Final Report in Optimization for Control and Signal Processing

Anders Blomqvist

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1 Introduction

In automatic control, pole placement (closed-loop poles) is a design tool with which one can achieve desired time responses and closed-loop damping, by keeping the poles away from the imaginary axis, and the origin in particular. By restricting the poles from getting too far from the origin, too fast controller dynamics can be avoided. Thus, pole placement is a useful tool in controller design. In [3] an algorithm for including pole placement constraints in H_∞-synthesis is introduced. The formulation produces a convex optimization problem with Linear Matrix Inequalities (LMIs). This report presents the LMI based design, and compares it to other methods, by comparing the design for a couple of problems from the literature.

1.1 Standard Problem of Robust Control

All design problems for multivariable linear systems that we are interested in can be restated as the "Standard Problem" of [4]. The system is described in Figure 1. Generally, all signals are vector-valued. The closed-loop transfer function of the system is:

$$\begin{aligned} z &= \left(G_{11} + G_{12}K\left(I - G_{22}K\right)^{-1} G_{21}\right) w \\ &= F_1 w \end{aligned} \tag{1}$$

The problem is to find a proper, real-rational compensator $K(s)$ that robustly stabilizes the plant $G(s)$. Optimal H_∞-design determines the compensator such that the H_∞-norm of F_1 is minimized among the family of controllers fulfilling the other requirements.

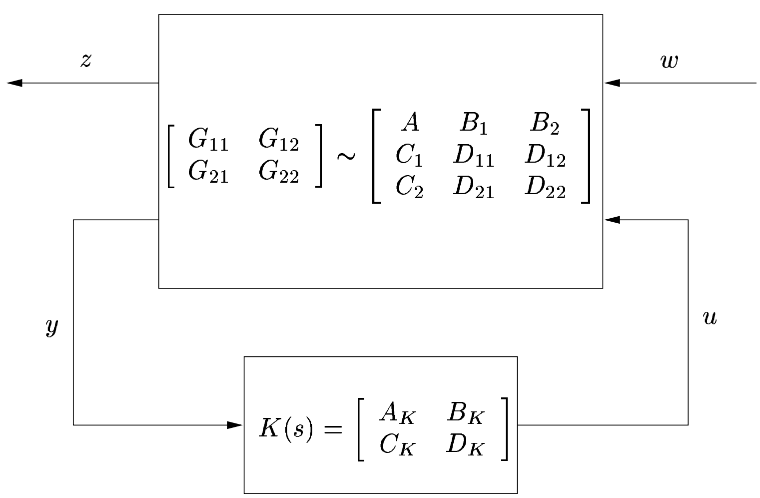

Figure 1: The standard system set-up in robust control

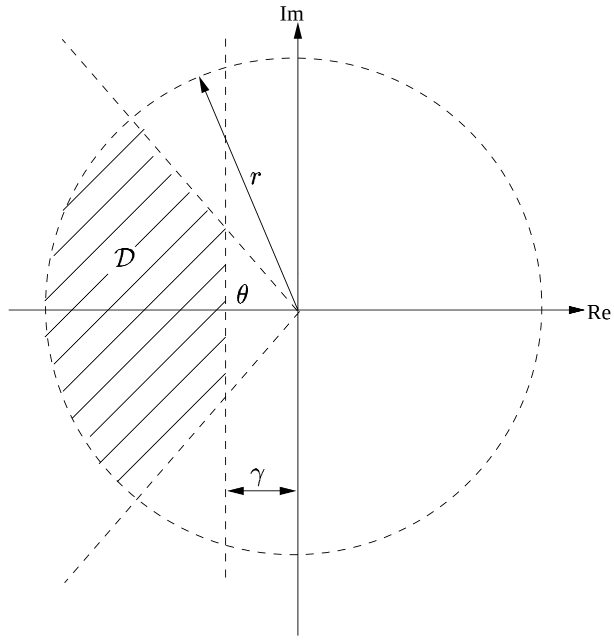

Figure 2: An example of an LMI-region in the complex plane, $S(r, \alpha, \theta)$

1.2 Using LMIs for Pole Placement

As discussed in the introduction section, it is desirable to be able to place the poles of the closed-loop system in a certain region, say $\mathcal{D}$, of the open left half-plane. That defines so called $\mathcal{D}$-stability. A general LMI-region can be written like

$$\mathcal{D} = \{z \in \boldsymbol{C} : f_{\mathcal{D}} < 0\} \tag{2}$$

where

$$f_{\mathcal{D}} = \alpha + \beta z + \beta^T \bar{z} = [\alpha_{kl} + \beta_{kl} z + \beta_{lk} \bar{z}]_{1 \leq k,l \leq m} \tag{3}$$

In [3], it is shown that, given an LMI-region of the form in equation (2), one can extend the Lyaponov stability to $\mathcal{D}$-stability by the means of the following theorem (theorem 2.2 in [3]):

Theorem 1.1 *The matrix A is $\mathcal{D}$-stable if and only if there exists a symmetric positive definite matrix X such that*

$$M_{\mathcal{D}}(A, X) = \left[\alpha_{kl} X + \beta_{kl} AX + \beta_{lk} X A^T\right]_{1 \leq k,l \leq m} \prec 0 \tag{4}$$

The proof of the theorem can be found in the appendix of [3]. In the LMI toolbox [7] there are functions for creating and intersecting LMI-regions. One typical such region in our applications is depicted in Figure 2. In this case we can think of $\mathcal{D}$ as the intersection of three regions:

1. A circle centered at the origin with radius r
2. A conic sector with opening angle (total) 2θ
3. A half-plane with upper real limit γ

We will here determine the corresponding functions $f_{\mathcal{D}}$. The intersection is determined by forming a block-diagonal matrix out of the former matrices. The matrices can also be determined with the help of the MATLAB LMI Toolbox. The circle criterion is determine like:

$$\begin{array}{rrcl}
 & |z| < r & \& & r > 0 \\
\Leftrightarrow & z\bar{z} < r^2 & \& & r > 0 \\
\Leftrightarrow & (-r) - z(-r)^{-1}\bar{z} < 0 & \& & -r < 0
\end{array}$$

$$\Leftrightarrow \begin{pmatrix} -r & z \\ \bar{z} & -r \end{pmatrix} \prec 0 \tag{5}$$

where we've used the strict Schur complement formula in the last equation. Similarly, for the conic sector we have

$$\begin{array}{rrcll} & \left|\frac{Im(z)}{Re(z)}\right| & < & \tan(\theta) & , \quad Re(z) < 0 \;\; \& \;\; 0 < \theta < \frac{\pi}{2} \\ \Leftrightarrow & \frac{(z-\bar{z})^2}{(z+\bar{z})^2} & < & \frac{\sin^2\theta}{\cos^2\theta} & , \quad z+\bar{z} < 0 \;\; \& \;\; 0 < \theta < \frac{\pi}{2} \\ \Leftrightarrow & (\sin\theta(z+\bar{z})) - \frac{\cos\theta(z-\bar{z})\cos\theta(z-\bar{z})}{(\sin\theta(z+\bar{z}))} & < & 0 & , \quad z+\bar{z} < 0 \;\; \& \;\; 0 < \theta < \frac{\pi}{2} \end{array}$$

$$\Leftrightarrow \begin{pmatrix} \sin\theta(z+\bar{z}) & \cos\theta(z-\bar{z}) \\ \cos\theta(\bar{z}-z) & \sin\theta(z+\bar{z}) \end{pmatrix} \prec 0 \tag{6}$$

where we again have used the strict Schur complement formula. And, finally, for the certain type of half-plane we get

$$\begin{array}{rrcl} & Re(z) & < & -\gamma \\ \Leftrightarrow & z+\bar{z}+2\gamma & < & 0 \end{array} \tag{7}$$

In order for all matrix inequalities, 5, 6 & 7 to hold we require that

$$\begin{pmatrix} -r & z & 0 & 0 & 0 \\ \bar{z} & -r & 0 & 0 & 0 \\ 0 & 0 & \sin\theta(z+\bar{z}) & \cos\theta(z-\bar{z}) & 0 \\ 0 & 0 & \cos\theta(\bar{z}-z) & \sin\theta(z+\bar{z}) & 0 \\ 0 & 0 & 0 & 0 & z+\bar{z}+2\gamma \end{pmatrix} \prec 0 \tag{8}$$

where we identify the matrices α and β:

$$\alpha = \begin{pmatrix} -r & 0 & 0 & 0 & 0 \\ 0 & -r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\gamma \end{pmatrix} \quad \& \quad \beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

2 Compensator Design Using LMIs

Using the fact that $\mathcal{D}$-stability can be recasted into an LMI, we can combine this with "regular" H_∞-control. This is the main result of [3]. Consider the following problem:

Problem 2.1 *Find a stabilizing controller $u = K(s)y$ with the H_∞-norm of the closed-loop transfer function from w to z, T_{wz}, less than $\gamma > 0$ and closed-loop poles in the region $\mathcal{D}$ for the system*

$$\begin{array}{rcrcrcr} \dot{x}(t) & = & Ax(t) & + & B_1w(t) & + & B_2u(t) \\ z(t) & = & C_1x(t) & + & D_{11}w(t) & + & D_{12}u(t) \\ y(t) & = & C_2x(t) & + & D_{21}w(t) & + & D_{22}u(t) \end{array} \tag{10}$$

The H_∞-constraint boils down to find a symmetric, positive definite matrix X_∞ which solves a pair of Riccati equations. The LMI constraint requires that there exists (another) symmetric, positive definite matrix $X_\mathcal{D}$ which fulfills an LMI. A conservative, but convex, formulation of this problem is to find the solution where the matrices coincide:

$$X = X^T = X_\infty = X_\mathcal{D} \succ 0 \tag{11}$$

This can be formulated as theorem 4.3 in [3]:

Theorem 2.1 *Let $\mathcal{D}$ be an arbitrary LMI region in the left half-plane with the characteristic function $f_{\mathcal{D}}$:*

$$\mathcal{D} = \{z \in \boldsymbol{C} : f_{\mathcal{D}} < 0\} \tag{12}$$

where

$$f_{\mathcal{D}} = \alpha + \beta z + \beta^T \bar{z} = [\alpha_{kl} + \beta_{kl} z + \beta_{lk} \bar{z}]_{1 \leq k,l \leq m} \tag{13}$$

Then, problem 2.1, is solvable if and only if the following system of LMIs is feasible

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \succ 0 \tag{14}$$

$$\begin{bmatrix} \Psi_{11} & \Psi_{21}^T \\ \Psi_{21} & \Psi_{22} \end{bmatrix} \succ 0 \tag{15}$$

$$\left[\alpha_{kl} \begin{pmatrix} R & I \\ I & S \end{pmatrix} + \beta_{kl} \Phi + \beta_{lk} \Phi^T \right]_{k,l} \prec 0 \tag{16}$$

where $R = R^T \in \boldsymbol{R}^{n \times n}$, $S = S^T \in \boldsymbol{R}^{n \times n}$ and Φ & Ψ are defined in the paper [3]. From R and S a compensator fulfilling the requirements is easily determined.

The proof is omitted here but can be found in the paper [3]. We note that the first two LMIs, 14 and 15, correspond to the constraints of H_∞while the last one, 16, originates from the LMI constraints. The algorithm is implemented in the MATLAB LMI Toolbox [7]. In the LMI toolbox also H_2-constraints and other optimization goals can be included. Here, however, we just consider the H_∞-control case. The functions of the toolbox will be evaluated in the following sections.

3 Examples from the Literature

In order to show how the technique described in the last section can be applied, we will consider two examples. For each example we will compare the LMI-based design with other common techniques.

3.1 Example 1: Two-mass-spring System

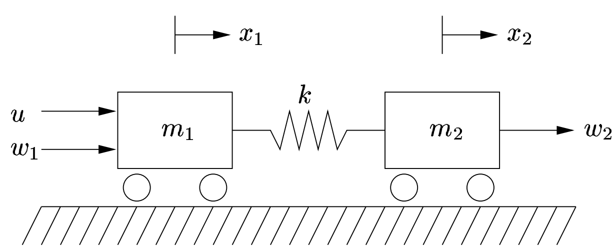

Figure 3: Example 1, the two-mass-spring

This example was constructed as a benchmark example in “Journal of Guidance, Control, and Dynamics” vol. 15 no. 5 1992 [8]. The problem, called problem 1 in [8], is to robustly stabilize a mechanical system consisting of two cars connected by a spring. The control acts on the first car and the position of the second car is measured with a noise. There are disturbances on both of the cars. The state space model of the system is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m_1} & \frac{k}{m_1} & 0 & 0 \\ \frac{k}{m_2} & -\frac{k}{m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \nu \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix} u$$

$$z = \begin{bmatrix} x_2 \\ u \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \nu \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \nu \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

where the disturbances w_1 and w_2 are introduced in Figure 3 while ν is a measurement noise on the output that goes into the controller. We will subsequently consider the nominal system, that is the system with $m_1 = m_2 = k = 1$ (with appropriate units). The design specifications are

1. Setting time of 15 s for impulse disturbance on bodies, that is w_1 and w_2.
2. Closed-loop stability for $0.5 \leq k \leq 2.0$ (robustness).
3. Insensitivity to high frequency noise.
4. Performance/stability robustness and gain/phase margins for reasonable bandwidth.
5. Reasonable control effort.
6. Low controller complexity.

3.1.1 Three Procedures -Three Compensators

We will analyze the problem in the same fashion as Wie, Liu and Byun do in [2]. They use a regular H_∞-design and we will reproduce their solution and denote it "Wie" from now on. We rewrite the system in order to get a better design. Firstly, we realize that any disturbances in k, Δk, will only appear in the system dynamics as multiplied with the *difference* between the position of the cars. Therefore, we introduce the $x_1 - x_2$ as an additional output and $\tilde{w}$ as a new input that will correspond to changes in k. Since smaller difference between $x_1 - x_2$ implies smaller effect of a change in k this will increase the robustness. Secondly, we introduce a new input in order to keep the control effort low, we also consider u as an output of our design problem. We get:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{w} \\ w_1 \\ \nu \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$z = \begin{bmatrix} x_1 - x_2 \\ x_2 \\ u \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{w} \\ w_1 \\ \nu \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{w} \\ w_1 \\ \nu \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

where we also have neglected the dependence on the disturbance w_2. This will simplify the design but we will still achieve the desired specifications. In order to get the desired features weights need to be put on each of the three outputs. This can be done by post-multiplying B_1 and D_{21} of Figure 1 by the matrix W. Choosing

$$W = diag\{0.1, 0.025, 0.025\} \tag{17}$$

and a H_∞-requirement

$$\gamma \leq 1 \tag{18}$$

we get the compensator of [2]:

$$K_{Wie}(s) = \frac{2.123s^3 + 5.514s^2 - 6.432s - 1.035}{s^4 + 4.671s^3 + 12.91s^2 + 18.3s + 12.63} \tag{19}$$

If we instead look for the H_∞-controller with the lowest upper bound we can find, $\gamma \leq 0.32$, we get the controller:

$$K_{H_\infty opt}(s) = \frac{-204.9s^3 + 406.9s^2 - 218s - 39.19}{s^4 + 22.17s^3 + 120.7s^2 + 340.2s + 500.7} \tag{20}$$

The third controller in our comparison is the LMI-based H_∞-controller. We choose an LMI-region

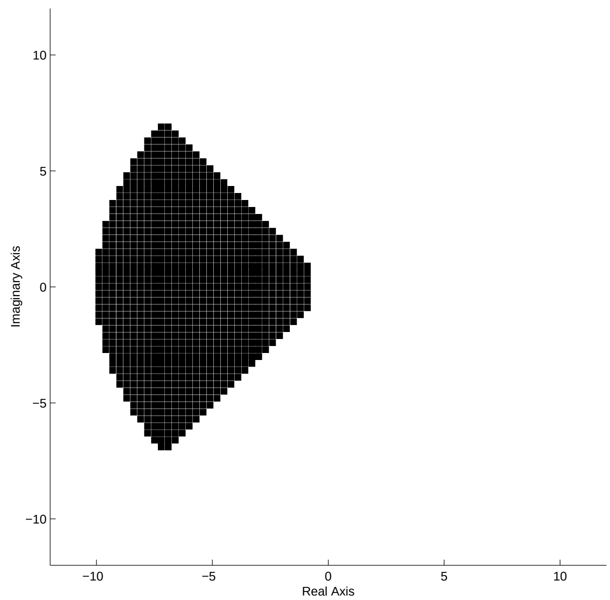

Figure 4: The LMI-region, plotted, and double-checked, with lmitest.m.

that is similar to that of [7] and is plotted in Figure 4. We chose $r = 12$, $\alpha = 0.5$ & $\theta = \frac{\pi}{4}$. The region is plotted in Figure 4 Looking for the lowest H_∞-performance we can get, we arrive at $\gamma \leq 1.9281$ and the controller

$$K_{LMI}(s) = \frac{-78.31s^3 + 498.7s^2 - 214.6s - 35.09}{s^4 + 19.96s^3 + 133.4s^2 + 400.9s + 538.8} \tag{21}$$

We note that our requirement of a strictly proper controller is fulfilled.

3.1.2 Comparison of controllers

We compare the controllers using some tools:

1. Pole and zero locations for the closed-loop systems.
2. Sensitivity and complementary sensitivity functions.
3. Bode plots for the open-loop systems.
4. Impulse responses.
5. A robustness measure for the spring constant k, that is the values of k for which the system is stable.

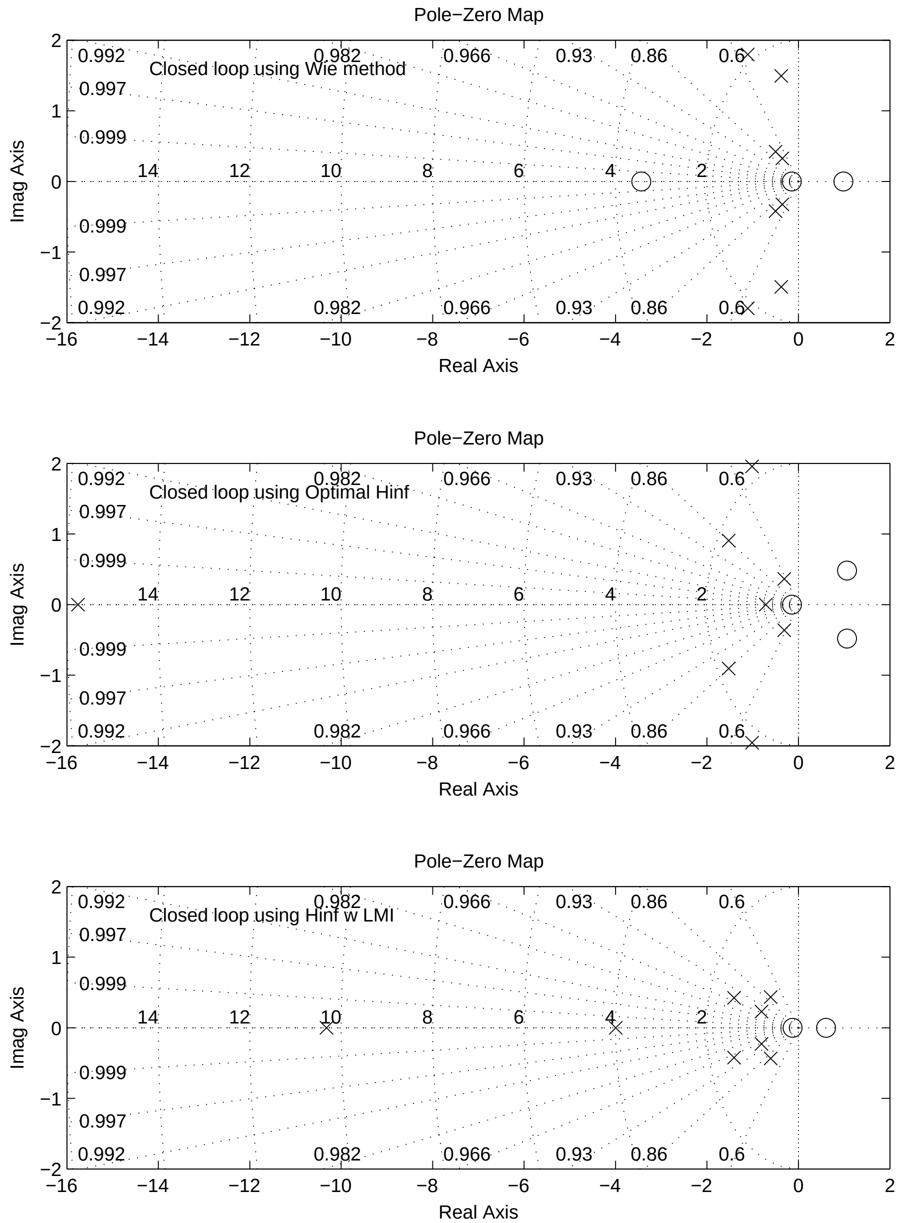

Figure 5: The pole and zero locations for the closed-loop systems with the different controllers. Comparing the pole locations with and without LMI-constraints we note that we have been able to move some of the poles away from the imaginary axis and push the pole at approximately -16 closer to the origin.

We will, using these tools, go though all the design specifications that we stated and make sure that all of them are fulfilled. In Figure 5, all poles and zeros of the closed-loop systems are plotted. We note that we have achieved the pole placement for the LMI based design according to our choice above. In this case the LMI-constraints are active since there are poles of the H_∞controller outside the region $\mathcal{D}$. The sensitivity functions for the different designs are quite similar, as one can see in Figure 6, and in none of the the systems there is amplification of high frequency noise. Thus requirement three is fulfilled. We note that all of them have a dramatic drop at the frequency where the original system has a poles on the imaginary axis. The complementary sensitivity functions are plotted in 7 and we see that the bandwidth is quite big for all of the controllers. The impulse responses can be found in Figure 8. All impulse responses behave nicely (the two upper subplots) and do set before 15 seconds as in specification one. The control effort is plotted in the last subplot and it stays below 1.5 in all

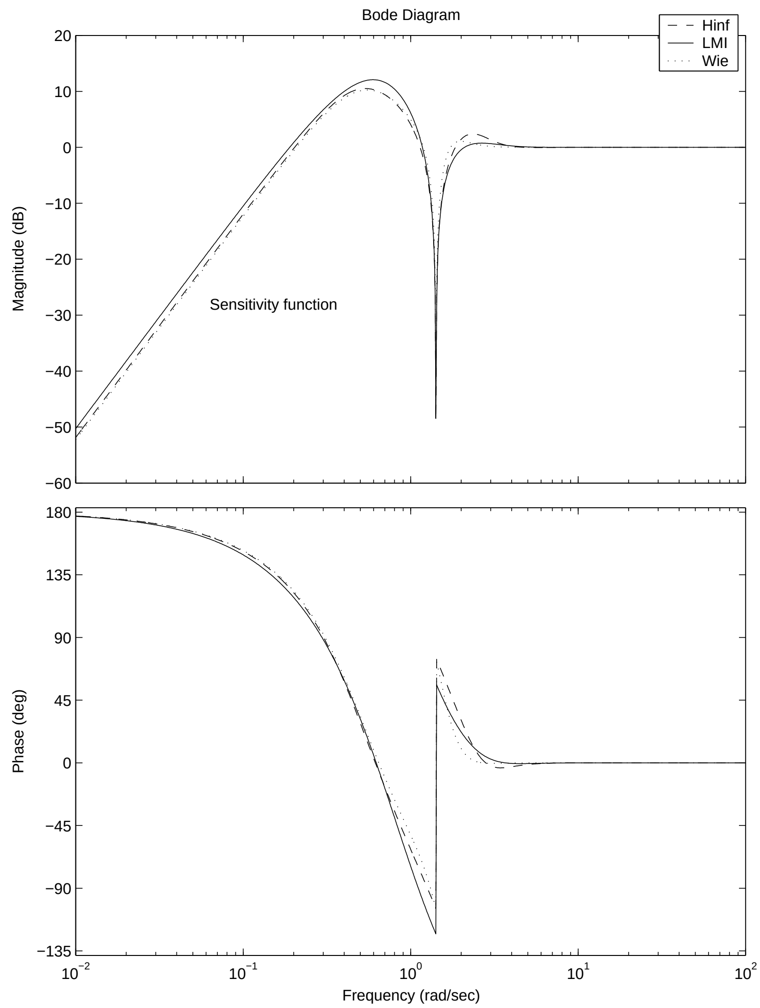

Figure 6: The sensitivity functions for the different controllers of the two-mass-spring problem

cases so specification five is fulfilled. One can also note that the control signal for the LMI based —H_∞-design is smoother than the regular H_∞-design. This is probably a result of moving the pole at approximately -16. In the Figure 9 the Bode plots for the open-loop systems corresponding to each controller are shown. From the plots we read the gain and phase margins of table 1. The margins are quite similar, with the smallest margins for the LMI based design. There is an annoying poles on the imaginary axis in the plant causing infinite gain at $\omega = 1.5$. This is at a higher frequency than our phase and gain margins. However a more comprehensive analysis is desirable. For further discussion whether these margins are sufficient, see [2].

Design	Gain margin dB(absolute)	Phase margin
Wie	3.36(1.47)	24.7
H_∞	3.20(1.45)	24.3
LMI	2.49(1.33)	22.9

Table 1: The gain and phase margins in the two-mass-spring example

We measure the robustness using μ-analysis as on page 3-25 in [7]. We get the stability regions in terms of values of k as in Table 2. They are all within .5 and 2.0 so specification two is fulfilled. Finally all controllers are of order 4 and strictly proper so specification six is fulfilled.

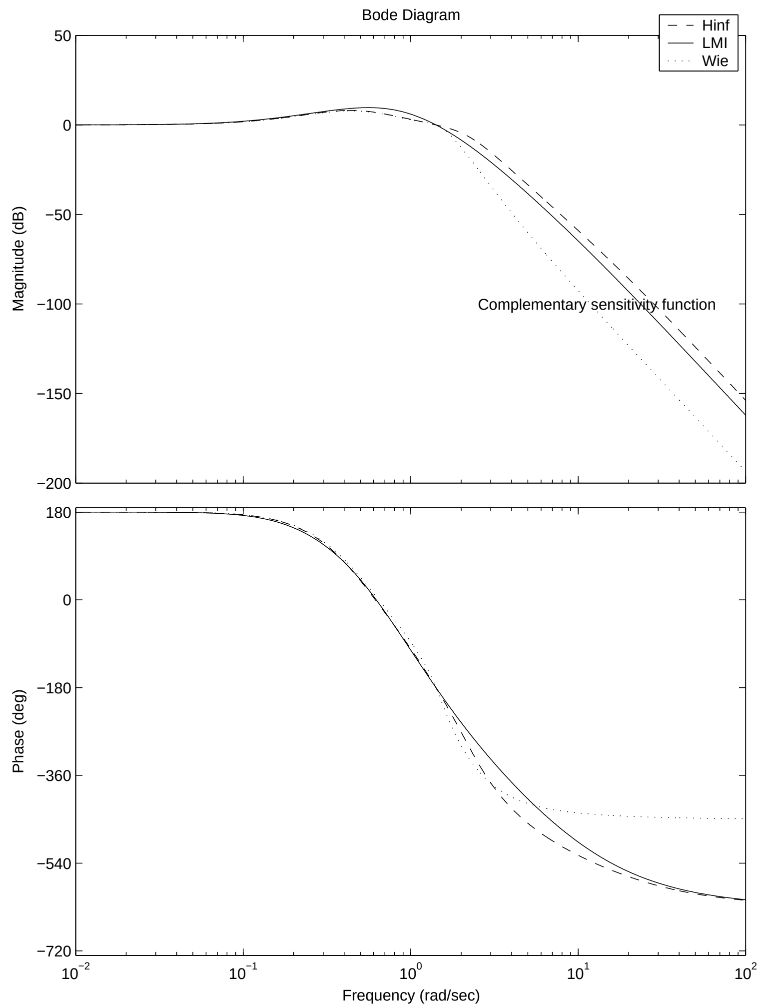

Figure 7: The complementary sensitivity functions for the different controllers of the two-mass-spring problem

Compensator	Lower limit	Upper limit
Wie	0.4497	2.0503
H_∞	0.4302	2.0698
LMI	0.5002	1.9998

Table 2: The values of k for which the closed-loop systems are stable in the two-mass-spring example

3.2 Example 2: Aircraft Model AIRC

This example we have taken from [1]. It considers an aircraft in a linearized vertical model. The model has three inputs, three outputs and five states. The problem has later been considered and a little bit modified in [5] and [6]. Unfortunately the B matrices are different in one component in the later references. We use the version of [1], but the difference between the systems does not change the problem that much. The state space model is:

$$A = \begin{bmatrix} 0 & 0 & 1.1320 & 0 & -1.000 \\ 0 & -0.0538 & -0.1712 & 0 & 0.0705 \\ 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0.0485 & 0 & -0.8556 & -1.0130 \\ 0 & -0.2909 & 0 & 1.0532 & -0.6859 \end{bmatrix}$$

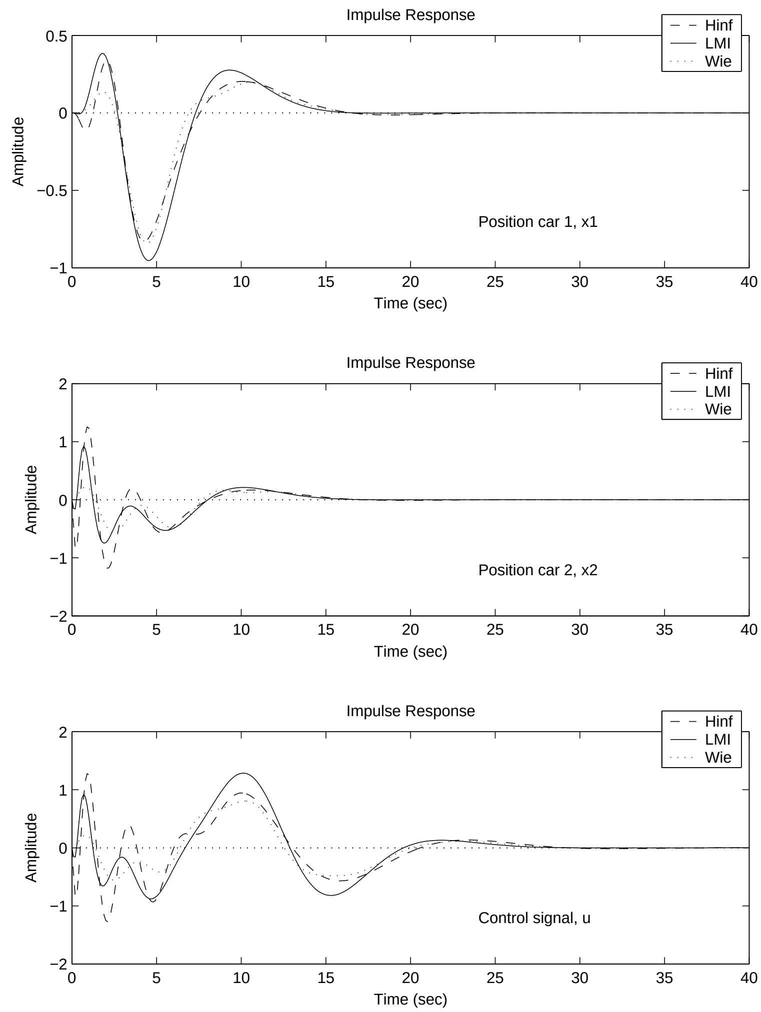

Figure 8: The impulse response for for disturbances on different components for the two-mass-spring problem

$$B = \begin{bmatrix} 0 & 0 & 0 \\ -0.0012 & 1.0000 & 0 \\ 0 & 0 & 0 \\ 4.4190 & 0 & -1.6650 \\ 1.5750 & 0 & -0.0732 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is a multidimensional example that requires a quite sophisticated design. We will consider the following specifications:

- Bandwidth of about 10 $rad\ s^{-1}$ for each loop.

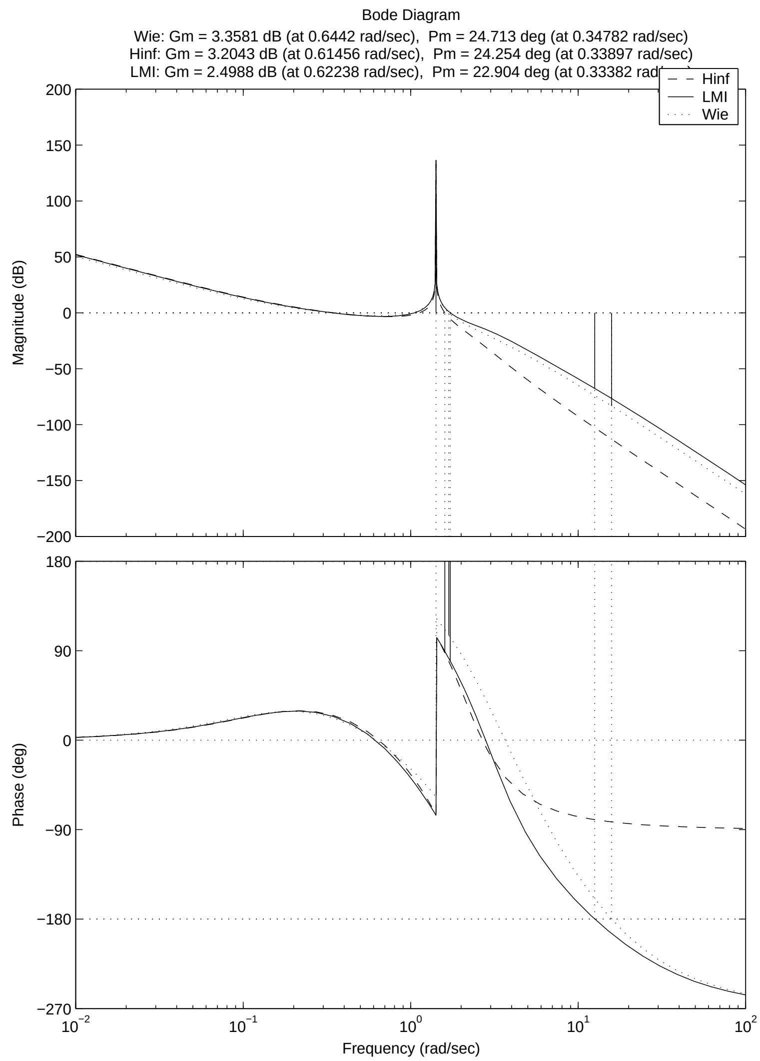

Figure 9: The Bode plots of the open-loop systems for the two-mass-spring problem

- Fast damping of step responses from all inputs to all outputs.
- Low complexity of compensator.

For further details and discussion of the specifications and properties of the system, see [6]. In the same book there are several design approaches and we will here mention two of them and compare them to the design that we achieved using LMIs.

3.2.1 Three Approaches - Three Compensators

The three approaches we will use are

1. Engineering approach with coupling of five compensators
2. H_∞optimal design
3. LMI-based optimal H_∞design

The first approach is described in [6] on pages 155-167. The resulting compensator is given by the state space realization

$$\mathcal{A}_{KMac} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1.0968 & 0.0016 & 0.2793 & -10.7225 & 0 \\ -0.6950 & -0.0008 & 0.3246 & 0 & -10.7225 \end{bmatrix}$$
$$\mathcal{B}_{KMac} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2.1937 & 0.0031 & 0.5587 \\ -1.3900 & -0.0017 & 0.6491 \end{bmatrix}$$
$$\mathcal{C}_{KMac} = \begin{bmatrix} -29.3803 & 0.0167 & 1.8067 & 164.6078 & -94.9601 \\ -3.5182 & 5.0006 & -0.2555 & 20.8859 & -11.1922 \\ -78.0162 & -0.0028 & -33.7400 & 743.9029 & 232.0489 \end{bmatrix}$$
$$\mathcal{D}_{KMac} = \begin{bmatrix} -58.7607 & 0.0333 & -.36134 \\ -7.0365 & 10.0011 & -0.5110 \\ -156.0324 & -0.0056 & -67.4800 \end{bmatrix}$$

In order for the second approach, H_∞, to work we need to modify the system that we use in the design procedure. We make the same modifications that are described in [6] on page 308ff. We move the pole in origin of the original system to -0.001 and replace D with an identity matrix scaled so that the eigenvalues are 10^{-5}. Both these changes will hardly affect the systems behavior in the region 0.001 to 100 $rad\ s^{-1}$. When we evaluate the controller we will of course consider the original system. We also introduce two shaping filters as in [6]. We use

$$\begin{aligned} w_1(s) &= \frac{(s+6)^2}{(s+0.00006)(s+0.6)} \\ w_2(s) &= \frac{2000(s+10)(s+50)}{(s+1000)^2} \end{aligned} \tag{22}$$

for shaping both sensitivity and complementary sensitivity. We will consider the H_∞-norm:

$$\left\| \begin{matrix} W_1 S \\ W_2 T \end{matrix} \right\|_\infty \tag{23}$$

This will produce a closed-loop system with both sensitivity and complementary sensitivity as we desire. We look for a compensator with the H_∞performance $\gamma \leq 5.00$. Due to the shaping filers we get a quite complex controller of dimension seventeen (five from plant and three times four from shaping filters). In order to reduce the complexity one could use model reduction but we don't consider that here. We end up with a controller with the state space representation of the appendix.The last design uses LMIs. It turned out to be very tricky to find an LMI region for which a good controller could be determined. The reason is probably that the constraints given by the LMIs are too conservative in combination with the stability and H_∞-constraints. This might be a substantial flaw of the LMI design procedure. Choosing $r = \infty$, $\theta = \frac{\pi}{4}$ and $\alpha = 0$ for the LMI-region in Figure 2 we get a reasonably good controller by *only* considering the pole placement constraints. The best controller we find is the one shown here. The singular values does not look satisfactory at all. For instance the maximal singular value of the complementary sensitivity function stays above 5 dB for $1 \leq \omega \leq 10$. The step responses look pretty good though. However, this might just be a lucky coincident. The controller is loser in our comparison and is shown in the appendix.

3.2.2 Comparison of compensators

We will compare the controllers comparing their sensitivity and complementary sensitivity functions and step responses. In Figure 10 the singular values of the closed-loop systems are plotted. We see that the $H_\infty$and the LMI based H_∞synthesis has similar or better amplification in the desired frequency region compared to the engineering design. Furthermore, we get a significantly smaller maximal amplification of the closed loop system. The same is the case for the complementary sensitivity function in Figure 11. The responses to step demands on each output are depicted in Figure 12. Since we have three inputs and three outputs there are altogether nine combinations for each controller. From the graph we conclude that all the controllers adjust to the step quickly and that the channels behave independently.

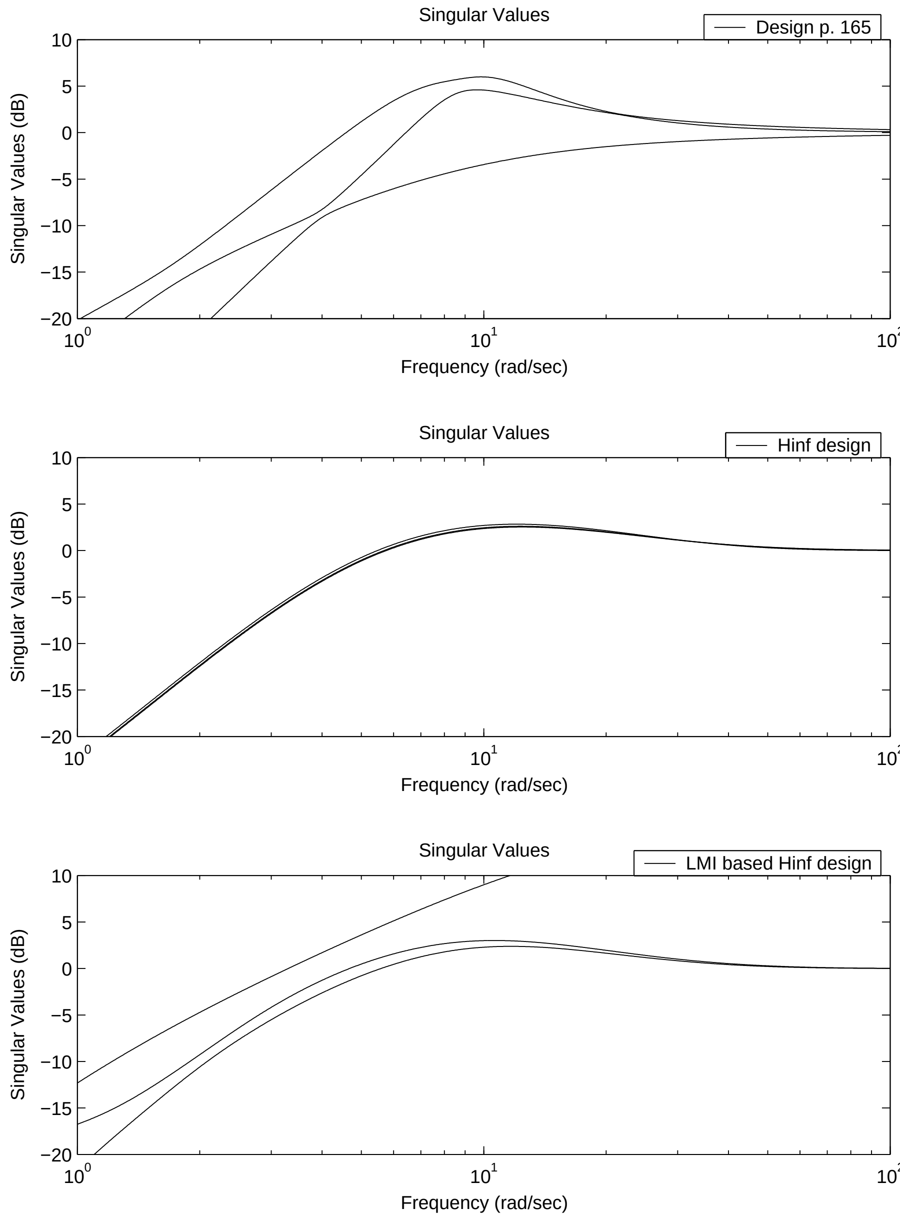

Figure 10: Singular value plots of the sensitivity functions for example 2

4 Summary and Conclusions

We have studied how one can include pole placement constraints in standard H_∞-control synthesis using LMIs. The problem can be formulated as a convex optimization problem. In the MATLAB LMI toolbox there are methods for solving these problems. Evaluating the achieved controller designs on a couple of design problems in the literature, and comparing with other design methods, we find out that the technique works ok and can be attractive. However, there are two main flaws:

- High complexity of the controller (as always in H_∞-design)
- Conservatism in method can lead to less or even no room for H_∞-specifications.

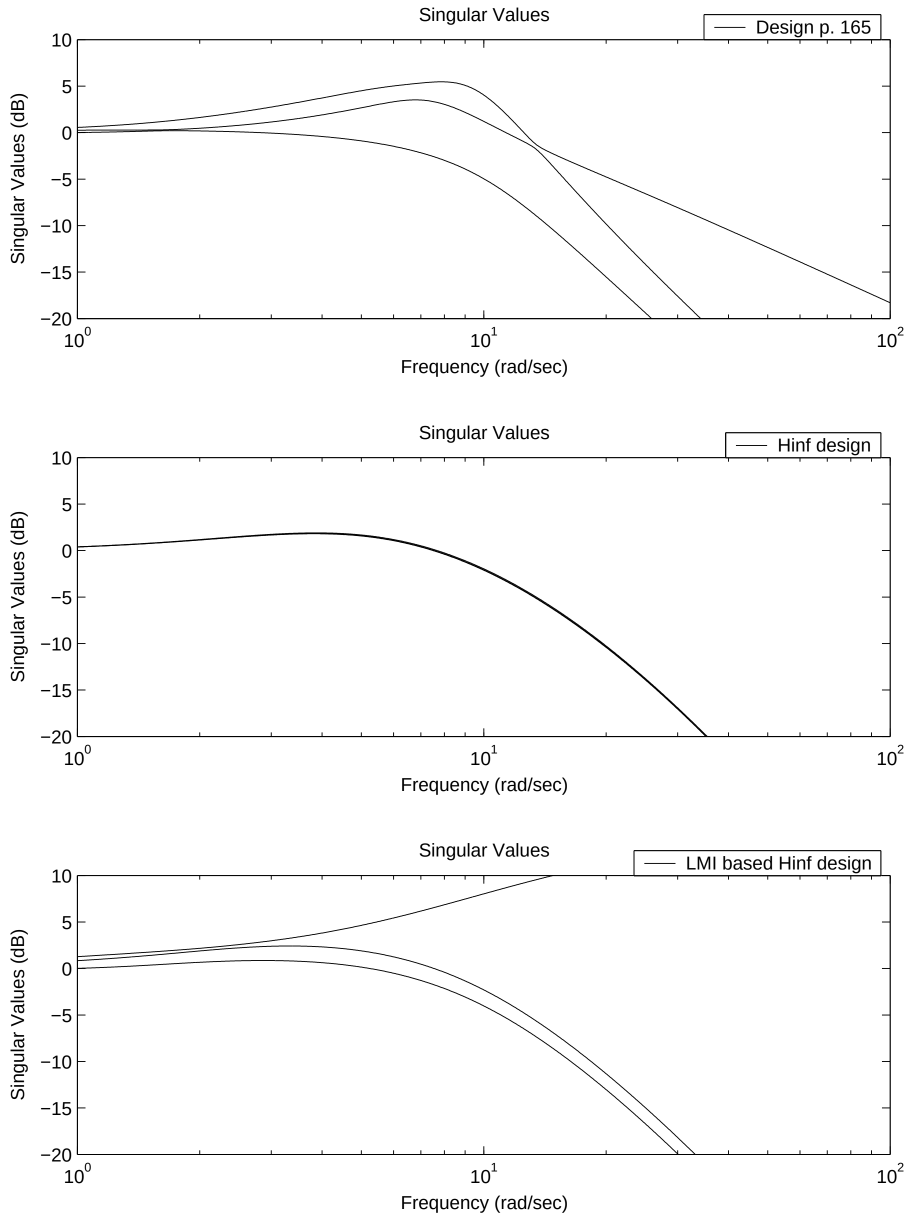

Figure 11: Singular value plots of the complementary sensitivity functions for example 2

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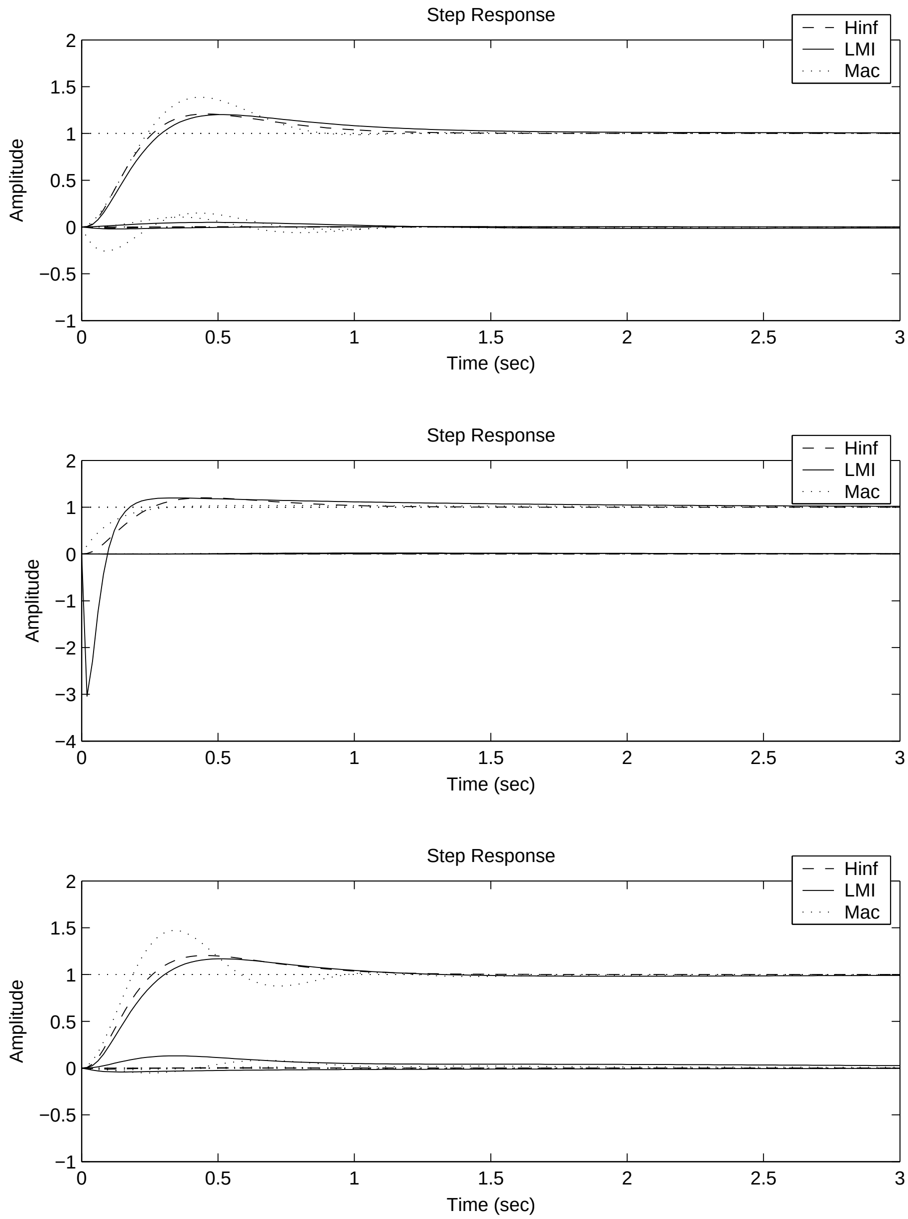

Figure 12: The responses for step demands on each output for example 2. The response step demand on output 2 using the LMI-based controller is not fullfilling our requirements.

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