Computation of bounded degree Nevanlinna-Pick interpolants by solving nonlinear equations

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***Abstract*— This paper provides a procedure for computing scalar real rational Nevanlinna-Pick interpolants of a bounded degree. It applies to a wider class of interpolation problems and seems numerically more reliable than previous, optimization-based, procedures. It is based on the existence and the uniqueness of the solution guaranteed by Georgiou's proof of bijectivity of a map between a class of nonnegative trigonometric polynomials and a class of numerator/denominator polynomial pairs of interpolants. Further analysis of this map suggests a numerical continuation method for determining the interpolant from a system of nonlinear equations. A numerical example illustrates the reliability of the proposed procedure.**

KEYWORDS. **Nevanlinna-Pick interpolation, positive realness, rationality, system of nonlinear equations, continuation method.**

I. INTRODUCTION

Nevanlinna-Pick interpolation theory has been recognized as a tool applicable to various areas in systems and control [5], [14]. Recent developments of the interpolation theory with complexity constraint have introduced new approaches in signal processing, circuit theory and robust control, see [3] and references therein. To fully exploit the theory in such applications, it is significant to devise a numerically efficient algorithm for computing any rational interpolant of a bounded degree.

The theory in [3] completely characterizes all the *strictly* positive real Nevanlinna-Pick interpolants of a bounded degree. More precisely, it shows the diffeomorphism between a class of *positive* trigonometric polynomials and a class of rational *strictly* positive real interpolants, a stronger assertion of Georgiou's conjecture of bijectivity in [7]. The problem of computing each such interpolant amounts to solving an optimization problem for which a procedure based on a continuation method has been developed in [2], [6], [10].

The optimization-based procedure may work even to determine non-strictly positive real interpolants unless the interpolant has poles on the unit circle, since that requires spectral factorization with spectral zeros on the unit circle, which is numerically infeasible. However, such interpolants are important in applications; e.g., it corresponds to spectral lines in spectral estimation. Furthermore, a reliable algorithm ought to be numerically stable for spectral zeros in the vicinity of the unit circle. This motivates the present paper.

Recently, Georgiou proved the bijectivity between a class of numerator/denominator polynomial pairs of (not necessarily strictly) positive real and a class of (not necessarily strictly) positive real interpolants in [8]. However, the proof in [8] does not offer any means of actually constructing such interpolants. In this paper, we solve a system of nonlinear equations that arises immediately from the bijectivity assertion. To solve the system, we use a *continuation method* [1] on a homotopy from the equations for the *central solution* and the system of our interest. This is fundamentally different from the optimization-based approach.

Apart from the reliability of a procedure, its numerical efficiency is of vital importance for some applications. In the present paper we will not discuss this issue in detail. However, at present the problem of computing arbitrary interpolants of bounded degree seems to require these general nonlinear methods.

The paper is organized as follows. In Section II, we will review the Nevanlinna-Pick interpolation problem with degree constraint while Section III is devoted to an exposition of properties of a map, which are important for the procedure proposed in this paper. In Section IV, we derive a system of nonlinear equation whose solution gives the unique positive real interpolant of a bounded degree. Section V proposes a procedure based on a continuation method to solve the system of nonlinear equations. An example is given in Section VI to illustrate the reliability of the procedure.

II. THE NEVANLINNA-PICK INTERPOLATION PROBLEM WITH DEGREE CONSTRAINT

In this section, we will formulate a Nevanlinna-Pick interpolation problem with degree constraint. We will review Theorem 2 in [8] concerning the complete characterization of a class of positive real interpolants which is pertinent to our result.

A. Nevanlinna-Pick interpolation with degree constraint

The core problem considered in this paper is the *Nevanlinna-Pick interpolation problem with degree constraint*, formulated as follows.

Problem 2.1: Given a set of self-conjugate complex number pairs $\{(z_j, w_j)\}_{j=0}^{n}$ with z_0=0 and distinct $\{z_j\}_{j=0}^{n} \subset \mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$, determine any function f which satisfies the following conditions:

- f is positive real, i.e., f is holomorphic in $\mathbb{D}$ and maps $\mathbb{D}$ into the closed right half-plane $\mathbb{C}_+$,

- f satisfies the interpolation conditions:

$$f(z_j) = w_j, \quad j = 0, 1, \ldots, n, \tag{1}$$

- f is real rational of degree at most n.

Remark 2.1: The results in this paper are still valid even in the case where the interpolation constraints include derivative constraints. The only difference appears in the matrix K that will be defined in (5).

The necessary and sufficient condition for the existence of rational interpolants of degree at most n is well-known to be expressed as the nonnegativity of the *Pick matrix* (see [8, p. 632]), which is the same as the classical result without degree constraint (see e.g. [13]). If the Pick matrix is nonnegative definite but singular, there is a unique solution to this problem. Hereafter, we assume that the Pick matrix is positive definite, leading to the case where there are infinitely many rational interpolants of degree at most n.

B. A complete characterization of all the solutions

In [7], for the covariance extension problem (which is essentially the same problem as the Nevanlinna-Pick interpolation problem), Georgiou conjectured that *the class of all the interpolants of degree at most n is completely characterized in terms of the class of nonnegative trigonometric polynomials of degree at most n.* This conjecture was proven to be true in [8, Theorem 2]. Even though the theorem in [8] deals with both complex and real rational interpolants, here, we restrict ourselves to real rational interpolants which are relevant to applications. In addition, we remove the normalization that was taken into account in [8].

Theorem 2.1: [8] For each element in the class of real nonnegative trigonometric polynomials of degree at most n:

$$\mathcal{D} := \left\{ \begin{array}{c} d(z, z^{-1}) := \displaystyle\sum_{j=0}^{n} d_j (z^j + z^{-j}) \geq 0, \forall\, |z| = 1 \\ d_j \in \mathbb{R},\ j = 0, 1, \ldots, n \end{array} \right\},$$

there is a unique pair of real polynomials (α, β) satisfying the following four conditions:

C1 $\deg \alpha \leq n$, $\deg \beta \leq n$ and $\alpha(0) > 0$,

C2 the rational function $f := \beta/\alpha$ satisfies the interpolation conditions (1),

C3 α and β satisfies

$$\alpha(z)\beta^*(z) + \beta(z)\alpha^*(z) = d(z, z^{-1}), \tag{2}$$

where $\alpha^*(z) := \alpha(z^{-1})$,

C4 $\alpha + \beta$ has all its roots in $\mathbb{D}^c := \{z \in \mathbb{C} :\ |z| \geq 1\}$.

Note that the combination of two conditions **C3** and **C4** is equivalent to the condition that a rational $f := \beta/\alpha$ is positive real (see [8]), and therefore, that the conditions **C1**–**C4** are the same as three conditions in Problem 2.1. Conversely, it is obvious that any Nevanlinna-Pick interpolant $f = \beta/\alpha$ of degree at most n corresponds to some nonnegative trigonometric polynomial $d(z, z^{-1})$ with (2). Consequently, the class $\mathcal{D}$ completely characterizes the class of all the solutions to the Nevanlinna-Pick interpolation problem with degree constraint.

Now, define the inner-product of the real functions a and b in L^2 space by

$$\langle a, b \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} a^*(e^{i\theta}) b(e^{i\theta}) d\theta.$$

Since, as will be shown later, the conditions **C1** and **C2** give rise to a linear relation between α and β, denoted by $\beta = \kappa(\alpha)$, the above theorem can be restated as follows.

Theorem 2.2: The map $G : \mathcal{A} \to \mathcal{D}$ defined by

$$G(\alpha) := \alpha^* [\kappa(\alpha)] + [\kappa(\alpha)]^* \alpha \tag{3}$$

is bijective. Here, the domain $\mathcal{A}$ is the class of all polynomials α which generate the Nevanlinna-Pick interpolants with degree constraint, defined as

$$\mathcal{A} := \{\alpha :\ \deg \alpha \leq n,\ \alpha(0) > 0,\ \kappa(\alpha)/\alpha \text{ is positive real}\}.$$

In fact, the map G has "stronger" properties than the bijectivity. This point will be discussed in the next section.

III. PROPERTIES OF THE MAP G

Now, we will discuss important properties of the map G in (3). As was stated in Theorem 2.2, the map is a bijection. However, we can actually show that *the map G is a homeomorphism*. In the region of *strictly* positive real interpolants, *the corresponding map is even a diffeomorphism*, as shown by Byrnes and Lindquist in [4]. Both of these properties will turn out to be vital in justifying a numerical continuation method. First we have the following theorem.

Theorem 3.1: The map $G : \mathcal{A} \to \mathcal{D}$ in (3) is a homeomorphism.

To prove this theorem, we show a normalized version of the theorem.

Lemma 3.1: The map $\tilde{G} : \tilde{\mathcal{A}} \to \tilde{\mathcal{D}}$ defined by

$$\tilde{G}(\alpha) := \alpha^* [\kappa(\alpha)] + [\kappa(\alpha)]^* \alpha$$

is a homeomorphism. Here, the domain $\tilde{\mathcal{A}}$ and the range $\tilde{\mathcal{D}}$ are normalized sets of $\mathcal{A}$ and $\mathcal{D}$ respectively, defined as

$$\begin{aligned} \tilde{\mathcal{A}} &:= \{\alpha \in \mathcal{A} :\ \langle \alpha, \kappa(\alpha) \rangle = 1\}, \\ \tilde{\mathcal{D}} &:= \{d \in \mathcal{D} :\ d_0 = 1\}. \end{aligned}$$

Note that the normalization $\langle \alpha, \kappa(\alpha) \rangle = 1$ is equivalent to

$$\left\langle \tilde{G}(\alpha), 1 \right\rangle = 2. \tag{4}$$

Proof: Since $\tilde{G}$ is known to be a bijection due to Theorem 2.2, it suffices to show that it is continuous and that the domain $\tilde{\mathcal{A}}$ is closed and bounded.

First we address the boundedness of $\tilde{\mathcal{A}}$. Let us take $\alpha \in \tilde{\mathcal{A}}$ and form the corresponding positive real function

$$f(z) := \frac{\kappa(\alpha(z))}{\alpha(z)}.$$

Note that the polynomials α and $\kappa(\alpha)$ can be written respectively as $\alpha_0\tilde{\alpha}$ and $\beta_0\tilde{\beta}$ using polynomials $\tilde{\alpha}$ and $\tilde{\beta}$ with constant terms equal one. Also note that we have a relation $\beta_0 = w_0\alpha_0$, since $f(z_0) = w_0$ with the assumption $z_0 = 0$. Hence, $f = w_0\tilde{\beta}/\tilde{\alpha}$.

Since f is positive real, the roots of $\tilde{\alpha}$ are in the closed unit disc. Thus, and since α_0 is taken positive, it suffices to show that α_0 is bounded from above. In order to prove the boundedness of α_0, we need to utilize the positivity of the Pick matrix. To this end, along the same line as in [9], we introduce the filter bank

$$H(z) := \begin{bmatrix} H_0(z) \\ \vdots \\ H_n(z) \end{bmatrix},$$

where $H_j(z) := 1/(1 - z_j z)$. Then the Pick matrix can be represented as

$$\begin{aligned} \Sigma &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(f + f^*)H^* d\theta \\ &= \frac{w_0}{2\pi}\int_{-\pi}^{\pi} H\left(\frac{\tilde{\alpha}}{\tau}\right)^{-1} \frac{\tilde{\alpha}\tilde{\beta}^* + \tilde{\beta}\tilde{\alpha}^*}{\tau\tau^*}\left(\frac{\tilde{\alpha}}{\tau}\right)^{-*} H^* d\theta. \end{aligned}$$

where we have suppressed the evaluation at $e^{i\theta}$, and τ is defined by $\tau(z) := \prod_{j=0}^{n}(1 - z_j z)$. Since $\tilde{\alpha}/\tau \in Span\{H_j\}$, there is a vector $k \in \mathbb{C}^{n+1}$ such that $\tilde{\alpha}/\tau = k^H H$. Therefore, and since the Pick matrix is positive definite, there is an $\varepsilon > 0$ such that

$$\begin{aligned} \alpha_0^2\varepsilon &\leq \alpha_0^2 k^H \Sigma k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\tilde{G}(\alpha)(e^{i\theta})}{\left|\tau(e^{i\theta})\right|^2} d\theta \\ &\leq \frac{M}{2\pi}\int_{-\pi}^{\pi} \tilde{G}(\alpha)(e^{i\theta}) d\theta = 2M, \end{aligned}$$

for some finite number $M \geq \max_{\theta\in[-\pi,\pi]} 1/\left|\tau(e^{i\theta})\right|^2$. Here, the finiteness of M follows from the assumption that all the interpolation points are in $\mathbb{D}$, and the last equality follows from (4). Since ε and M are independent of α_0, we have shown that α_0 is bounded from above, and thus $\tilde{\mathcal{A}}$ is bounded.

Secondly, we deal with the continuity of $\tilde{G}$. For arbitrary two elements α_1 and α_2 in $\tilde{\mathcal{A}}$, we have

$$\begin{aligned} &\tilde{G}(\alpha_1) - \tilde{G}(\alpha_2) \\ &\quad = (\alpha_1 - \alpha_2)^*\kappa(\alpha_1) + \alpha_2^*\kappa(\alpha_1 - \alpha_2) \\ &\qquad + [\kappa(\alpha_1)]^*(\alpha_1 - \alpha_2) + [\kappa(\alpha_1 - \alpha_2)]^*\alpha_2. \end{aligned}$$

Therefore, since the linear map κ is bounded and the set $\tilde{\mathcal{A}}$ is bounded, it is easy to show that there exists a constant $C > 0$ satisfying

$$\|\tilde{G}(\alpha_1) - \tilde{G}(\alpha_2)\| \leq C\|\alpha_1 - \alpha_2\|.$$

Since α_1 and α_2 are arbitrary in $\tilde{\mathcal{A}}$, we have proven the continuity of $\tilde{G}$ in $\tilde{\mathcal{A}}$.

Finally the closedness of $\tilde{\mathcal{A}}$ follows immediately by using the property of continuity and bijectivity of $\tilde{G}$ and the closedness of $\tilde{\mathcal{D}}$ [11, p. 87]. This completes the proof. ■

Proof: [Proof of Theorem 3.1] The map G can be written as a composite of three maps:

$$G = M \circ \tilde{G}_{ex} \circ N,$$

where

$$\begin{aligned} &N : \mathcal{A} \mapsto \tilde{\mathcal{A}} \times \mathbb{R}^+ :\ N(\alpha) := \begin{bmatrix} \dfrac{\alpha}{(\langle\alpha, \kappa(\alpha)\rangle)^{1/2}} \\ \langle\alpha, \kappa(\alpha)\rangle \end{bmatrix}, \\ &\tilde{G}_{ex} : \tilde{\mathcal{A}} \times \mathbb{R}^+ \mapsto \tilde{\mathcal{D}} \times \mathbb{R}^+ :\ \tilde{G}(\alpha, r) := \begin{bmatrix} \tilde{G}(\alpha) \\ r \end{bmatrix}, \\ &M : \tilde{\mathcal{D}} \times \mathbb{R}^+ \mapsto \mathcal{D} :\ M(d, r) := rd. \end{aligned}$$

The maps N and M play roles of normalization and inverse scaling, respectively. Due to Lemma 3.1, the map $\tilde{G}_{ex}$ is homeomorphic. Since the maps N and M are also homeomorphic, so is the composite map G. ■

Now we phrase a smoothness result from the theory developed for strictly positive real interpolants. Here we consider the subsets of $\mathcal{A}$ and $\mathcal{D}$ as:

$$\begin{aligned} \mathcal{A}^+ &:= \{\alpha \in \mathcal{A} : \kappa(\alpha)/\alpha \text{ is strictly positive real}\}, \\ \mathcal{D}^+ &:= \{d \in \mathcal{D} : d(z, z^{-1}) > 0\ \forall |z| = 1\}. \end{aligned}$$

The following theorem was proven by Byrnes and Lindquist:

Theorem 3.2: [4] The map $G^+ : \mathcal{A}^+ \to \mathcal{D}^+$, defined by $G^+(\alpha) := G(\alpha)$ for all $\alpha \in \mathcal{A}^+$, is a diffeomorphism.

IV. DERIVATION OF A SYSTEM OF NONLINEAR EQUATIONS

In this section we will rewrite the conditions **C1**–**C4** in terms of coefficient vectors of polynomials α and β. This will yield a system of nonlinear equations.

First, due to the condition **C1**, we can parameterize the polynomials α and β as

$$\alpha(z) := \boldsymbol{\alpha}^T \boldsymbol{z}, \quad \beta(z) := \boldsymbol{\beta}^T \boldsymbol{z},$$

where the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{z}$ are defined by

$$\begin{aligned} \boldsymbol{\alpha} &:= \begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_n \end{bmatrix}^T, \quad \alpha_0 > 0, \\ \boldsymbol{\beta} &:= \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_n \end{bmatrix}^T, \\ \boldsymbol{z} &:= \begin{bmatrix} 1 & z & \cdots & z^n \end{bmatrix}^T. \end{aligned}$$

Next, due to the condition **C2**, we have a linear relation between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as

$$\boldsymbol{\beta} = K\boldsymbol{\alpha}, \tag{5}$$

where $K := \Gamma^{-1} W \Gamma$ with

$$\Gamma := \begin{bmatrix} 1 & z_0 & \cdots & z_0^n \\ 1 & z_1 & \cdots & z_1^n \\ \vdots & \vdots & & \vdots \\ 1 & z_n & \cdots & z_n^n \end{bmatrix}, \ W = \begin{bmatrix} w_0 & & & \\ & w_1 & & \\ & & \ddots & \\ & & & w_n \end{bmatrix}.$$

Note that the Vandermonde matrix Γ is nonsingular because of the distinct assumption of $\{z_j\}$. To express the condition

C3 in terms of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we introduce a Hankel + Toeplitz operator defined for a vector.

Definition 4.1: For a vector $\boldsymbol{v} := \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix}^T$, a linear continuous operator $S : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is defined by

$$S(\boldsymbol{v}) := \begin{bmatrix} v_1 & \cdots & v_m \\ \vdots & \iddots & \\ v_m & & \end{bmatrix} + \begin{bmatrix} v_1 & \cdots & v_m \\ & \ddots & \vdots \\ & & v_1 \end{bmatrix}.$$

Remark 4.1: The continuity of the linear operator is due to [12, Lemma 1.20].

With this operator, the relation (2) can be written as $S(\boldsymbol{\alpha})\boldsymbol{\beta} = \boldsymbol{d}$ where $\boldsymbol{d} := \begin{bmatrix} 2 & d_1 & \cdots & d_n \end{bmatrix}^T$. The condition **C4** is described by means of the parameter vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as $\boldsymbol{\alpha} + \boldsymbol{\beta} \in \mathcal{S}$ where the Schur stability region $\mathcal{S}$ in $\mathbb{R}^{n+1}$ is defined by

$$\mathcal{S} := \left\{ \boldsymbol{v} \in \mathbb{R}^{n+1} : \ \boldsymbol{v}^T \boldsymbol{z} \neq 0, \forall z \in \mathbb{D} \right\}$$

From the vector expressions of the conditions **C1**–**C4**, Theorem 2.1 can be rephrased as follows.

Corollary 4.1: For each vector $\boldsymbol{d}$ in the region

$$\hat{\mathcal{D}} := \left\{ \boldsymbol{d} \in \mathbb{R}^{n+1} : \sum_{j=0}^{n} d_j (z^j + z^{-j}) \in \mathcal{D} \right\},$$

the system of nonlinear equations

$$\boldsymbol{g}(\boldsymbol{\alpha}) := S(\boldsymbol{\alpha}) K \boldsymbol{\alpha} - \boldsymbol{d} = 0 \tag{6}$$

has a unique solution in the region

$$\hat{\mathcal{A}} := \left\{ \boldsymbol{\alpha} \in \mathbb{R}^{n+1} : \ \alpha_0 > 0, \ (I + K)\boldsymbol{\alpha} \in \mathcal{S} \right\}.$$

The remaining issue is, for each specified vector $\boldsymbol{d}$ in $\hat{\mathcal{D}}$, to find the unique solution to the system of nonlinear equations (6) in $\hat{\mathcal{A}}$. However, because of the nonlinearity of $\boldsymbol{g}$, this is a non-trivial issue, and we need in general to rely on some numerical methods. In order to find the unique solution, we shall use a continuation method, which is one of the standard numerical tools for solving a system of nonlinear equations. The performance of the continuation method crucially depends on properties of the map G in (3), which were discussed in Section III.

V. A PROCEDURE BASED ON A CONTINUATION METHOD

In the following, we will introduce a homotopy that defines a trajectory to be traced numerically, discuss the properties of the trajectory crucial for the use of the continuation method, and explain the numerical method of tracing the trajectory, that is, the predictor-corrector steps.

A. Construction of a homotopy

To construct a homotopy, we notice that if $\boldsymbol{d}$ is chosen as $\boldsymbol{\tau} := [2\tau_0, \tau_1, \cdots, \tau_n]^T \in \hat{\mathcal{D}}$ which consists of coefficients of the trigonometric polynomial

$$\sum_{j=0}^{n} \tau_j (z^j + z^{-j}) := \frac{1}{M} \prod_{j=0}^{n} (1 - z_j z)(1 - z_j z)^*,$$

where M is a scaling to make $\tau_0 = 1$, then the corresponding system of nonlinear equation

$$\boldsymbol{g}_\tau(\boldsymbol{\alpha}) := S(\boldsymbol{\alpha}) K \boldsymbol{\alpha} - \boldsymbol{\tau} = 0$$

is easy to solve.

We design a convex homotopy $\boldsymbol{h} : \mathbb{R}^{n+1} \times [0,1] \to \mathbb{R}^{n+1}$ as

$$\begin{aligned} \boldsymbol{h}(\boldsymbol{\alpha}, \nu) &:= (1-\nu)\boldsymbol{g}_\tau(\boldsymbol{\alpha}) + \nu \boldsymbol{g}(\boldsymbol{\alpha}), \\ &= S(\boldsymbol{\alpha}) K \boldsymbol{\alpha} - \boldsymbol{\tau} + \nu(\boldsymbol{\tau} - \boldsymbol{d}), \quad \nu \in [0,1]. \end{aligned}$$

Note that $\boldsymbol{h}(\boldsymbol{\alpha}, 0) = \boldsymbol{g}_\tau(\boldsymbol{\alpha})$ and $\boldsymbol{h}(\boldsymbol{\alpha}, 1) = \boldsymbol{g}(\boldsymbol{\alpha})$. Therefore, $\boldsymbol{h}(\boldsymbol{\alpha}, 0) = 0$ is easy to solve, while $\boldsymbol{h}(\boldsymbol{\alpha}, 1) = 0$ is our problem at hand.

For each $\nu \in [0,1]$, the system $\boldsymbol{h}(\boldsymbol{\alpha}, \nu) = 0$ has a unique solution in $\hat{\mathcal{A}}$, due to Corollary 4.1 and the convexity of $\hat{\mathcal{D}}$. Let us denote the unique solution of the system $\boldsymbol{h}(\boldsymbol{\alpha}, \nu) = 0$ as $\hat{\boldsymbol{\alpha}}(\nu)$, and we call the class $\{\hat{\boldsymbol{\alpha}}(\nu)\}_{\nu=0}^{1}$ the *trajectory*. Our objective is to trace this implicitly defined trajectory numerically from $\nu = 0$ to $\nu = 1$, and to obtain $\hat{\boldsymbol{\alpha}}(1)$. For this purpose, we use a numerical continuation method with predictor-corrector steps. Before proceeding the exposition of the predictor-corrector steps, we shall analyze the properties of the trajectory.

B. Properties of the trajectory

To apply a continuation method to a trajectory tracing, the trajectory should enjoy some favorable properties, since otherwise, the method is likely to break down or end up with a wrong solution. For example, the trajectory should have sufficient smoothness (such as continuity and differentiability), but neither a bifurcation nor a turning point (see [1]). We shall next study these properties for the trajectory $\{\hat{\boldsymbol{\alpha}}(\nu)\}_{\nu=0}^{1}$.

First, due to Lemma 3.1 and Theorem 3.2, the property of $\{\hat{\boldsymbol{\alpha}}(\nu)\}_{\nu=0}^{1}$ concerning the smoothness follows immediately.

Proposition 5.1: The trajectory $\{\hat{\boldsymbol{\alpha}}(\nu)\}_{\nu=0}^{1}$ is continuously differentiable in the interval $[0,1)$. In addition, it is continuous at $\nu = 1$.

A direct consequence of Proposition 5.1 is the following.

Corollary 5.1: The trajectory $\{\hat{\boldsymbol{\alpha}}(\nu)\}_{\nu=0}^{1}$ does not have any turning point in $[0,1)$.

Because of the uniqueness of $\hat{\boldsymbol{\alpha}}(\nu)$ for each $\nu \in [0,1]$, we also have that

Corollary 5.2: The trajectory $\{\hat{\boldsymbol{\alpha}}(\nu)\}_{\nu=0}^{1}$ does not have any bifurcation.

The three properties above justify the use of a continuation method to trace the trajectory numerically.

Owing to the continuous differentiability of the trajectory, we can express the trajectory as a solution of an ordinary differential equation with an initial value. When we take a derivative of $\boldsymbol{h}(\hat{\boldsymbol{\alpha}}(\nu), \nu) = 0$ with respect to ν, we have

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{\alpha}}(\hat{\boldsymbol{\alpha}}(\nu), \nu) \cdot \frac{d\hat{\boldsymbol{\alpha}}}{d\nu}(\nu) + \frac{\partial \boldsymbol{h}}{\partial \nu}(\hat{\boldsymbol{\alpha}}(\nu), \nu) = 0,$$

or equivalently,

$$\frac{d\hat{\boldsymbol{\alpha}}}{d\nu}(\nu) = \boldsymbol{v}(\hat{\boldsymbol{\alpha}}(\nu)),$$
$$\boldsymbol{v}(\hat{\boldsymbol{\alpha}}(\nu)) := -\left[\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{\alpha}}(\boldsymbol{\alpha},\nu)\right]^{-1} \frac{\partial \boldsymbol{h}}{\partial \nu}(\boldsymbol{\alpha},\nu)\Bigg|_{\boldsymbol{\alpha}=\hat{\boldsymbol{\alpha}}(\nu)}.$$

The invertibility of the Jacobian $\partial \boldsymbol{h}/\partial \boldsymbol{\alpha}$ is guaranteed on the trajectory in the interval $[0,1)$ because of the differentiability of $\hat{\boldsymbol{\alpha}}$. Since we can easily compute $\hat{\boldsymbol{\alpha}}(0)$, the problem to solve is the ordinary differential equation with an initial value:

$$\begin{cases} \frac{d\hat{\boldsymbol{\alpha}}}{d\nu}(\nu) &= \boldsymbol{v}(\hat{\boldsymbol{\alpha}}(\nu)), \\ \hat{\boldsymbol{\alpha}}(0) &: \text{given.} \end{cases}$$

To solve this initial value problem numerically, we use predictor-corrector steps.

C. The predictor step

In the predictor step, given a point $\hat{\boldsymbol{\alpha}}(\nu)$ on the trajectory, we move the point to a new point as

$$\boldsymbol{\alpha}(\nu+\delta\nu) := \hat{\boldsymbol{\alpha}}(\nu) + \delta\nu \cdot \boldsymbol{v}(\hat{\boldsymbol{\alpha}}(\nu)).$$

This point may not be on the trajectory, and hence we use the notation $\boldsymbol{\alpha}(\nu+\delta\nu)$ instead of $\hat{\boldsymbol{\alpha}}(\nu+\delta\nu)$. To determine the new point $\boldsymbol{\alpha}(\nu+\delta\nu)$, we need to compute $\boldsymbol{v}(\hat{\boldsymbol{\alpha}}(\nu))$ and to provide $\delta\nu$.

The directional vector $\boldsymbol{v}(\hat{\boldsymbol{\alpha}}(\nu))$ consists of two factors: one is the inverse of $\partial \boldsymbol{h}/\partial \boldsymbol{\alpha}$, which is the Jacobian of $\boldsymbol{g}$ and can be written explicitly as

$$\nabla \boldsymbol{g}(\boldsymbol{\alpha}) = S(\boldsymbol{\alpha})K + \sum_{j=0}^{n} \alpha_j S(\boldsymbol{k}_j),$$

where $\boldsymbol{k}_j$ is the $(j+1)$-th column of the matrix K in (5). Note that, due to the differentiability of $\hat{\boldsymbol{\alpha}}$, the Jacobian is nonsingular for all ν in the interval $[0,1)$. The other factor is

$$\frac{\partial \boldsymbol{h}}{\partial \nu} = \boldsymbol{\tau} - \boldsymbol{d},$$

which is independent of both $\boldsymbol{\alpha}$ and ν, and can thus be computed off-line.

The step size must be chosen in a careful way. This is because a small step size will cause a long time to arrive at $\nu = 1$ and to obtain $\hat{\boldsymbol{\alpha}}(1)$, while a large step size might ruin the convergence of the corrector step that follows the predictor step. Next, we propose a reasonable way to make this trade-off.

On the trajectory, we have $\boldsymbol{h}(\hat{\boldsymbol{\alpha}}(\nu),\nu) = 0$, and in particular,

$$\begin{aligned} e_1^T \boldsymbol{h}(\hat{\boldsymbol{\alpha}}(\nu),\nu) &= e_1^T S(\hat{\boldsymbol{\alpha}}(\nu))K\hat{\boldsymbol{\alpha}}(\nu) - e_1^T\boldsymbol{\tau}, \\ &= 2\hat{\boldsymbol{\alpha}}(\nu)^T K\hat{\boldsymbol{\alpha}}(\nu) - 2 = 0. \end{aligned}$$

A criterion for not deviating too far from the trajectory is to require

$$1-\mu \le \boldsymbol{\alpha}(\nu+\delta\nu)^T K \boldsymbol{\alpha}(\nu+\delta\nu) \le 1+\mu,$$

for some small number $\mu > 0$. For a given direction $\boldsymbol{v}(\hat{\boldsymbol{\alpha}}(\nu))$, we can compute the maximal step length $\delta\nu_*$ as

$$(\hat{\boldsymbol{\alpha}} + \delta\nu_* \boldsymbol{v}(\hat{\boldsymbol{\alpha}}))^T K(\hat{\boldsymbol{\alpha}} + \delta\nu_* \boldsymbol{v}(\hat{\boldsymbol{\alpha}})) = 1 \pm \mu,$$

which yields

$$\delta\nu_*^2 + \delta\nu_* \underbrace{\frac{\boldsymbol{v}(\hat{\boldsymbol{\alpha}})^T K\hat{\boldsymbol{\alpha}} + \hat{\boldsymbol{\alpha}}^T K\boldsymbol{v}(\hat{\boldsymbol{\alpha}})}{\boldsymbol{v}(\hat{\boldsymbol{\alpha}})^T K\boldsymbol{v}(\hat{\boldsymbol{\alpha}})}}_{=:p} \pm \underbrace{\frac{-\mu}{\boldsymbol{v}(\hat{\boldsymbol{\alpha}})^T K\boldsymbol{v}(\hat{\boldsymbol{\alpha}})}}_{=:q} = 0.$$

We pick the smallest positive solution:

$$\delta\nu_* = \begin{cases} -\frac{p}{2} - \sqrt{\frac{p^2}{4} - |q|} & \text{if } p<0 \ \&\ \frac{p^2}{4} > |q|, \\ -\frac{p}{2} + \sqrt{\frac{p^2}{4} + |q|} & \text{otherwise.} \end{cases}$$

D. The corrector step

In the corrector step, given a point $\boldsymbol{\alpha}(\nu+\delta\nu)$ which is obtained in the predictor step, we pull the point back to the trajectory by fixing ν-value, and obtain $\hat{\boldsymbol{\alpha}}(\nu+\delta\nu)$. This is equivalent to solving the system of nonlinear equations

$$\tilde{\boldsymbol{h}}(\boldsymbol{\alpha}) := \boldsymbol{h}(\boldsymbol{\alpha}, \nu+\delta\nu) = 0.$$

We use Newton's method with an initial point $\boldsymbol{\alpha}(\nu+\delta\nu)$ to solve this system numerically. The Newton's method uses iterations

$$\boldsymbol{\alpha}_{k+1} = \boldsymbol{\alpha}_k - \nabla\tilde{\boldsymbol{h}}(\boldsymbol{\alpha}_k)^{-1} \cdot \tilde{\boldsymbol{h}}(\boldsymbol{\alpha}_k), \quad k = 0,1,2,\ldots.$$

The Jacobian $\nabla\tilde{\boldsymbol{h}}(\boldsymbol{\alpha}_k)$ is nonsingular if $\boldsymbol{\alpha}_k$ is close enough to the trajectory, due to the continuity of the Jacobian.

VI. AN ILLUSTRATING EXAMPLE

In this section, we will consider an example which indicates the potential of the proposed procedure for computing rational Nevanlinna-Pick interpolants. The example is of a quite academic character but illustrates the ability of the procedure to compute interpolants with spectral zeros on the unit circle as well as the continuity of the map G in (3).

Example 6.1: Consider the positive real function

$$f(z) = 1 + \frac{1+e^{i\theta_0}z}{1-e^{i\theta_0}z} + \frac{1+e^{-i\theta_0}z}{1-e^{-i\theta_0}z}.$$

The associated spectral density $\Phi = 2\,\mathrm{Re}\, f$ corresponds to a sinusoid in (Gaussian) white noise. Let us take $\theta_0 = 0.5$ and interpolation points at 0 and $0.5e^{\pm i\theta_0}$. We can then compute the interpolation data $\{w_k\}$. Now, for various choices of trigonometric polynomial $d(z,z^{-1})$, we will recover corresponding positive real functions $f(z)$. Associated with each interpolant is the spectral factorization $w(z)w^*(z) := f(z) + f^*(z)$. The zeros of w will also be the zeros of d. Now we choose a family of these zeros converging to the true one at $e^{i\theta_0}$. In Fig. 1, the poles and the (chosen) zeros of the spectral factor are plotted. We see that as the spectral zeros approach the unit circle, the poles approach the same points. When the zero is chosen to be exactly $e^{i\theta_0}$, we get a cancellation in the spectral factor and we have recovered the

true function f. In Fig. 2, the corresponding spectral densities Φ are plotted. We note that densities gradually tend to the singular density corresponding to the true f.

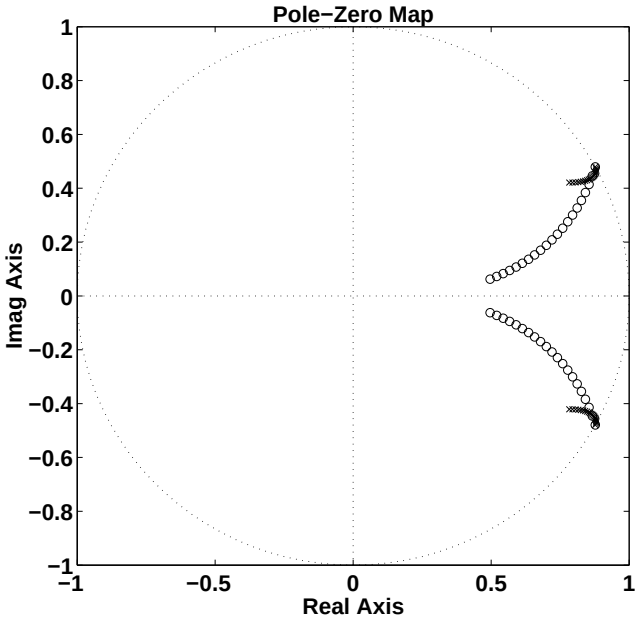

Fig. 1. The given spectral zeros (o) and corresponding poles (x) of the positive real functions consistent with the interpolation data in Example 6.1

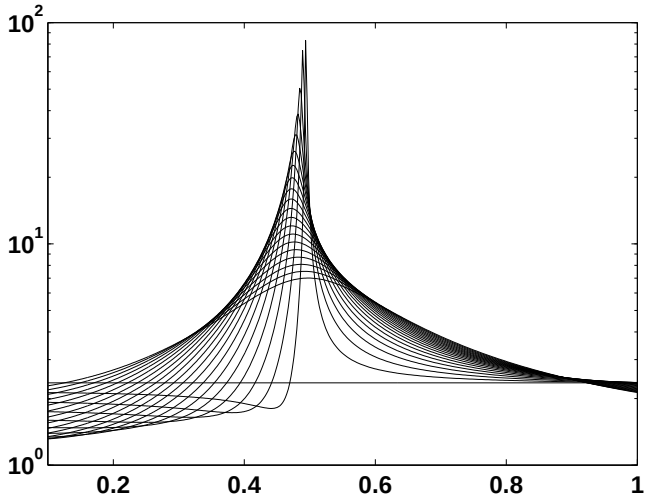

Fig. 2. The spectral densities associated with the interpolants in Example 6.1

VII. CONCLUSIONS

In this paper, we have developed a procedure to compute any real rational, positive real Nevanlinna-Pick interpolant with a degree bound. The new approach, compared to previously developed solvers based on convex optimization, enhances the numerical reliability.

Even though we can see experimentally that the developed procedure converges well, a theoretical proof of the convergence is missing. Besides, real-world engineering problems, for instance the spectral estimation problem, should be examined by utilizing the proposed procedure. Furthermore, it will be important to extend this procedure to the multivariable cases, that is, to both matrix-valued and tangential Nevanlinna-Pick interpolation problems. These will be future research subjects.

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