the power moments $\mathcal{U}(k)$ of the designed controls. Moreover, the sampling strategy ensures that the system state $x(k)$ and the control input $u(k)$ are independent from each other, hence (11) is satisfied. Then the problem comes to putting forward a sampling strategy. We consider using the acceptance-rejection sampling [4] strategy for this task.

The idea of acceptance-rejection sampling is that even it is not feasible for us to directly sample from $\hat{\rho}$, there exists another density $\tilde{\rho}$, from which it is easy to sample from. The task can be reduced to sampling from $\tilde{\rho}$ directly and then rejecting the samples in a strategic way to make the remaining samples seemingly drawn from $\hat{\rho}$. We call the density $\tilde{\rho}$ as the "candidate density" and $\hat{\rho}$ as the "target density".

We assume that

$$c = \sup_{x \in \operatorname{supp}(\hat{\rho})} \frac{\hat{\rho}}{\tilde{\rho}} < \infty \tag{25}$$

and that we can calculate c. In our paper, the support of both $\hat{\rho}$ and $\tilde{\rho}$ is $\mathcal{C}$. As to satisfy (25), the candidate density need to have heavier tails than the target density. Then we give the sampling algorithm in the following algorithm.

Algorithm 2 Sample $u_i(k)$ from the realized control.

Input: The number of agents $N \in \mathbb{N}_0$; the realized control $\hat{\rho}_k(u)$; the candidate density $\tilde{\rho}_k(u)$
Output: The controls $u_i(k), i = 1, \cdots, N$

1: $i \Leftarrow 1$
2: **while** $i \leq N$ **do**
3:     Sample an r_i from a uniform distribution $U[0, 1]$
4:     Sample a $u \in \operatorname{supp}(\hat{\rho})$ from the candidate density $\tilde{\rho}$
5:     **if** $u \leq \frac{\hat{\rho}}{c\tilde{\rho}}$ **then**
6:         $u_i(k) \Leftarrow u$
7:     **else**
8:         back to step 3
9:     **end if**
10:     $i \Leftarrow i + 1$
11: **end while**

We note that in real applications, the controls $u_i(k), i = 1, \cdots, N$ are sometimes bounded. In the previous results, it has not been treated by the feedback control laws given the domain of the system state being the whole $\mathbb{R}$. By the proposed Algorithm 2, we simply need to choose $\hat{\rho}$ and the corresponding $\tilde{\rho}$ to be truncated densities supported on specified bounded intervals.

We now adopt the proposed acceptance-rejection sampling strategy to update Algorithm 1 as to treat the occupation measure steering problem, which is given in Algorithm 3.

Algorithm 3 Occupation measure steering of a colossal group of agents.

Input: The number of agents $N \in \mathbb{N}_0$; the maximal time step K; the parameter of the system $a(k)$ for $k = 0, \cdots, K-1$; the initial occupation measure $dq_0(x)$; the specified terminal occupation measure $dq_T(x)$.
Output: The control inputs for the i_{th} target $u_i(k)$, $k = 0, \cdots, K-1$, $i = 1, \cdots, N$.

1: $k \Leftarrow 0$
2: **while** $k < K$ and $e(k) \notin \mathbb{V}_{++}^{2n}$ **do**
3:     Calculate $\mathcal{X}(k)$ by (4) if $k > 0$ or by (5) if $k = 0$
4:     Calculate $e(k)$ by (12)
5:     **if** $e(k) \in \mathbb{V}_{++}^{2n}$ **then**
6:         Calculate the states of the moment system $\mathcal{X}(i)$ for $i = k+1, \cdots, K-1$ by (13) with $\omega_k = \cdots = \omega_{K-1}$
7:         Calculate the controls of the moment system $\mathcal{U}(i)$ for $i = k, \cdots, K-1$ by (4)
8:         **if** $\exists i, \mathcal{U}(i) \notin \mathbb{V}_{++}^{2n}$ **then**
9:             Back to Step 6, adjust $\omega_k, \cdots, \omega_{K-1}$
10:         **end if**
11:         Optimize the cost function (18) and obtain the analytic estimates of the densities $\hat{p}_i(u)$ for $i = k, \cdots, K-1$
12:         Do Algorithm 2 and obtain the control inputs $u_i(j)$ of all agents at time step $j = k, \cdots, K-1$
13:     **else**
14:         $u_i(k) = 0, i = 1, \cdots, N$
15:     **end if**
16:     Calculate the power moments of the system state $x(k+1)$, i.e., $\mathcal{X}(k+1)$
17:     $k \Leftarrow k + 1$
18: **end while**

4 Numerical examples

In the previous sections, we proposed algorithms for steering a colossal group of agents, either characterized as a probability density function or aN occupation measure. In this section, we perform numerical simulations on different types of distributions, supported on $\mathbb{R}$ or a compact subset of $\mathbb{R}$, with multiple modes or a single mode, to validate our proposed algorithms.

We first simulate the steering of a colossal group of agents as a probability density function. We begin with unconstrained control, i.e., the control inputs $u(k)$ are not constrained.

4.1 Unconstrained density steering of a colossal group of agents

In Example 1, we simulate a typical problem which is to steer a Gaussian density to another in 4 steps. The initial Gaussian density is chosen as

$$q_0(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}}, \tag{26}$$

and the terminal density is specified as

$$q_T(x) = \frac{1}{\sqrt{2\pi} \cdot 2} e^{\frac{(x-1)^2}{2 \cdot 2^2}}. \tag{27}$$

The system parameters $a(k), k = 0, \cdots, 3$ are i.i.d. samples drawn from the uniform distribution $U[0.5, 0.7]$. The states of the moment system, i.e., $\mathcal{X}(k)$ for $k = 0, 1, 2, 3$ are given in Figure 1. The controls of the moment system, i.e., $\mathcal{U}(k)$ for $k = 0, 1, 2, 3$ are given in Figure 2. We note that by our proposed algorithm, $\mathcal{X}(k), \mathcal{U}(k) \in \mathbb{V}_{++}^{2n}$, which makes it feasible for us to realize the controls. The realized control inputs are given in Figure 3. The transition of the control inputs is smooth, which is satisfactory.

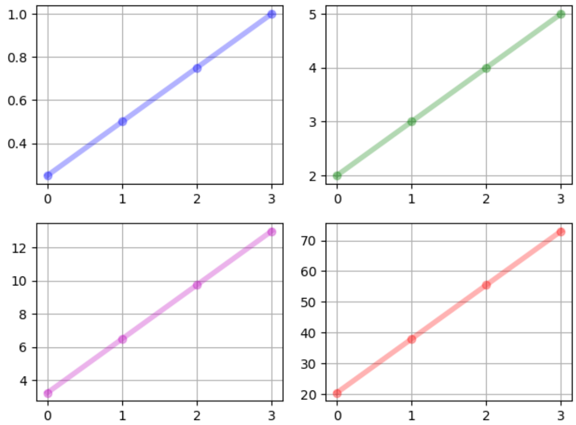

Fig. 1. $\mathcal{X}(k)$ at time steps $k = 0, 1, 2, 3$. The upper left figure shows $\mathbb{E}[x(k)]$. The upper right one shows $\mathbb{E}\left[x^2(k)\right]$. The lower left one shows $\mathbb{E}\left[x^3(k)\right]$ and the lower right one shows $\mathbb{E}\left[x^4(k)\right]$.

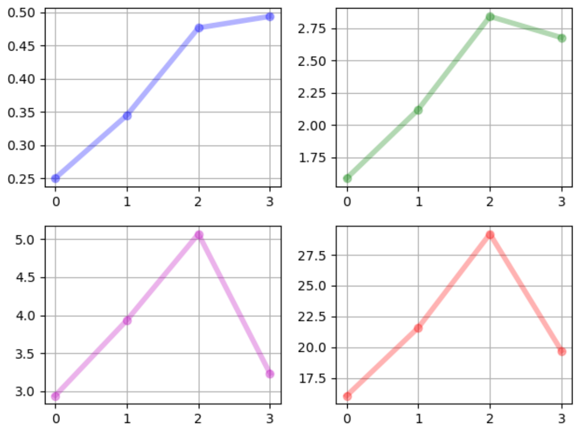

Fig. 2. $\mathcal{U}(k)$ at time steps $k = 0, 1, 2, 3$. The upper left figure shows $\mathbb{E}[u(k)]$. The upper right one shows $\mathbb{E}\left[u^2(k)\right]$. The lower left one shows $\mathbb{E}\left[u^3(k)\right]$ and the lower right one shows $\mathbb{E}\left[u^4(k)\right]$.

In Example 2, we simulate a steering problem in 3 steps where the initial density function is a Gaussian and the terminal density function is a mixture of Gaussians with two modes. The initial one is chosen as

$$q_0(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}}.$$

and the terminal one is specified as

$$q_T(x) = \frac{0.4}{\sqrt{2\pi}} e^{\frac{(x-1)^2}{2}} + \frac{0.6}{\sqrt{2\pi}} e^{\frac{(x+1)^2}{2}}. \tag{28}$$

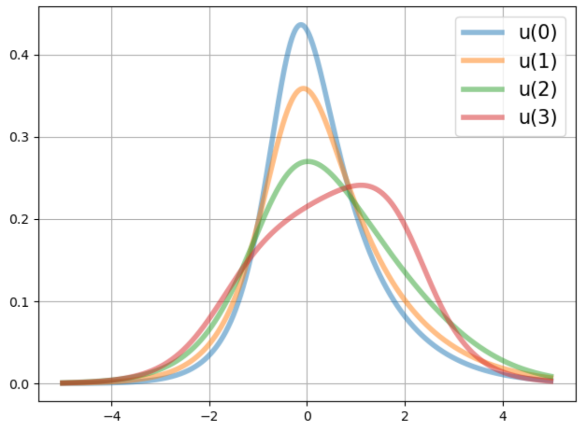

Fig. 3. Realized control inputs $u(k)$ by $\mathcal{U}(k)$ for $k = 0, 1, 2, 3$, which are obtained by our proposed control scheme.

The system parameters $a(k), k = 0, \cdots, 3$ are i.i.d. samples drawn from the uniform distribution $U[0.5, 0.7]$. The states of the moment system, i.e., $\mathcal{X}(k)$ for $k = 0, 1, 2, 3$ are given in Figure 4. The controls of the moment system, i.e., $\mathcal{U}(k)$ for $k = 0, 1, 2, 3$ are given in Figure 5. The realized controls in Figure 6 also show that the transition of the control inputs is smooth, even the specified terminal density has two modes.

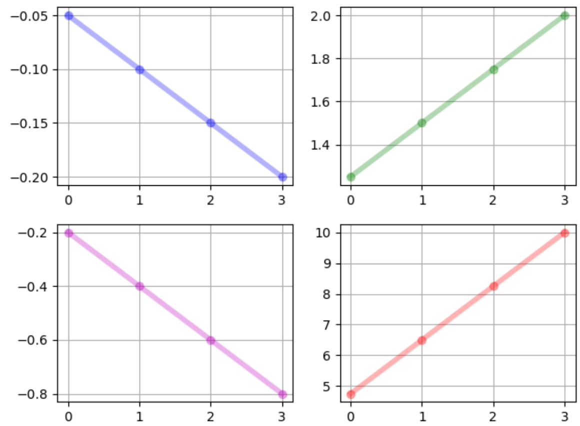

Fig. 4. $\mathcal{X}(k)$ at time steps $k = 0, 1, 2, 3$.

In Example 3, we will simulate a steering problem in 4 steps which has not been treated in the previous papers. The initial density is chosen as a Gaussian distribution

$$q_0(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}}.$$

However the terminal density function is specified as a multi-modal density which is a mixture of two Laplacians

$$q_T(x) = \frac{0.7}{2} e^{|x-1|} + \frac{0.3}{2} e^{-|x+3|}.$$

The system parameter $a(k), k = 0, \cdots, 3$ are i.i.d. sam-

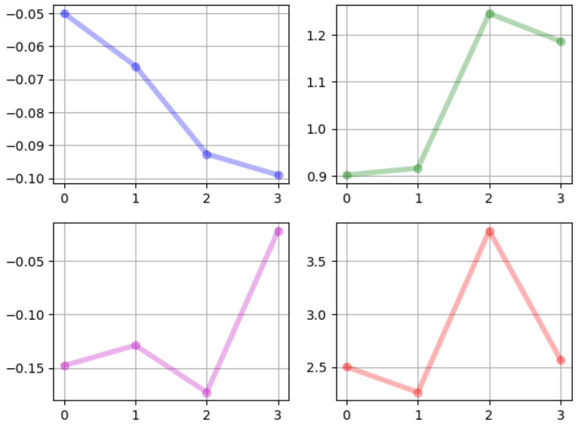

Fig. 5. $\mathcal{U}(k)$ at time steps $k = 0, 1, 2, 3$.

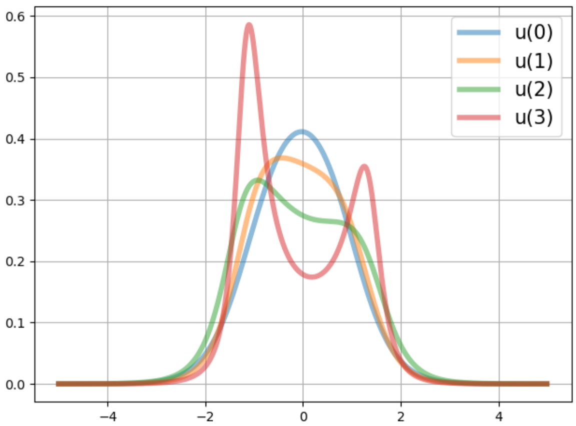

Fig. 6. Realized control inputs $u(k)$.

ples drawn from the uniform distribution $U[0.5, 0.7]$. The results are given in Figure 7, 8 and 9. In this example, the two modes are not close to each other as in Example 2. However the realized control inputs are still smooth, which validates the performance of our algorithm in steering a group to separate groups relatively far from each other.

4.2 Constrained density steering of a colossal group of agents

In some scenarios, there are boundary conditions on the control inputs. A common constraint is that they are bounded by a compact interval $[a, b]$ on $\mathbb{R}$. To the best of our knowledge, there has not been a feedback law for a colossal group of agents with bounded controls. With our proposed algorithm, we are able to treat the density steering problem with bounded control inputs.

In Example 4, we will simulate a problem which is sort of problematic however important in real practice. The initial and the terminal densities are both chosen as a mixture of two Gaussian densities, which are

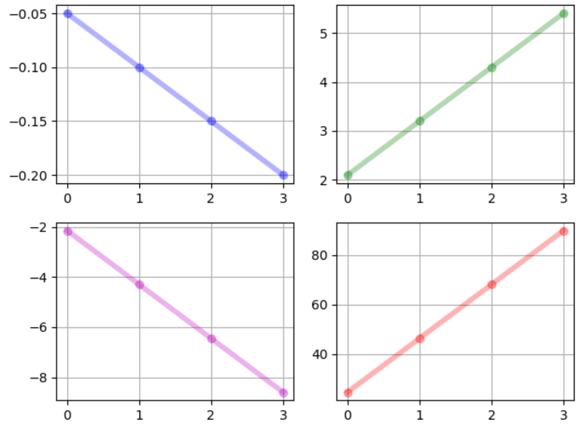

Fig. 7. $\mathcal{X}(k)$ at time steps $k = 0, 1, 2, 3$.

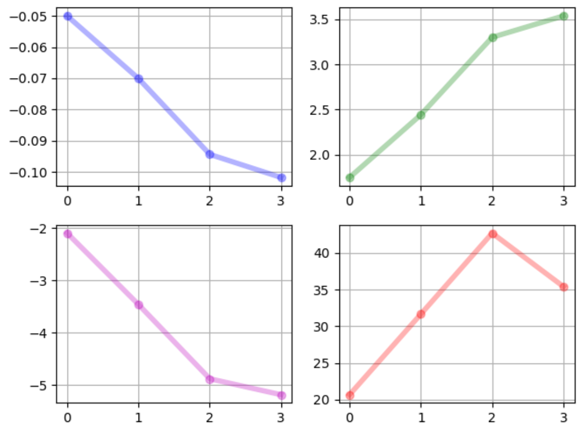

Fig. 8. $\mathcal{U}(k)$ at time steps $k = 0, 1, 2, 3$.

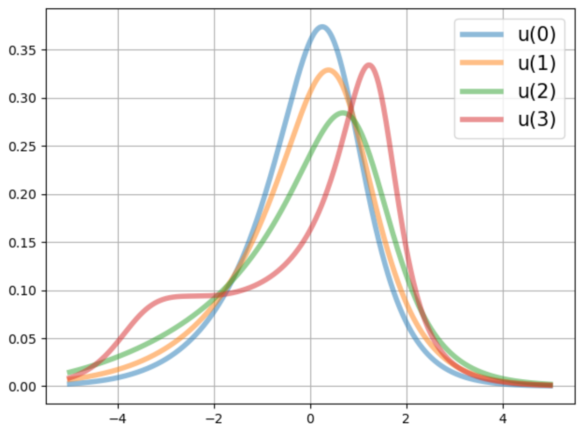

Fig. 9. Realized control inputs $u(k)$.

$$q_0(x) = \frac{0.5}{\sqrt{2\pi}\cdot 2} e^{\frac{x^2}{2\cdot 2^2}} + \frac{0.5}{\sqrt{2\pi}\cdot 2} e^{\frac{(x+1)^2}{2\cdot 2^2}},$$

and

$$q_T(x) = \frac{0.4}{\sqrt{2\pi}} e^{\frac{(x-1)^2}{2}} + \frac{0.6}{\sqrt{2\pi}} e^{\frac{(x+1)^2}{2}}.$$

The density steering is performed in 5 steps. The control inputs are bounded on the interval $\mathcal{C} = [-2, 2]$. The system parameter $a(k), k = 0, \cdots, 4$ are i.i.d. samples drawn from the uniform distribution $U[0.5, 0.7]$.

The results are given in Figure 10, 11 and 12. We note that at time step $k = 0$, $\mathfrak{X}_T - \mathfrak{X}(0) \notin \mathbb{V}^{2n}_{++}$. Then by the proposed algorithm, we don't apply control inputs to the agents, which is shown in Figure 11. At time step $k = 1$, we have $\mathfrak{X}_T - \mathfrak{X}(1) \notin \mathbb{V}^{2n}_{++}$. Hence we start applying controls starting from this step. It takes 3 steps for us to steer the density to the specified one.

The goal of the steering in this problem is to steer two distinct groups of agents to specified terminal two groups. The boundary of control inputs and the multiple modality of both initial and terminal densities make the problem a challenging one. By our proposed algorithm, we give a solution to this problem and the realized control inputs are still smooth, which is a satisfactory performance.

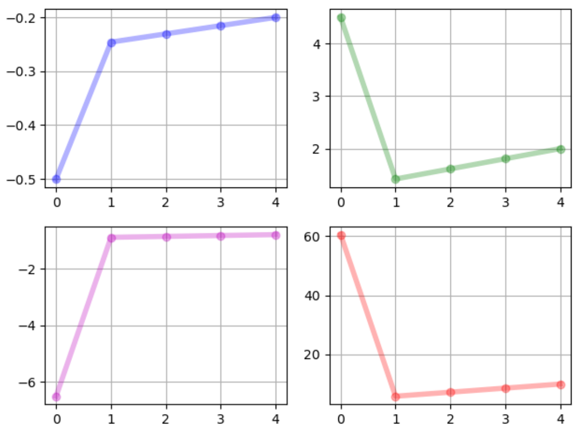

Fig. 10. $\mathfrak{X}(k)$ at time steps $k = 0, 1, 2, 3, 4$.

However, we may notice that we have not provided the terminal densities by the effects of the proposed control inputs, even the control inputs $u(k), k = 0, 1, 2, 3$ we obtained have an analytic form of function. Because of (3), we are not able to obtain an analytic $u(K)$. Luckily, our proposed algorithm by using the moments is able to treat the occupation measure steering problem. The analytic function $u(k)$ at each time step is realized as control inputs $u_i(k)$ of each agent i. We can therefore obtain a terminal occupation measure to compare to the specified one. In the following part of this section, we simulate the occupation measure steering examples.

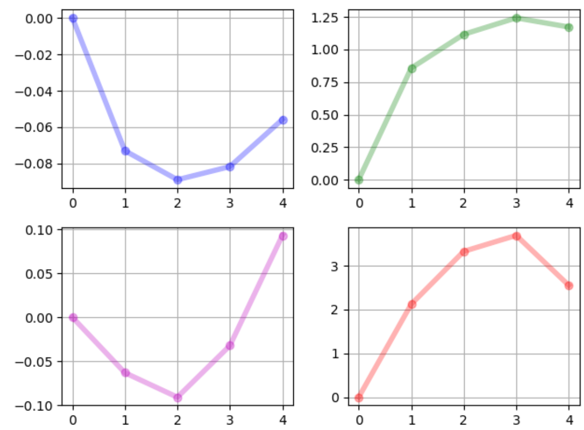

Fig. 11. $\mathcal{U}(k)$ at time steps $k = 0, 1, 2, 3, 4$.

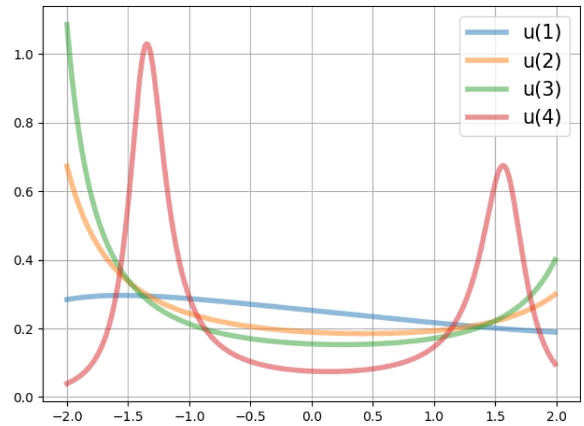

Fig. 12. Realized control inputs $u(k)$ at time steps $k = 1, 2, 3, 4$. The agents are uncontrolled at $k = 0$, i.e., $u(0) = 0$.

4.3 Occupation measure steering of a colossal group of agents

We simulate examples on occupation measure steering in this part of section. In Example 5, we steer 1000 agents to a specified occupation measure. The initial states of each agent x_i is drawn i.i.d. from the Gaussian distribution (26). The specified terminal occupation measure consists of 1000 i.i.d. samples drawn from the terminal distribution (27). The system parameter $a(k), k = 0, \cdots, 3$ are i.i.d. samples drawn from the uniform distribution $U[0.5, 0.7]$.

The histograms of $u_i(k)$ at time step k for each agent i are given in Figure 13. They are 1000 i.i.d samples drawn from the realizations of $\mathcal{U}(k)$ in Figure 3. Figure 14 gives the histogram of the terminal occupation measure of the states of the 1000 agents. The power moments of order 1 to 4 of the terminal occupation measure by our proposed algorithm are 1.21, 5.41, 14.44, 78.77 respectively. We note that it is quite close to the desired terminal dis-

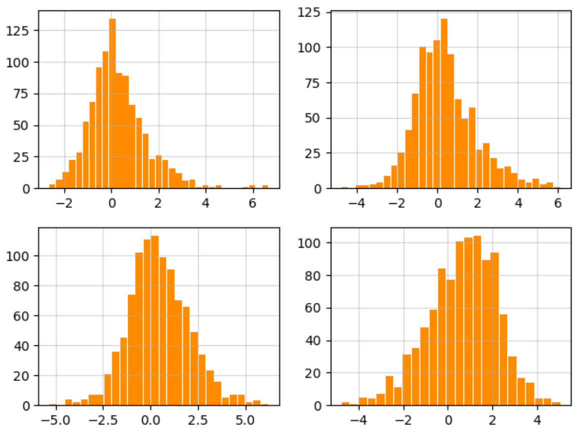

Fig. 13. The histograms of $u_i(k)$ at time step k for each agent i. The upper left and right figures are $u_i(0)$ and $u_i(1), i = 1, \cdots, 1000$ respectively. The lower left and right figures are $u_i(2)$ and $u_i(3)$ respectively.

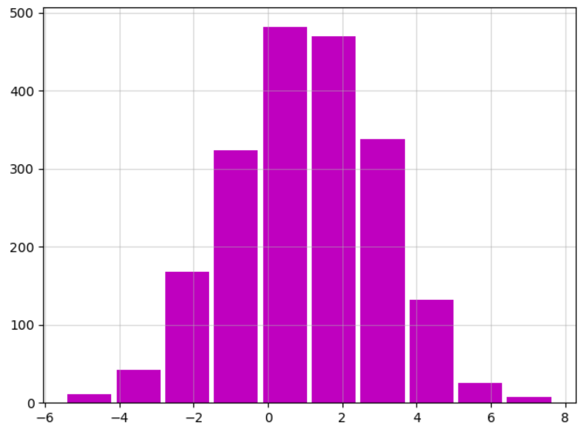

Fig. 14. The histogram of the terminal occupation measure of $x(K)$ at time step $K = 4$. It is close to the specified terminal distribution (27).

tribution, of which the power moments of order 1 to 4 are $1, 5, 13, 73$ respectively. It validates our algorithm of steering the occupation measure.

In Example 6, we steer 1000 agents to a specified occupation measure. The initial states of each agent x_i is drawn i.i.d. from the Gaussian distribution (26). The specified terminal occupation measure consists of 1000 i.i.d. samples drawn from the terminal distribution (28), of which the density is a mixture of two Gaussian densities. The system parameter $a(k), k = 0, \cdots, 3$ are i.i.d. samples drawn from the uniform distribution $U[0.5, 0.7]$.

The histograms of $u_i(k)$ at time step k for each agent i are given in Figure 15. They are 1000 i.i.d samples drawn from the realizations of $\mathcal{U}(k)$ in Figure 6. Figure 16 gives the histogram of the terminal occupation measure of the states of the 1000 agents. We note that the power moments of order 1 to 4 of (28) are $-0.2, 2, -0.8, 10$ respectively. The power moments of order 1 to 4 of the terminal occupation measure by our proposed algorithm are $-0.19, 2.07, -0.84, 10.39$ respectively, which are quite close to the specified ones. Moreover, there are two distinguishable modes in the histogram of $u(K)$ by our proposed algorithm as shown in Figure 16, which shows that the algorithm maintains the number of modes of the specified terminal distribution (28).

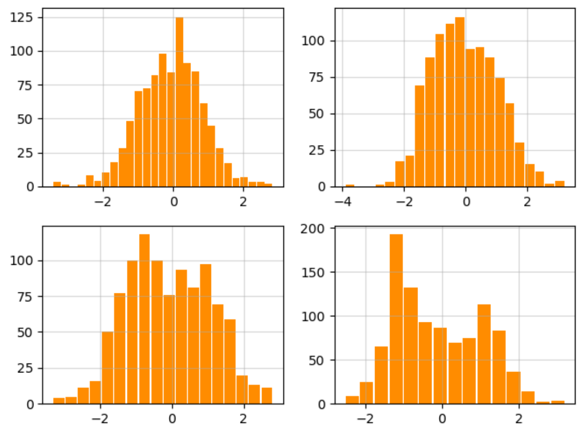

Fig. 15. The histograms of $u_i(k)$ at time step k for each agent i. The upper left and right figures are $u_i(0)$ and $u_i(1), i = 1, \cdots, 1000$ respectively. The lower left and right figures are $u_i(2)$ and $u_i(3)$ respectively.

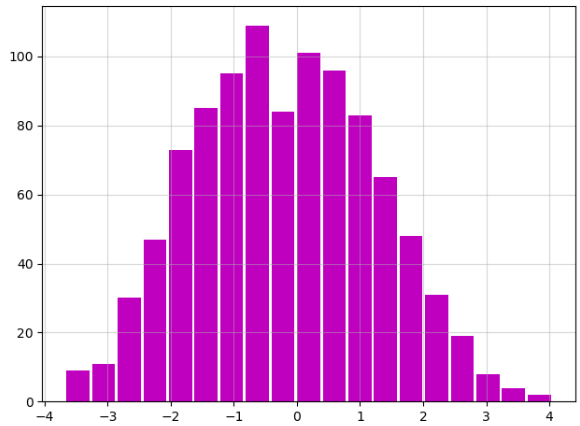

Fig. 16. The histogram of the terminal occupation measure of $x(K)$ at time step $K = 4$. It is close to the specified terminal distribution (28).

5 A concluding remark

In this paper, we propose to use moments for the problem of steering a colossal group of agents. The colossal group of agents are characterized as probability density functions and occupation measures. We first treat the density steering problem. Without assuming the initial and terminal density to fall within specific function classes, the original problem is infinite-dimensional and intractable. As to treat this problem, we propose a moment system representation of the original system, and reduce the original problem to the control of the moment system. Different from the conventional control problems, the elements of the control inputs are in the system matrix, and we have to ensure the Hankel matrix of the control inputs at each time step to be positive definite. Since it is not treatable by the existing control schemes including the commonly used optimal control, we propose an empirical control scheme to treat this problem. By doing this, it is feasible for us to realize the control inputs for the original system. Since the problem is infinite-dimensional, where error of the terminal density from the specified one is inevitable, we propose an error upper bound of the terminal density by using our proposed algorithm. Our algorithm of steering an arbitrary initial density to another arbitrary one in limited steps for the discrete-time system, without assuming their function classes, is the first one in the literature.

Based on the proposed density-steering algorithm, we put forward an algorithm steering an arbitrary occupation measure representing the colossal group of agents to another arbitrary one. We use acceptance-rejection sampling to draw i.i.d. samples from the realized control inputs $u(k)$ and assign them to each agents. By doing this, the control inputs are independent of the current states of each agent. Our proposed algorithm is again the first one in the literature which is able to steer an arbitrary occupation measure to another one of which the power moments are as specified within limited time steps.

This paper treats the group steering problem of the first-order time-variant linear system. However, the problem is much more complicated for multi-order and multivariate systems. For first-order systems, the control inputs $u(k)$'s are one-dimensional density functions. Hence the positive definiteness of the Hankel matrix of each $\mathcal{U}(k)$ is the necessary and sufficient condition of the existence of $u(k)$ [16]. However for multi-dimensional densities, it is no longer valid. To derive control inputs $\mathcal{U}(k)$'s for the moment system then to realize them will be a challenging problem. Results in subjects e.g. multi-dimensional moment problem and real algebraic geometry shall be used to treat the group steering problem for those systems.

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Guangyu Wu received the B.E. degree from Northwestern Polytechnical University, Xi'an, China, in 2013, and two M.S. degrees, one in control science and engineering from Shanghai Jiao Tong University, Shanghai, China, in 2016, and the other in electrical engineering from the University of Notre Dame, South Bend, USA, in 2018.

He is currently pursuing the Ph.D. degree at Shanghai Jiao Tong University. His research interests are the moment problem and its applications to stochastic filtering, stochastic control and statistics.

Anders Lindquist received the Ph.D. degree in optimization and systems theory from the Royal Institute of Technology, Stockholm, Sweden, in 1972, and an honorary doctorate (Doctor Scientiarum Honoris Causa) from Technion (Israel Institute of Technology) in 2010.

He is currently a Zhiyuan Chair Professor at Shanghai Jiao Tong University, China, and Professor Emeritus at the Royal Institute of Technology (KTH), Stockholm, Sweden. Before that he had a full academic career in the United States, after which he was appointed to the Chair of Optimization and Systems at KTH. Dr. Lindquist is a Member of the Royal Swedish Academy of Engineering Sciences, a Foreign Member of the Chinese Academy of Sciences, a Foreign Member of the Russian Academy of Natural Sciences, a Member of Academia Europaea (Academy of Europe), an Honorary Member the Hungarian Operations Research Society, a Fellow of SIAM, and a Fellow of IFAC. He received the 2003 George S. Axelby Outstanding Paper Award, the 2009 Reid Prize in Mathematics from SIAM, and the 2020 IEEE Control Systems Award, the IEEE field award in Systems and Control.