Sats: (L'Hôpital)

Om f och g är kontinuerliga och deriverbara i ett intervall kring $x=a$ och $g'(x) \neq 0$ i det intervallet och $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, då är

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

om det senare gränsvärdet existerar.

Motivation: $\lim_{x\to a} \frac{f(x)}{g(x)} \quad \lim_{x\to a} \frac{f(a)+f'(a)(x-a)+O((x-a)^2)}{g(a)+g'(a)(x-a)+O((x-a)^2)}$

$$= \lim_{x\to a} \frac{0+f'(a)(x-a)+O((x-a)^2)}{0+g'(a)(x-a)+O((x-a)^2)} =$$

$$= \lim_{x\to a} \frac{f'(a)+O((x-a))}{g'(a)+O((x-a))} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Ex: $\lim\limits_{x\to 0} \frac{\sin x}{x} = \lim\limits_{x\to 0} \frac{D\sin x}{Dx} =$

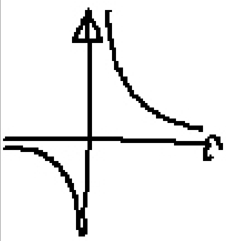
$= \lim\limits_{x\to 0} \frac{\cos x}{1} = 1$

Ex: $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \frac{D(1-\cos x)}{D\,x^2} =$

$= \lim\limits_{x\to 0} \frac{\sin x}{2x} = \lim\limits_{x\to 0} \frac{D\sin x}{D2x} =$

$= \lim\limits_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$

Varianter



I: $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \text{"}\frac{0}{0}\text{"} = \lim\limits_{x\to\infty} \frac{f'(x)}{g'(x)}$

$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \left[\begin{array}{l} x = \frac{1}{t} \\ x\to\infty \\ t\to 0^+ \end{array}\right] = \lim\limits_{t\to 0^+} \frac{f(\frac{1}{t})}{g(\frac{1}{t})} =$

$= \lim\limits_{t\to 0^+} \frac{f'(\frac{1}{t})(-\frac{1}{t^2})}{g'(\frac{1}{t})(-\frac{1}{t^2})} = \lim\limits_{t\to 0^+} \frac{f'(\frac{1}{t})}{g'(\frac{1}{t})} = \lim\limits_{x\to\infty} \frac{f'(x)}{g'(x)}$

Ex: $\lim\limits_{x\to\infty} x\sin\frac{1}{x} = \lim\limits_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \left[\begin{array}{l} x = \frac{1}{t} \\ x\to\infty \\ t\to 0^+ \end{array}\right]$

$= \lim\limits_{t\to 0^+} \frac{\sin t}{t} = 1$

$$\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{D\sin\frac{1}{x}}{D\frac{1}{x}} =$$

$$= \lim_{x\to\infty}\frac{\cos\frac{1}{x}\cdot\left(-\cancel{\frac{1}{x^2}}\right)}{-\cancel{\frac{1}{x^2}}} = \lim_{x\to\infty}\frac{\cos\frac{1}{x}}{1} = 1$$

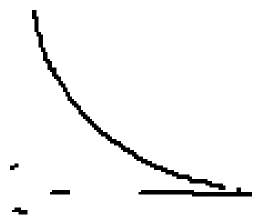
II: $\lim_{x\to a}\frac{f(x)}{g(x)} = \text{"}\frac{\infty}{\infty}\text{"} \Rightarrow \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

Ex: $$\lim_{x\to\infty}\frac{x^2+2x-1}{2x^2+3x+4} = \lim_{x\to\infty}\frac{D(x^2+2x-1)}{D(2x^2+3x+4)} =$$

$$= \lim_{x\to\infty}\frac{2x+2}{4x+3} = \lim_{x\to\infty}\frac{D(2x+2)}{D(4x+3)} = \lim_{x\to\infty}\frac{2}{4} = \frac{1}{2}$$

OBS! Att gränsvärdet

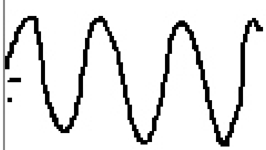
$\lim_{x\to a}\frac{f'(x)}{g'(x)}$ inte existerar



betyder inte nödvändigtvis

att $\lim_{x\to a}\frac{f(x)}{g(x)}$ ej existerar.

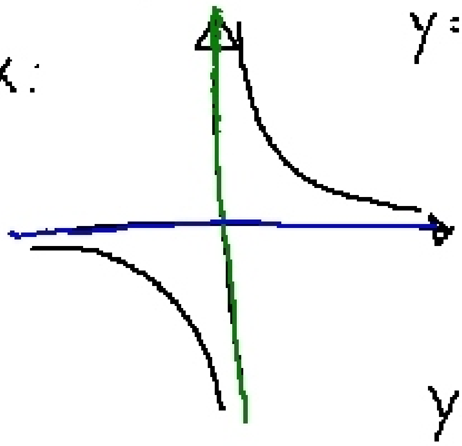
Ex: $$\lim_{x\to\infty}\frac{x+\sin x}{x} = \lim_{x\to\infty} 1 + \overbrace{\frac{\sin x}{x}}^{\to 0} = 1$$



$\lim_{x\to\infty}\frac{D(x+\sin x)}{Dx} = \lim_{x\to\infty}\frac{1+\cos x}{1}$ existerar ej

Asymptoter

Ex:

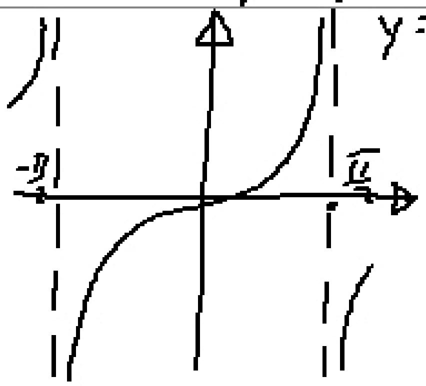


$y = \frac{1}{x}$

$x = 0$ är en lodrät asymptot

$y = 0$ är en sned asymptot

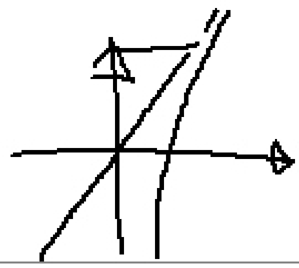
Def: Linjen $x = c$ kallas lodrät asymptot till kurvan $y = f(x)$ om $\lim_{x \to c^+} f(x) = \pm\infty$ eller $\lim_{x \to c^-} f(x) = \pm\infty$



$y = \tan x$

Kurvan har lodräta asymptoter i $x = \frac{\pi}{2}$ och $x = -\frac{\pi}{2}$.

Def: Linjen $y = ax + b$ kallas sned asymptot till $y = f(x)$ om $\lim_{x \to \pm\infty} f(x) - b - ax = 0$

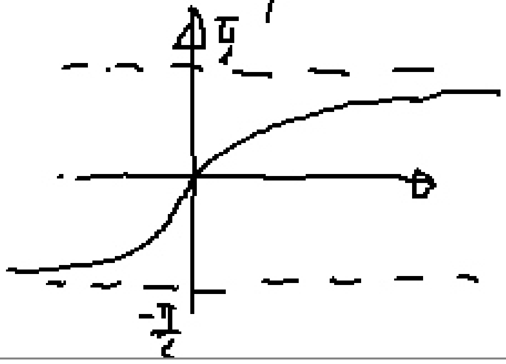


Ex. Linjen $y=0$ är en sned asymptot till $y=\frac{1}{x}$.

$$\lim_{x\to\infty} \frac{1}{x} - 0 = 0$$

$$\lim_{x\to-\infty} \frac{1}{x} - 0 = 0$$

Ex: $y = \arctan x$



$y=\frac{\pi}{2}$ är en sned asymptot:

$$\lim_{x\to+\infty} \arctan x - \frac{\pi}{2} = 0$$

Ex: Linjen $y = x+1$ är en sned asymptot till $y=\sqrt{x^2+2x}$:

$$\lim_{x\to\infty} \sqrt{x^2+2x} - x - 1 =$$

$$\lim_{x\to\infty} \frac{\left(\sqrt{x^2+2x} - x - 1\right)\left(\sqrt{x^2+2x} + x + 1\right)}{\sqrt{x^2+2x} + x + 1} =$$

$$= \lim_{x\to\infty} \frac{x^2+2x-(x+1)^2}{\sqrt{x^2+2x}+x+1} = \lim_{x\to\infty} \frac{\cancel{x^2}+\cancel{2x}-\cancel{x^2}-\cancel{2x}-1}{\sqrt{x^2+2x}+x+1}$$

$$= \lim_{x\to\infty} \frac{-1}{x\left(\sqrt{1+\underbrace{\frac{2}{x}}_{\to 0}}+1+\underbrace{\frac{1}{x}}_{\to 0}\right)} = 0$$

$$\lim_{x \to \infty} f(x) - ax - b = 0$$

$$\lim_{x \to \infty} \frac{f(x)}{x} - a = 0 \Rightarrow$$

$$\Rightarrow \quad a = \lim_{x \to \infty} \frac{f(x)}{x}$$

Vi får sedan

$$b = \lim_{x \to \infty} f(x) - ax$$

Ex: $\lim_{x \to \infty} \frac{\sqrt{x^2+2x}}{x} = \lim_{x \to \infty} \frac{x\sqrt{1+\frac{2}{x}}}{x} = 1$ $\quad \frac{2}{x} \to 0$

$\lim_{x \to \infty} \sqrt{x^2+2x} - x = 1 \Rightarrow x+1$ sned asymptot

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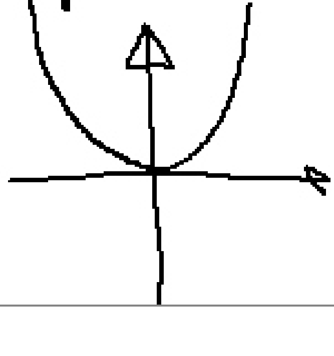
Ex: Bestäm alla sneda asymptoter till $y = x^2$.

$$\lim_{x \to \infty} \frac{x^2}{x} = \lim_{x \to \infty} x = \infty$$

$$\lim_{x \to -\infty} \frac{x^2}{x} = \lim_{x \to -\infty} x = -\infty$$

$\Rightarrow$ Det finns inga sneda asymptoter till $y = x^2$



Ex: $y^2 - x^2 = 1$ hyperbel

$y = \sqrt{1+x^2}$

Har kurvan några sneda asymptoter?

$$\lim_{x \to \infty} \frac{\sqrt{1+x^2}}{x} = \lim_{x \to \infty} \frac{\cancel{x}\sqrt{\frac{1}{x^2}+1}}{\cancel{x}} = 1$$

$$\lim_{x \to \infty} \sqrt{1+x^2} - x = \lim_{x \to \infty} \frac{(\sqrt{1+x^2} - x)(\sqrt{1+x^2} + x)}{(\sqrt{1+x^2} + x)}$$

$$= \lim_{x \to \infty} \frac{1+x^2-x^2}{\sqrt{1+x^2}+x} = \lim_{x \to \infty} \frac{1}{\sqrt{1+x^2}+x} = 0$$

$\Rightarrow y = x$ är en sned asymptot.

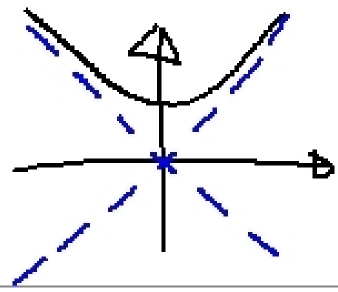
$$\lim_{x \to -\infty} \frac{\sqrt{1+x^2}}{x} = \lim_{x \to -\infty} \frac{\sqrt{x^2(1+\frac{1}{x^2})}}{x} =$$

$$= \lim_{x \to -\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}}{x} = -1$$

$$\lim_{x \to -\infty} \sqrt{1+x^2} + x = \lim_{x \to -\infty} \frac{(\sqrt{1+x^2}+x)(\sqrt{1+x^2}-x)}{\sqrt{1+x^2} - x}$$

$$= \lim_{x \to -\infty} \frac{1+x^2-x^2}{\sqrt{1+x^2} - x} = \lim_{x \to -\infty} \frac{1}{\sqrt{1+x^2} - x} = 0$$

$\Rightarrow y = -x$ är en sned asymptot



$y = \sqrt{1+x^2}$

$x^3 - y^3 = (x-y)(x^2+xy+y^2)$

8.15 g: $\quad x^3 - y^3 + 6x^2 = 0$

$$y = (x^3+6x^2)^{\frac{1}{3}}$$

$$\lim_{x\to\infty} \frac{(x^3+6x^2)^{\frac{1}{3}}}{x} = \lim_{x\to\infty} \frac{x\left(1+\frac{6}{x}\right)^{\frac{1}{3}}}{x} = 1$$

$$\lim_{x\to\infty} (x^3+6x^2)^{\frac{1}{3}} - x =$$

$$= \lim_{x\to\infty} \frac{\left((x^3+6x^2)^{\frac{1}{3}} - x\right)\left((x^3+6x^2)^{\frac{2}{3}} + (x^3+6x^2)^{\frac{1}{3}}x + x^2\right)}{(x^3+6x^2)^{\frac{2}{3}} + (x^3+6x^2)^{\frac{1}{3}}x + x^2}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^3}+6x^2-\cancel{x^3}}{(x^3+6x^2)^{\frac{2}{3}} + (x^3+6x^2)^{\frac{1}{3}}x + x^2} =$$

$$= \lim_{x\to\infty} \frac{6\cancel{x^2}}{\cancel{x^2}\left(\left(1+\underset{\downarrow\,0}{\frac{6}{x}}\right)^{\frac{2}{3}} + \left(1+\underset{\downarrow\,0}{\frac{6}{x}}\right)^{\frac{1}{3}} + 1\right)} = 2$$

$x+2$ är en sned asymptot.