An extension of a Nevanlinna-Pick interpolation solver to cases including derivative constraints

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Abstract

This paper extends a previously proposed solver for scalar Nevanlinna-Pick interpolation problems with degree constraint to the ones including derivative constraints. The solver computes any real rational interpolant with a degree bound by solving an optimization problem of the same type as encountered in the problem without derivative constraint. A robust homotopy continuation method, previously devised by the second author for the problem without derivative constraint, can be applied to solve the new optimization problem.

1 Introduction

The main purpose of this paper is to develop a solver for the scalar *Nevanlinna-Pick interpolation problem with degree constraint* (we write *NPDC* for short), allowing for derivative constraints, formulated as follows. Suppose that two sets of complex numbers are given:

$$\begin{aligned}\mathcal{Z} &:= \{z_j : j = 0, 1, \ldots, n, \ z_i \neq z_j \text{ if } i \neq j\}, \\ \mathcal{W} &:= \{w_{jk} : j = 0, 1, \ldots, n, \ k = 0, 1, \ldots, m_j - 1\},\end{aligned}$$

where $\mathcal{Z} \subset \mathbb{D} := \{z : |z| < 1\}$. The problem is to determine any function f that satisfies the following:

C1 f fulfills the interpolation constraints:

$$\frac{f^{(k)}(z_j)}{k!} = w_{jk}, \qquad \begin{array}{l} j = 0, 1, \ldots, n, \\ k = 0, 1, \ldots, m_j - 1, \end{array} \tag{1}$$

C2 f is *strictly positive real*, i.e., f is analytic in an open region containing the closed unit disc $\overline{\mathbb{D}}$ and $\operatorname{Re} f(z) > 0$ for all $z \in \overline{\mathbb{D}}$,

C3 f is rational and $\deg f \leq m := \sum_{j=0}^{n} m_j - 1$.

We stress that this problem is different from the classical analytic interpolation problems because of **C3**. In applications the degree restriction is important since it corresponds to a low degree of dynamical systems. To handle degree of interpolants, the analytic interpolation theory with degree constraint that has been developed in recent years is most powerful and promising (see the survey in [7] and references therein). The theory gives a complete parameterization of the set

$$\mathcal{S}_{NPDC} := \{f : \ f \text{ satisfies } \mathbf{C1}, \mathbf{C2} \text{ and } \mathbf{C3}\},$$

whose elements smoothly depend on spectral zeros of the interpolant. It turned out that the problem of computing any function in $\mathcal{S}_{NPDC}$ amounts to solving a convex optimization problem. To solve the optimization problem in a numerically robust way, algorithms based on a homotopy continuation approach have been devised for two special cases; one presented in [11] is for Carathéodory extension with interpolation conditions $f^{(k)}(0)/k! = w_k$, $k = 0, 1, \ldots, m-1$, and the other in [17] is for Nevanlinna-Pick interpolation with interpolation conditions $f(z_j) = w_j$, $j = 0, 1, \ldots, n$.

The goal of this paper is to extend the algorithms in [11, 17] to the one that can treat more general interpolation conditions (1) directly. This extension is important since it enables us to smoothly handle the H^∞ control problem with multiple unstable poles and/or zeros in the plant, which the classical interpolation approach cannot (see [15, p. 18]).

The paper is organized as follows. In Section 2, we discuss the solvability condition for *NPDC* including derivative constraints by introducing a generalized Pick matrix. Section 3 briefly reviews the theory on *NPDC* presented in [4, 5, 8] and state a convex optimization problem for computing each interpolant. Section 4 reduces the convex optimization problem, which is difficult to solve accurately, to a non-convex optimization problem that has attractive properties. We apply the same homotopy continuation method as in [11, 17] to solve it in a numerically robust way. We give an example from the control literature to illustrate the efficiency of our solver in Section 5. Appendix explains bilinear transformations of the domain and range.

2 Solvability and a generalized Pick matrix

The classical Nevanlinna-Pick interpolation problem including derivative constraints considers interpolation conditions **C1** and a condition

C2' f is *positive real*, i.e., f is analytic in the open unit disc $\mathbb{D}$ and $\operatorname{Re} f(z) \geq 0$ for all $z \in \mathbb{D}$.

This problem is a generalization of both the classical Carathéodory extension and the classical Nevanlinna-Pick interpolation problem [18, Section 2.6],[12, p. 298].

We refer to Theorem 1 and Theorem 2 in [14] for a useful formulation of the necessary and sufficient condition for the existence of a positive real interpolant.

To present this condition, we introduce some notation. First, from the data set $\mathcal{Z}$, we construct a block diagonal matrix

$$A := \begin{bmatrix} A_0 & & \\ & \ddots & \\ & & A_n \end{bmatrix}, \; A_j := \begin{bmatrix} z_j & & & \\ 1 & z_j & & \\ & \ddots & \ddots & \\ & & 1 & z_j \end{bmatrix},$$

where each block A_j is of size $m_j \times m_j$. In addition, we define a vector as

$$b := [e_1^{m_0}, e_1^{m_1}, \cdots, e_1^{m_n}]^T, \quad e_1^{m_j} := [1, 0, \cdots, 0],$$

where $e_1^{m_j}$ is of size $1 \times m_j$. Next, from the data set $\mathcal{W}$, introduce another block diagonal matrix

$$W := \begin{bmatrix} W_0 & & & \\ & W_1 & & \\ & & \ddots & \\ & & & W_n \end{bmatrix},$$

$$W_j := \begin{bmatrix} w_{j0} & & & \\ w_{j1} & w_{j0} & & \\ \vdots & \ddots & \ddots & \\ w_{j,m_j-1} & \cdots & w_{j1} & w_{j0} \end{bmatrix}.$$

Theorem 2.1 *There exists an interpolant for the Nevanlinna-Pick interpolation problem including derivative constraints if and only if a Hermitian matrix*

$$P := WE + EW^H \tag{2}$$

is nonnegative definite. Here, the matrix E is a unique positive definite solution to the Lyapunov equation:

$$E - AEA^H = bb^T. \tag{3}$$

The matrix P in (2) is called a *generalized Pick matrix.* If P is nonnegative definite but singular, the interpolant is unique and it is not strictly positive real. On the other hand, if P is positive definite, there exist infinitely many interpolants. The class of these interpolants can be represented in a linear fractional transformation form with a free H^∞ function whose H^∞ norm is bounded by one [19]. In particular, this class contains strictly positive real functions of degree at most m whenever P is positive definite. One of such functions is the so-called *central solution* which can be obtained by setting the free H^∞ function equal zero. Consequently, we have the following fact.

Corollary 2.2 *NPDC including derivative constraints is solvable if and only if P is positive definite.*

3 Review of the theory for *NPDC*

In this section, we will briefly review the theory for *NPDC* [5, 4, 8]. In particular, we follow the approach by Byrnes and Lindquist in [8], where the interpolation problem was seen as a generalized moment problem. The main results there are that the set $\mathcal{S}_{NPDC}$ is completely parameterized in terms of spectral zeros of the interpolant, and that the computation of each interpolant amounts to solving a convex optimization problem. We will present these results next.

Hereafter, we assume that (**A1**) P is positive definite, (**A2**) $z_0 = 0$ and (**A3**) $(\bar{z}_j, \bar{w}_{jk})$ is in $\mathcal{Z} \times \mathcal{W}$ whenever (z_j, w_{jk}) is. (**A1**) guarantees that the set $\mathcal{S}_{NPDC}$ is nonempty, due to Corollary 2.2, (**A2**) is for mathematical convenience, and (**A3**) is assumed since it leads to *real* interpolants which are relevant to applications.

The complete parameterization of the set $\mathcal{S}_{NPDC}$ in the generality of the present paper was given by Byrnes and Lindquist in [8] as follows.

Theorem 3.1 *[9, 13, 5, 4, 8] There is a bijective map between the set of pairs of real polynomials*

$$\left\{ (\alpha, \beta) : \begin{array}{l} \deg \alpha \le m, \; \deg \beta \le m, \\ \alpha(0) \ne 0, \; \beta(0) \ne 0, \end{array} \; f := \frac{\beta}{\alpha} \in \mathcal{S}_{NPDC} \right\},$$

and the set of real Schur polynomials

$$\{\rho : \; \deg \rho = m, \; \rho(z) \ne 0, \forall\, |z| \ge 1\}.$$

The bijectivity implies that the Schur polynomials are the characterizing factor of the set $\mathcal{S}_{NPDC}$. The computation of an interpolant f from ρ amounts to an optimization problem $\min_{q \in \mathcal{Q}_+} \mathbb{J}_\rho(q)$, where

$$\mathbb{J}_\rho(q) := \langle q + q^*, w + w^* \rangle - \left\langle \log(q + q^*), \frac{\rho\rho^*}{\tau\tau^*} \right\rangle. \tag{4}$$

Here, w is an arbitrary function in $\mathcal{H}^2$ that satisfies interpolation conditions (1), $\tau := \prod_{j=1}^n (1 - \bar{z}_j z)^{m_j}$ is a fixed polynomial depending on the data $\mathcal{Z}$, and the domain $\mathcal{Q}_+$ is defined by

$$\mathcal{Q}_+ := \left\{ q : \begin{array}{l} \text{real rational, } q \in \operatorname{span}\{G_{z_j,k}\}_{j,k} \\ q(z) + q(z^{-1}) > 0, \; \forall\, |z| = 1 \end{array} \right\},$$

where $G_{pk}(z) := z^k / (1 - \bar{p}z)^{k+1}$. For two functions f and g in $\mathcal{L}^2$, the inner product is defined by $\langle f, g \rangle := \int_{-\pi}^{\pi} f^*(e^{i\theta}) g(e^{i\theta}) d\theta / 2\pi$, where $f^*(z) := \overline{f(\bar{z}^{-1})}$.

This optimization problem is convex, that is, the domain $\mathcal{Q}_+$ is a convex region and the objective function $\mathbb{J}_\rho$ is a strictly convex function. After obtaining the minimizer q in $\mathcal{Q}_+$, the real polynomials α and β can be calculated respectively by spectral factorization:

$$q(z) + q(z^{-1}) = a(z)a(z^{-1}), \quad a(z) := \frac{\alpha(z)}{\tau(z)}, \tag{5}$$

and by solving a system of linear equations:

$$\alpha(z)\beta(z^{-1}) + \alpha(z^{-1})\beta(z) = \rho(z)\rho(z^{-1}). \qquad (6)$$

The first breakthrough about the same type of convex optimization to (4) was done by Byrnes, Gusev and Lindquist in [6] (see also [7]) for the Carathéodory (covariance) extension problem with degree constraint, followed by the work for the Nevanlinna-Pick interpolation problem with degree constraint in [4, 5]. However, optimization solvers proposed there were not quite robust numerically. Especially, the solvers are not able to obtain interpolants with poles in close vicinity of the unit circle accurately. Such interpolants are often required in applications, and hence the solvers needed to be modified. To remove the drawback, the solvers have been modified with a homotopy continuation method by Enqvist for Carathéodory extension in [11], followed by the second author for Nevanlinna-Pick interpolation in [17]. We will take the same approach as these results.

4 A new optimization problem for *NPDC* including derivative constraints

The approach taken in [11, 17] first translates the optimization problem (4) with respect to q into an optimization problem with respect to α by substituting the relation (5) into the objective function in (4). By using the same idea, we will transform the optimization problem $\min_{q\in\mathcal{Q}_+} \mathbb{J}_\rho(q)$ into another optimization problem with respect to a real vector

$$\boldsymbol{\alpha} := [\alpha_0, \alpha_1, \cdots, \alpha_m]^T \in \mathbb{R}^{m+1}, \qquad (7)$$

which consists of coefficients of a polynomial $\alpha(z) =: \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m$. When we substitute (5) into the function $\mathbb{J}_\rho(q)$ in (4), we obtain a function of $\boldsymbol{\alpha}$ as

$$\hat{J}_\rho(\boldsymbol{\alpha}) := \langle aa^*, w + w^*\rangle - \left\langle \log aa^*, \frac{\rho\rho^*}{\tau\tau^*}\right\rangle, \qquad (8)$$

where $a(z) := \alpha(z)/\tau(z)$.

The first term in (8) can be written as a quadratic form of $\boldsymbol{\alpha}$ containing the generalized Pick matrix. To prove this, we first state the following proposition that represents the first term of $\hat{J}_\rho$ as a function of a complex vector $\boldsymbol{\gamma} \in \mathbb{C}^{m+1}$, where

$$\boldsymbol{\gamma} := \left[\boldsymbol{\gamma}_0^T, \cdots, \boldsymbol{\gamma}_n^T\right]^T, \ \boldsymbol{\gamma}_j = \left[\gamma_{j0}, \cdots, \gamma_{j,m_j-1}\right]^T,$$

and the scalars γ_{jk} are defined by

$$a(z) =: \sum_{j=0}^{n}\sum_{k=0}^{m_j-1} \gamma_{jk} G_{z_j,k}(z). \qquad (9)$$

In addition, this proposition implies that $\hat{J}_\rho$ is independent of the choice of w.

Proposition 4.1 *For any $w \in \mathcal{H}^2$ which satisfies the interpolation constraints, the following holds:*

$$\langle aa^*, w + w^*\rangle = \boldsymbol{\gamma}^H P \boldsymbol{\gamma}, \qquad (10)$$

where the matrix P is the generalized Pick matrix defined in (2), and the vector $\boldsymbol{\gamma}$ is defined in (9).

Proof: Since $\langle aa^*, w + w^*\rangle = \langle a, aw\rangle + \overline{\langle a, aw\rangle}$, we examine only the term $\langle a, aw\rangle$. Using the expression (9), this term is transformed as

$$\langle a, aw\rangle = \sum_{j=0}^{n}\sum_{k=0}^{m_j-1}\sum_{p=0}^{n}\sum_{q=0}^{m_p-1} \bar{\gamma}_{jk}\gamma_{pq}\left\langle G_{z_j,k}, wG_{z_p,q}\right\rangle.$$

Using the relation $\langle G_{pk}, f\rangle = f^{(k)}(p)/k!$, we have

$$\left\langle G_{z_k,j}, wG_{z_p,q}\right\rangle = [w_{kj}, \cdots, w_{k0}] \begin{bmatrix} \left\langle G_{z_k,0}, G_{z_p,q}\right\rangle \\ \vdots \\ \left\langle G_{z_k,j}, G_{z_p,q}\right\rangle \end{bmatrix}.$$

By using this equation, we can derive $\langle a, aw\rangle = \boldsymbol{\gamma}^H WE\boldsymbol{\gamma}$, where

$$E := \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{i\theta})G^*(e^{i\theta})d\theta\right)^T,$$

and $G(z) := (I - z\bar{A})^{-1}b$. The matrix E is the controllability Gramian that is a positive definite solution of the Lyapunov equation $E - AEA^H = bb^T$. The other half becomes $\overline{\langle a, aw\rangle} = \boldsymbol{\gamma}^H EW^H\boldsymbol{\gamma}$. ■

Now, we clarify the relation between vectors $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$ in (7) in order to transform (10) into a function of $\boldsymbol{\alpha}$. To this end, we express τ by $\tau(z) =: 1 + \tau_1 z + \cdots + \tau_m z^m$, with $\tau_k = 0$ for all $k \geq m - m_0 + 2$. By the similar arguments to [17, Lemma 3.1], we can derive the linear relation between $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$ as

$$\boldsymbol{\alpha} = L_m V\boldsymbol{\gamma}. \qquad (11)$$

Here, the $(m+1)\times(m+1)$ nonsingular matrices L_m and V are written as

$$L_m := \begin{bmatrix} 1 & & & \\ \tau_1 & 1 & & \\ \vdots & \ddots & \ddots & \\ \tau_m & \cdots & \tau_1 & 1 \end{bmatrix}, \quad V := [V_0\ V_1\ \cdots\ V_n],$$

where the $(m+1)\times m_k$ block matrices V_k is defined by

$$V_k := \begin{bmatrix} 1 & & & \\ \bar{z}_k & 1 & & \\ \bar{z}_k^2 & \binom{2}{1}\bar{z}_k & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ \vdots & \vdots & & \binom{m_k}{1}\bar{z}_k \\ \vdots & \vdots & & \vdots \\ \bar{z}_k^m & \binom{m}{m-1}\bar{z}_k^{m-1} & & \binom{m}{m+1-m_k}\bar{z}_k^{m+1-m_k} \end{bmatrix}.$$

The notation $\binom{n}{k}$ means the binomial coefficients.

Due to Proposition 4.1 and the relation (11), the first term of $\hat{J}_\rho$ has a representation with the vector $\boldsymbol{\alpha}$ as

$$\langle aa^*, w + w^* \rangle = \boldsymbol{\alpha}^T K \boldsymbol{\alpha},$$

where $K := L_m^{-T} V^{-H} P V^{-1} L_m^{-1}$. Note that K is positive definite due to the assumption (**A1**) and the non-singularity of L_m and V.

Since α and τ are real polynomials, the second term of $\hat{J}_\rho$ in (8) becomes

$$\left\langle \log aa^*, \frac{\rho\rho^*}{\tau\tau^*} \right\rangle = \left\langle 2\log\alpha, \frac{\rho\rho^*}{\tau\tau^*} \right\rangle - \left\langle 2\log\tau, \frac{\rho\rho^*}{\tau\tau^*} \right\rangle .$$

Hence, the function $\hat{J}_\rho$ in (8) can be written by

$$\hat{J}_\rho(\boldsymbol{\alpha}) = \boldsymbol{\alpha}^T K \boldsymbol{\alpha} - 2\left\langle \log\alpha, \frac{\rho\rho^*}{\tau\tau^*} \right\rangle + 2\left\langle \log\tau, \frac{\rho\rho^*}{\tau\tau^*} \right\rangle ,$$

where $\boldsymbol{\alpha}$ should be in the region:

$$\mathcal{S}_m := \left\{ \boldsymbol{\alpha} : \begin{array}{l} \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m \neq 0, \forall z \in \overline{\mathbb{D}} \\ \alpha_0 > 0 \end{array} \right\}.$$

Since the last term does not contain α, the new optimization problem equivalent to the original one is

$$\min_{\boldsymbol{\alpha} \in \mathcal{S}_m} J_\rho(\boldsymbol{\alpha}), \quad J_\rho(\boldsymbol{\alpha}) := \boldsymbol{\alpha}^T K \boldsymbol{\alpha} - 2\left\langle \log\alpha, \frac{\rho\rho^*}{\tau\tau^*} \right\rangle .$$

Since K is positive definite, this is exactly the same kind of problem that has appeared in [11, 17]. More precisely, the function J_ρ has a unique stationary point in the open region $\mathcal{S}_m$, and it is locally strictly convex around the unique stationary point. Thus, we can use the same homotopy continuation method (see [1]) as in [11, 17] to solve this optimization problem. The details of the optimization has been presented in [17]. With the optimal $\boldsymbol{\alpha}$, the interpolant f is calculated by solving the system of linear equations (6) with respect to β.

5 A control example

This example is taken from a book written by Doyle, Francis & Tannenbaum (DFT) [10, Section 10.3 & 12.4]. A plant is given as

$$P(s) = \frac{-6.4750s^2 + 4.0302s + 175.7700}{s(5s^3 + 3.5682s^2 + 139.5021s + 0.0929)},$$

that is, it has one unstable pole at the origin and unstable zeros at infinity (multiplicity 2) and 5.5308. For this plant, our goal is to design a strictly proper controller $C(s)$ in the feedback structure of Figure 1, fulfilling internal stability and, for a step reference $r(t)$, settling time of at most 8 seconds, overshoot of at most 10 % and control signal $u(t)$ of at most magnitude 0.5. Our approach is to find an appropriate sensitivity function

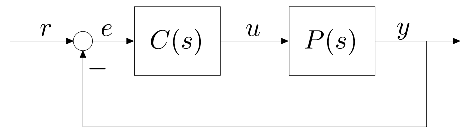


Figure **1:** The closed-loop system

$S := 1/(1 + PC)$ so that the closed-loop system fulfills all the specifications. Afterwards, we calculate the corresponding controller C.

For internal stability, S must be in RH^∞ (the set of real rational proper stable functions) and satisfy the interpolation conditions:

$$S(0) = 0,\ S(\infty) = 1,\ S'(\infty) = 0,\ S(5.5308) = 1.$$

In order to design a strictly proper controller, we require PC to have another zero at infinity. This is achieved by imposing another constraint $S''(\infty) = 0$. We note that this derivative constraint can easily be incorporated in our procedure. In addition, our procedure requires an upper bound of the H^∞ norm.

An admissible set of degree-bounded interpolants is

$$\mathcal{S}_{NPDC} := \left\{ S \in RH^\infty : \begin{array}{l} S(0) = 0,\ S(\infty) = 1, \\ S'(\infty) = S''(\infty) = 0, \\ S(5.5308) = 1, \\ \|S\|_\infty < \gamma,\ \deg S \leq 4 \end{array} \right\}.$$

The degree bound of S is a consequence of having the total number of interpolation constraints as five. This corresponds to a controller degree of at most $\deg P - 1 = 4$ (see [16, Proposition 4.1]). This also implies that we get four spectral zeros as design parameters. Note that this set of constraints includes derivative constraints, and hence we cannot use the solver in [17] directly. In addition, we apply bilinear transformations to both domain and range of S in order for the interpolation problem to become the form in this paper. See Appendix A.

In the frequency domain, time domain specifications correspond to the sensitivity function (see [10, p. 181]):

$$S_{ideal}(s) := \frac{s(s + 1.2)}{s^2 + 1.2s + 1}. \qquad (12)$$

Therefore, we will aim at obtaining an interpolant in the set $\mathcal{S}_{NPDC}$ which has similar frequency domain characteristics to that of S_{ideal}, by tuning the spectral zeros. We choose the value γ comparable to that of S_{ideal}.

We choose the design parameters based on the follow reasoning. We set an H^∞ norm $\gamma = 1.8$ causing the peak of the sensitivity function to be slightly lower than that of the DFT-design. We put two spectral zeros at $s = \pm 1.7i$ since we want the peak of the magnitude of the sensitivity at that frequency (see [16] for the shaping technique). We place the remaining two spectral zeros at $s = 7$ and $s = \infty$, bringing the magnitude of u down. However, it should be noted that the effect of spectral zeros away from the imaginary axis is quite unclear. The resulting controller is

$$C_{NPDC}(s) = \frac{12.63s^3 + 9.016s^2 + 352.5s + 0.2347}{s^4 + 20.15s^3 + 139.2s^2 + 448.8s + 650.7}.$$

We compare the closed-loop performance of the controller C_{NPDC} with that of the controller in [10, Section 12.4]. The performance of the different designs, both in the frequency and time domain, is summarized in Table 1 and Figure 2. We have clearly found an at least as good design but of the half degree compared with the DFT-design.

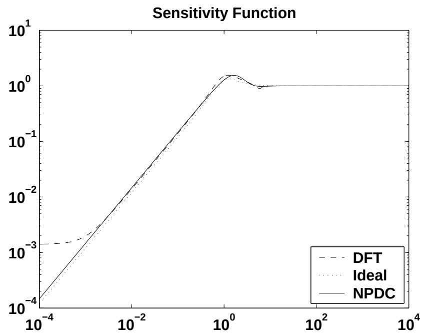
Table **1:** The time and frequency domain performance.

	DFT	**NPDC**
Controller degree	8	4
Peak Gain	1.56	1.55
Bandwidth (Hz)	0.48	0.52
Rise Time (sec)	1.55	1.46
Overshoot	1.11	1.02
Settling Time (sec)	5.41	2.49
Max $\lvert u \rvert$	0.48	0.48

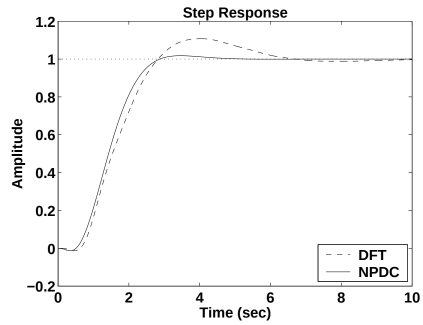
6 Conclusions

In this paper, we have shown that the Nevanlinna-Pick interpolation problem with degree constraint including derivative constraints can be treated in the same framework as the problem without derivative constraint. We can obtain each rational strictly positive real interpolant by solving the same kind of optimization problem as the one that appears in the plain Nevanlinna-Pick interpolation, as well as Carathéodory extension, with degree constraint. The major difference is in the construction of a positive definite matrix in the objective function, which contains the generalized Pick matrix. We have demonstrated that the solver is quite convenient for a sensitivity shaping in control when a plant has multiple unstable poles/zeros, yielding derivative constraints, as well as when we want to design a strictly proper controller of low degree. A MATLAB implementation of the solver is available at `http://www.math.kth.se/~andersb/software.html`.

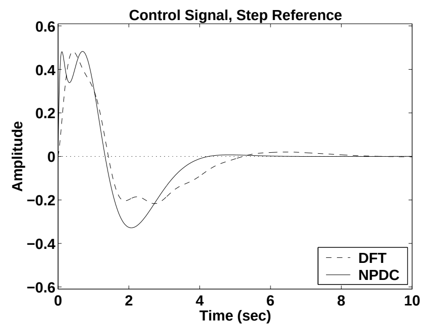
The solver developed in this paper is applicable only for *scalar* problems. However, to treat multivariable control problems by a similar interpolation approach, it is important to consider the problem dealing with the classical multivariable (matrix-valued or bitangential) analytic interpolation (see e.g. [2, 12, 18]) plus certain complexity constraint. One such generalization will be presented in a forthcoming paper [3].



(a) Bode diagram of S



(b) Step responses $y(t)$



(c) Control signals $u(t)$

Figure **2:** DFT-design and our design.

A Transformation of domain and range

The Nevanlinna-Pick problem considered in this paper is assumed to find the mapping f from the unit disc into the right half-plane, even though problems originating from applications may have different domain and/or range. In addition, the first interpolation point is assumed to be origin in our formulation, which might not always be the case. However, these assumptions are without loss of generality, since the linear fractional (bilinear) transformation $u = (az+b)/(cz+d)$, $ad-bc \neq 0$, can be applied to both the variable and the function in order for domain and range to have the desired form.

Furthermore, one of the interpolation points can always be transformed to the origin. The next two lemmas provide formula of the interpolation constraints after bilinear transformations of domain and range.

Lemma A.1 *Under the bilinear transformation $u(z) := (az+b)/(cz+d)$, $ad-bc \neq 0$, the interpolation constraints (1) of f transform to those for the function $g(u) := f(z(u))$ with $z(u) = (-du+b)/(cu-a)$ as*

$$\frac{g^{(k)}(u_j)}{k!} = v_{jk}, \qquad \begin{array}{l} j = 0,1,\ldots,n, \\ k = 0,1,\ldots,m_j-1, \end{array}$$

where for each j, $u_j = (az_j+b)/(cz_j+d)$ and

$$v_{jk} = \begin{cases} w_{j0}, \\ \displaystyle \frac{1}{k!}\sum_{l=1}^{k} s_k^l w_{j,k-l+1}(k-l+1)!(z')^{k-l}z^{(l)}\Big|_{z=z(u_j)}, \\ \qquad\qquad k = 1,2,\ldots,m_j-1. \end{cases}$$

The coefficients s_k^l fulfills the recursive formula: $s_k^1 = s_k^k = 1$, $k = 1,\ldots,m_j-1$, and

$$s_{k+1}^l = \frac{2k-l+2}{l}s_k^{l-1} + s_k^l, \quad l = 2,\ldots,k.$$

The term $z^{(l)}(u_j)$ is obtained by

$$z^{(l)}(u_j) = (-1)^{l+1}l!c^{l-1}\frac{ad-bc}{(cu_j-a)^{l+1}}, \quad l = 1,\ldots,k.$$

Lemma A.2 *Under the bilinear transformation $g(z) := (af(z)+b)/(cf(z)+d)$, $ad-bc \neq 0$, the interpolation constraints (1) of f transform to those of the function g as*

$$\frac{g^{(k)}(z_j)}{k!} = v_{jk}, \qquad \begin{array}{l} j = 0,1,\ldots,n, \\ k = 0,1,\ldots,m_j-1, \end{array}$$

where for each $j = 0,1,\ldots n$,

$$v_{jk} := \begin{cases} \dfrac{aw_{j0}+b}{cw_{j0}+d}, \\ \displaystyle \frac{1}{k!}\frac{1}{cw_{j0}+d}\left(ak!w_{jk} - \sum_{l=0}^{k-1}\binom{k}{l}cl!(k-l)!v_{jl}w_{j,k-l}\right), \\ \qquad\qquad k = 1,2,\ldots,m_j-1. \end{cases}$$

Acknowledgments

The authors would like to thank their advisor Prof. A. Lindquist for valuable comments. This research was partly supported by a grant from the Swedish Research Council (VR).

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