Deriverings regler

1. $(f \pm g)'(x) = f'(x) \pm g'(x)$

$$\lim_{\Delta x \to 0} \frac{(f+g)(x+\Delta x) - (f+g)(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) + g(x+\Delta x) - f(x) - g(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x) + g(x+\Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

2. $(cf)'(x) = c\,f'(x)$

$$\lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x} = c \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

3 $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{(f(x+\Delta x) - f(x))\,g(x+\Delta x) + f(x)(g(x+\Delta x) - g(x))}{\Delta x}$$

$$= \underbrace{\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}_{\downarrow \; f'(x)} \cdot \underbrace{g(x+\Delta x)}_{\downarrow \; g(x)} + \underbrace{\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}}_{\downarrow \; g'(x)} f(x)$$

4. $$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{(g(x))^2}, \quad g(x) \neq 0$$

$$\lim_{\Delta x \to 0} \frac{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{g(x)f(x+\Delta x) - g(x+\Delta x)f(x)}{g(x+\Delta x)g(x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - g(x+\Delta x)f(x)}{\Delta x} \frac{1}{g(x+\Delta x)g(x)}$$

$$= \lim_{\Delta x \to 0} \left(g(x) \underbrace{\frac{f(x+\Delta x) - f(x)}{\Delta x}}_{\downarrow \atop f'(x)} + f(x) \underbrace{\frac{g(x) - g(x+\Delta x)}{\Delta x}}_{\downarrow \atop -g'(x)} \right) \underbrace{\frac{1}{g(x+\Delta x)g(x)}}_{\downarrow \atop g(x)}$$

5 $\quad (f \circ g)'(x) = f'(g(x))\, g'(x)$

$$\lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x) - g(x)} \; \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

g kontinuerlig $\Rightarrow$ $g(x+\Delta x) - g(x) \to 0$

$$y = g(x)$$

$$\Delta y = g(x+\Delta x) - g(x)$$

$$\lim_{\Delta y \to 0} \underbrace{\frac{f(y+\Delta y) - f(y)}{\Delta y}}_{f'(y) = f'(g(x))} \cdot \frac{\Delta y}{\Delta x}$$

$$6. \quad (g^{-1})'(x) = \frac{1}{g'(g^{-1}(x))}$$

$g^{-1}(x+\Delta x) = y + \Delta y$

$x + \Delta x = g(y+\Delta y)$

$\Delta x = (x+\Delta x) - x$

$$\lim_{\Delta x \to 0} \frac{g^{-1}(x+\Delta x) - g^{-1}(x)}{\Delta x} = \left[\begin{array}{ll} y = g^{-1}(x) & \Leftrightarrow \; x = g(y) \\ \Delta y = g^{-1}(x+\Delta x) - g^{-1}(x) & \Delta x = g(y+\Delta y) - g(y) \end{array}\right]$$

$$= \lim_{\Delta y \to 0} \frac{\Delta y}{g(y+\Delta y) - g(y)} = \frac{1}{g'(y)} = \frac{1}{g'(g^{-1}(x))}$$

V: vet att

$$g \circ g^{-1}(x) = x$$

VL deriverat blir enl (5) $\quad g'(g^{-1}(x))(g^{-1})'(x)$

HL deriverat blir 1

$$g'(g^{-1}(x))(g^{-1})'(x) = 1$$

Deriveringsregler för elementära funktioner

7. $DC = 0$ $\quad C$ konstant

8. $Dx^{\alpha} = \alpha x^{\alpha - 1}$

9. $Da^{x} = \ln a \; a^{x}$

speciellt $De^{x} = e^{x}$

10. $D \sin x = \cos x$

$D \cos x = -\sin x$

$D \cos x = D \sin\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right) \cdot 1$

$= -\sin x$

$$D \tan x = \frac{1}{\cos^2 x}$$

$$* \quad D\,\frac{\sin x}{\cos x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$D \cot x = -\frac{1}{\sin^2 x}$$

$$* \quad D\,\frac{\cos x}{\sin x} = \frac{\sin x(-\sin x) - \cos x \cos x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

11. $D \log_a x = \dfrac{1}{\ln a\, x}$ speciellt $D \ln x = \dfrac{1}{x}$

$\log_a x$ invers funktion till a^x så

$$D \log_a x = \frac{1}{\ln a\, a^{\log_a x}} = \frac{1}{\ln a\, x}$$

17. $D \arcsin x = \dfrac{1}{\sqrt{1-x^2}}$

$$\left(D\arcsin x = \frac{1}{D\sin(\arcsin x)} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}} \right)$$

$D \arccos x = -\dfrac{1}{\sqrt{1-x^2}}$

$$\left(D\arccos x = \frac{1}{D\cos(\arccos x)} = -\frac{1}{\sin(\arccos x)} = -\frac{1}{\sqrt{1-x^2}} \right)$$

$$D \arctan x = \frac{1}{1+x^2}$$

$$\left(D \arctan x = \frac{1}{D \tan(\arctan x)} = \frac{1}{\frac{1}{\cos^2(\arctan x)}} = \right.$$

$$\left. = \cos^2(\arctan x) = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2} \right)$$

$$D \operatorname{arccot} x = -\frac{1}{1+x^2}$$

Övn: Visa det.

Bevis av 8: $Dx^\alpha = \alpha x^{\alpha-1}$

$$Dx^\alpha = D e^{\alpha \ln x} = e^{\alpha \ln x} \cdot D(\alpha \ln x) = \alpha e^{\alpha \ln x} D \ln x$$

$$= \alpha e^{\alpha \ln x} \frac{1}{x} = \alpha x^{\alpha-1}$$

Bevis av $D\sin x = \cos x$

$$\lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \sin \Delta x \cos x - \sin x}{\Delta x}$$

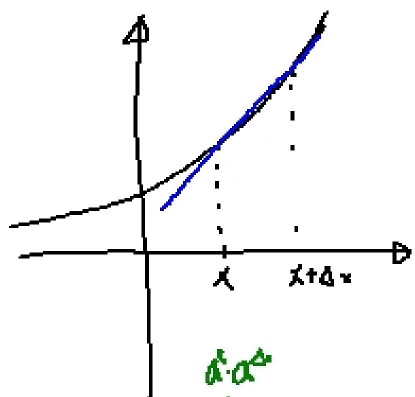
$$= \lim_{\Delta x \to 0} \frac{\sin x (\cos \Delta x - 1) + \cos x \sin \Delta x}{\Delta x}$$

$$= \underbrace{\lim_{\Delta x \to 0} \sin x \left(\frac{\cos \Delta x - 1}{\Delta x}\right)}_{=0} + \underbrace{\lim_{\Delta x \to 0} \cos x \frac{\sin \Delta x}{\Delta x}}_{= \cos x}$$

$$\to \frac{\cos \Delta x - 1}{\Delta x} = \frac{-\sin^2 \frac{\Delta x}{2}}{\Delta x} = -\underbrace{\sin \frac{\Delta x}{2}}_{\to 0} \underbrace{\frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}}}_{\to 1} \frac{1}{2} \to 0$$

$$Da^x = \ln a \; a^x$$

Vi antar $a > 1$



Grafen för a^x är konvex (K.?) (grafen ligger under de rita linjerna) så lutning $\dfrac{a^{x+\Delta x} - a^x}{x+\Delta x - x}$ är växande.

$$\Rightarrow \lim_{\Delta x \to 0^+} \frac{\overset{a^x \cdot a^{\Delta x}}{\overset{\parallel}{a^{x+\Delta x}}} - a^x}{\Delta x} \quad \text{och} \quad \lim_{\Delta x \to 0^-} \frac{a^{x+\Delta x} - a^x}{\Delta x} \quad \text{existerar}$$

$$\lim_{\Delta x \to 0^+} \frac{a^{x+\Delta x} - a^x}{\Delta x} = a^x \lim_{\Delta x \to 0^+} \frac{a^{\Delta x} - 1}{\Delta x}$$

$$\lim_{\Delta x \to 0^-} \frac{a^{\Delta x} - 1}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{a^{-\Delta x} - 1}{-\Delta x} = \lim_{\Delta x \to 0^+} a^{-\Delta x}\left(\frac{1 - a^{\Delta x}}{-\Delta x}\right) =$$

$$= \lim_{\Delta x \to 0^+} \underbrace{a^{-\Delta x}}_{\downarrow\, 1} \left(\frac{a^{\Delta x} - 1}{\Delta x}\right) = \lim_{\Delta x \to 0^+} \frac{a^{\Delta x} - 1}{\Delta x}$$

Vänstergränsvärdet lika med högergränsvärdet, dvs a^x är deriverbar.

$$\lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = \left[\begin{array}{l} s = a^{\Delta x} - 1 \\ \Delta x = \log_a(1+s) \\ \Delta x \to 0,\ s \to 0 \end{array}\right] = \lim_{s \to 0} \frac{s}{\log_a(1+s)} =$$

$$= \lim_{s \to 0} \frac{1}{\log_a (1+s)^{\frac{1}{s}}} = \frac{1}{\log_a\Big(\underbrace{\lim_{s \to 0} (1+s)^{\frac{1}{s}}}_{=e}\Big)} = \frac{1}{\log_a e} = \ln a$$