Ex: MacLaurinutveckling
av sin x till ordning 1.

$$f(x) = \overbrace{f(0) + f'(0)\,x}^{\text{grad } 1} + f''(\xi)\frac{x^2}{2}$$

där $\xi$ ligger mellan 0 och x.

$$\sin x = 0 + \cos 0 \cdot x + (-\sin\xi)\frac{x^2}{2}$$
$$= x - \sin\xi\,\frac{x^2}{2}$$

$$R_1(x) = -\sin\xi\,\frac{x^2}{2} \quad \left(\xi \text{ kan bero av } x\right)$$

$$|R_1(x)| = \left|-\sin\xi\,\frac{x^2}{2}\right| \leq \frac{x^2}{2} \quad \left(\text{för } |-\sin\xi| \leq 1\right)$$

# Ordo räkning

I exemplet hade vi $|f''(\xi)| \leq 1$

Om $|f''(\xi)| \leq C$, $|R_1(x)| = \left|f''(\xi)\frac{x^2}{2}\right| \leq C\frac{x^2}{2}$

Def av ordo: $f(x) = O((x-a)^n) \iff \left|\frac{f(x)}{(x-a)^n}\right| \leq C$

när $x$ går mot $a$

Ex: $\sin x = x + O(x^2)$ $\quad \sin x = x + \left(-\sin\xi\frac{x^2}{2}\right)$

Ex: $\lim\limits_{x\to 0}\frac{\sin x}{x} = \lim\limits_{x\to 0}\frac{x+O(x^2)}{x} = \lim\limits_{x\to 0} 1+O(x) = 1$

Ex: $e^x = 1 + x + \ldots + \frac{x^n}{n!} + O(x^{n+1})$ $\quad f^{(n+1)}(\xi)\frac{x^{n+1}}{(n+1)!}$

Enligt MacLaurin

(Taylorutveckling i $x=0$)

$$e^x = e^0 + xDe^x\Big|_{x=0} + \ldots + \underbrace{\frac{x^n}{n!}}D^n e^x\Big|_{x=0} + D^{n+1}e^x\Big|_{x=\xi}\frac{x^{n+1}}{(n+1)!}$$

$$= 1 + e^0 x + \ldots + e^0\frac{x^n}{n!} + e^{\xi}\frac{x^{n+1}}{(n+1)!}$$

$$= 1 + \ldots + \frac{x^n}{n!} + O(x^{n+1})$$

Ex: $\sin x = x - \frac{x^3}{3!} + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \mathcal{O}(x^{2n+1})$

$D^{4n} \sin x \big|_{x=0} = 0 \Leftarrow \quad D^{4n} \sin x = \sin x$

$D^{1+4n} \sin x \big|_{x=0} = 1 \quad D^{1+4n} \sin x = \cos x$

$D^{2+4n} \sin x \big|_{x=0} = 0 \quad D^{2+4n} \sin x = -\sin x$

$D^{3+4n} \sin x \big|_{x=0} = -1 \quad D^{3+4n} \sin x = -\cos x$

eftersom vi tar MacLaurin utvecklingen ($\sin 0 = 0$)

Alla koefficienter för jämna potenser blir 0. Koefficienterna framför $x^{1+4n}$ blir $\frac{1}{(1+4n)!}$ och de framför $x^{3+4n}$ blir $\frac{-1}{(3+4n)!}$.

$$\cos x = 1 - \frac{x^2}{2!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + O(x^{2n+2})$$

$$D \cos x = -\sin x \Rightarrow D^{1+4n} \cos x = -\sin x$$

$$D^2 \cos x = -\cos x \Rightarrow D^{2+4n} \cos x = -\cos x$$

$$D^3 \cos x = \sin x \Rightarrow D^{3+4n} \cos x = \sin x$$

$$D^4 \cos x = \cos x \Rightarrow D^{4n} \cos x = \cos x$$

När vi stoppar in $x = 0$

$$D^{4n} \cos x \Big|_{x=0} = \cos 0 = 1, \quad D^{2+4n} \cos x \Big|_{x=0} = -\cos 0 = -1$$

$$D^{1+4n} \cos x \Big|_{x=0} = -\sin 0 = 0, \quad D^{3+4n} \cos x \Big|_{x=0} = \sin 0 = 0$$

Ex: $(1+x)^{\alpha} = 1 + \alpha x + \alpha(\alpha-1)\frac{x^2}{2!} + O(x^3)$

$D(1+x)^{\alpha} = \alpha(1+x)^{\alpha-1}$

$D^2(1+x)^{\alpha} = D\left(\alpha(1+x)^{\alpha-1}\right) = \alpha(\alpha-1)(1+x)^{\alpha-2}$

$(1+x)^{\alpha} = (1+0)^{\alpha} + D(1+x)^{\alpha}\Big|_{x=0} x + D^2(1+x)^{\alpha}\Big|_{x=0}\frac{x^2}{2!} + O(x^3)$

$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \ldots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + O(x^{n+1})$

Specialfall: $\alpha = 2$

$O(x^{n+1})$

$(1+x)^2 = 1 + 2x + \frac{2\cdot 1}{2}x^2 + 0$

Sats (MacLaurin utvecklingens entydighet)

Om $f$ har derivator av godtycklig ordning och

$$f(x) = p(x) + O(x^{n+1})$$

där $p(x)$ har grad högst $n$, så är $p(x)$ MacLaurin polynomet av ordning $n$ till $f$.

Bevis: Om $q(x)$ är MacLaurin polynomet till $f$ får vi $0 = f(x) - f(x) = q(x) + O(x^{n+1}) - p(x) + O(x^{n+1})$

så $q(x) - p(x) = O(x^{n+1})$

men $\deg(q(x) - p(x)) \leq n$

polynom grad

så $q(x) - p(x) = 0$.

Ex: $e^{2x} = (e^x)^2 = \left(1 + x + \frac{x^2}{2} + O(x^3)\right)^2 =$

$= 1 + 2x + x^2 + 2\frac{x^2}{2} + O(x^3)$

$= 1 + 2x + 2x^2 + O(x^3)$

Enligt satsen är detta MacLaurin-utvecklingen av $e^{2x}$ till ordning 2.

Ex: Bestäm MacLaurinutvecklingen av $\sin x^2$ till ordning 14.

$$\sin x^2 = \left[t = x^2\right] = \sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + O(t^8)$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + O(x^{16})$$

Ex: $\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + O(x^{n+1})$

(Övn: Visa detta!)

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \left[t = -x\right] = \frac{1}{1-t} = 1 + t + t^2 + \ldots t^n + O(t^{n+1})$$

$$= 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + O(x^{n+1})$$

Ex: MacLaurin utveckla $e^{\sin x}$ till ordning 3

$$e^{\sin x} = \left[t = \sin x\right] = e^{t} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + O(t^4)$$

$$= 1 + \left(x - \frac{x^3}{3!} + O(x^5)\right) + \frac{\left(x - \frac{x^3}{3!} + O(x^5)\right)^2}{2!} +$$

$$+ \frac{\left(x - \frac{x^3}{3!} + O(x^5)\right)^3}{3!} + O(x^4) = 1 + x - \frac{x^3}{3!} +$$

$$+ \frac{x^2}{2} + \frac{x^3}{3!} + O(x^4) = 1 + x + \frac{x^2}{2} + O(x^4)$$

Test:

$$e^{\sin x} = e^{\sin 0} + De^{\sin x}\Big|_{x=0} x + D^2 e^{\sin x}\Big|_{x=0} \frac{x^2}{2} + D^3 e^{\sin x}\Big|_{x=0} \frac{x^3}{3!} +$$

$$+ O(x^4) = 1 + \left(e^{\sin x}\cos x\right)\Big|_{x=0} x + \Big(\left(De^{\sin x}\right)\cos x +$$

$$+ e^{\sin x} D(\cos x)\Big)\Big|_{x=0} \frac{x^2}{2} + D\Big(\left(De^{\sin x}\right)\cos x + e^{\sin x} D\cos x\Big)\Big|_{x=0} \frac{x^3}{3!} +$$

$$+ O(x^4) = 1 + x + \left(e^{\sin 0}\cos^2 0 + e^{\sin 0}(-\sin 0)\right)\frac{x^2}{2} +$$

$$+ \Big(e^{\sin x}\cos^3 x + e^{\sin x} 2\cos x(-\sin x) + e^{\sin x}\cos x(-\sin x) +$$

$$+ e^{\sin x}(-\cos x)\Big)\Big|_{x=0} \frac{x^3}{3!} + O(x^4) = 1 + x + \frac{x^2}{2} + 0\frac{x^3}{3!} + O(x^4)$$