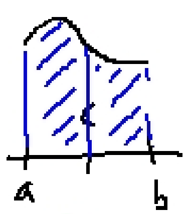
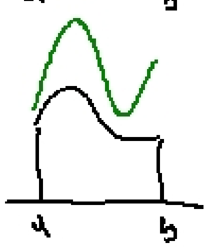
Sats Låt f och g vara två integrerbara funktioner på intervallet $[a,b]$. Då gäller

1) $\displaystyle\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$



2) $\displaystyle\int_a^b C f(x)\,dx = C\int_a^b f(x)\,dx$ $\quad C$ konstant

3) $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$



4) Om $f(x)\geq g(x)$ för alla x i intervallet $[a,b]$ så är $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

5) $\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$

$(\; |a+b| \leq |a| + |b| \;)$

Ex: $\int \sin^2 x \, dx$ (Primitiva funktioner till $\sin^2 x$)

$$\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx =$$

$$= \int \frac{1}{2} \, dx - \int \frac{\cos 2x}{2} \, dx = \frac{1}{2} \int 1 \, dx - \frac{1}{2} \int \cos 2x \, dx$$

$$= \frac{x}{2} - \frac{1}{2} \frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

Ex: $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

$$\left(D \ln|f(x)| = \frac{f'(x)}{f(x)} \qquad D \cos x = -\sin x \right)$$

$$\int \frac{\sin x}{\cos x} \, dx = -\int \frac{D \cos x}{\cos x} \, dx = -\ln|\cos x| + C$$

T ex $\int e^{ax^2} \, dx$ går ej att lösa elementärt

Ex: $\displaystyle\int_0^2 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_0^2 \frac{2x}{1+x^2}\,dx =$

$\left(\quad D(1+x^2) = 2x \quad \right)$

$\displaystyle = \frac{1}{2}\int_0^2 \frac{D(1+x^2)}{1+x^2}\,dx = \frac{1}{2}\Big[\ln|1+x^2|\Big]_0^2 =$

$\displaystyle = \frac{1}{2}\left(\ln(1+2^2) - \ln(1+0^2)\right) = \frac{1}{2}\ln 5 - 0$

$\displaystyle = \frac{1}{2}\ln 5.$

Substitution

$$D(f \circ g)(x) = f'(g(x))\, g'(x)$$

$$\int_a^b f'(g(x))\, g'(x)\, dx = \int_a^b D(f \circ g)(x)\, dx =$$

$$= \Big[f \circ g(x) \Big]_a^b = f(g(b)) - f(g(a)) = \Big[f(t) \Big]_{g(a)}^{g(b)}$$

$$= \int_{g(a)}^{g(b)} f'(t)\, dt$$

Ex: $\int_0^{\pi} \sin^3 x \, dx = \int_0^{\pi} \sin x \sin^2 x \, dx =$

$= \int_0^{\pi} \sin x \left(1 - \cos^2 x\right) dx = \int_0^{\pi} \sin x \, dx +$

$+ \int_0^{\pi} (-\sin x) \cos^2 x \, dx = \Big[-\cos x\Big]_0^{\pi} + \int_0^{\pi} \cos^2 x (-\sin x) \, dx$

$= 2 + \int_0^{\pi} \underbrace{\cos^2 x}_{g(x)^2} \underbrace{D \cos x}_{g'(x)} \, dx = \begin{bmatrix} g(x) = \cos x \\ f(x) = \frac{x^3}{3} \\ f'(x) = x^2 \end{bmatrix} =$

$= 2 + \int_0^{\pi} f'(g(x)) \, g'(x) \, dx = 2 + \Big[f(g(x))\Big]_0^{\pi} =$

$$= 2 + \Big[f(g(x))\Big]_0^{\pi} = 2 + \left[\frac{\cos^3 x}{3}\right]_0^{\pi} =$$

$$= 2 + \frac{\cos^3 \pi}{3} - \frac{\cos^3 0}{3} = 2 - \frac{1}{3} - \frac{1}{3} =$$

$$= 2 - \frac{2}{3} = \frac{4}{3}.$$

Alternativt

$$\int_0^{\pi} \cos^2 x \, D\cos x \, dx = \left[\begin{array}{l} t = \cos x \\ dt = D\cos x \, dx \end{array}\right] = \int_1^{-1} t^2 \, dt =$$

$$= -\int_{-1}^{1} t^2 dt = -\left[\frac{t^3}{3}\right]_{-1}^{1} = -\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}$$

Ex: $$\int_0^1 x e^{x^2} dx = \int_0^1 e^{x^2} \frac{D(x^2)}{2} dx$$

$\frac{dt}{dx} = 2x$

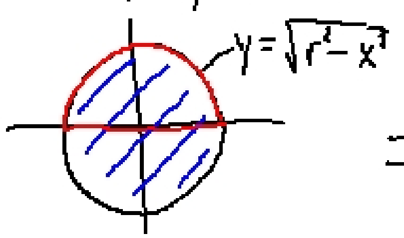
$$= \begin{bmatrix} t = x^2 \\ dt = 2x\,dx \end{bmatrix} = \frac{1}{2} \int_0^1 e^t \, dt = \frac{1}{2} \left[e^t \right]_0^1 =$$

$$= \frac{1}{2}\left(e^1 - e^0\right) = \frac{e}{2} - \frac{1}{2}$$

Ex: Beräkna arean av cirkelskivan

$x^2 + y^2 \leq r^2$

$$\text{Arean} = 2 \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx =$$



$$= \begin{bmatrix} x = r\cos\alpha \\ dx = r(-\sin\alpha)\,d\alpha \end{bmatrix} =$$

$$= 2 \int_{\pi}^{0} \sqrt{r^2 - r^2 \cos^2 \alpha}\,(-r \sin \alpha)\,d\alpha$$

$$= 2 \int_{\pi}^{0} r^2 \sqrt{1 - \cos^2 x}\,(-\sin x)\,d\alpha =$$

$$= 2r^2 \int_{\pi}^{0} \sin \alpha\,(-\sin \alpha)\,d\alpha =$$

$$= 2r^2 \left(- \int_{\pi}^{0} \sin^2 \alpha\,d\alpha\right) = 2r^2 \int_{0}^{\pi} \sin^2 \alpha\,d\alpha$$

$$= 2r^2 \left[\frac{\alpha}{2} - \frac{\sin 2\alpha}{4}\right]_0^{\pi} = 2r^2 \left(\frac{\pi}{2} - \frac{0}{2}\right) = \pi r^2$$

Partiell integration

$$D\big(f(x)\cdot g(x)\big) = f'(x)g(x) + f(x)g'(x)$$

$$\int_a^b f'(x)g(x) + f(x)g'(x)\,dx = \int_a^b D\big(f(x)\cdot g(x)\big)\,dx =$$

$$= \big[f(x)g(x)\big]_a^b$$

$$\int_a^b f'(x)g(x)\,dx = \big[f(x)g(x)\big]_a^b - \int_a^b f(x)g'(x)\,dx.$$

Ex: $\int_0^1 x^3 e^{x^2} dx = \int_0^1 (xe^{x^2}) x^2 dx =$

$= \left[\frac{e^{x^2}}{2} x^2\right]_0^1 - \int_0^1 \frac{e^{x^2}}{2} 2x\, dx =$

$= \left(\frac{e^1 \cdot 1^2}{2} - 0\right) - \left[\frac{1}{2} e^{x^2}\right]_0^1 =$

$= \frac{e}{2} - \frac{e}{2} + \frac{1}{2} = \frac{1}{2}$

Ex: $\int \ln|x|\, dx = \int 1 \cdot \ln|x|\, dx = x \ln|x| - \int x \frac{1}{x} dx =$

$= x \ln|x| - x + C$

Ex: $\int_0^{\frac{\pi}{4}} x \sin x \, dx = \left[-x\cos x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} -\cos x \, dx =$

$= -\frac{\pi}{4}\cos\frac{\pi}{4} + \left[\sin x\right]_0^{\frac{\pi}{4}} = -\frac{\pi}{4}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} =$

$= \frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{4}\right).$

Ex: $\int e^{2x} \sin 3x \, dx = \frac{e^{2x}}{2}\sin 3x - \int \frac{e^{2x}}{2} 3\cos 3x \, dx =$

$= \frac{e^{2x}}{2}\sin 3x - \frac{3}{2}\left(\frac{e^{2x}}{2}\cos 3x - \int \frac{e^{2x}}{2}(-3\sin 3x)\, dx\right) =$

$= \frac{e^{2x}}{2}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}\int e^{2x}\sin 3x \, dx$

$\Rightarrow \int e^{2x}\sin 3x \, dx = \frac{4}{13}\left(\frac{e^{2x}}{2}\sin 3x - \frac{3}{4}e^{2x}\cos 3x\right).$

427 c) Visa att $\cos x \geq 1 - \frac{x^2}{2}$ för alla x.

Lösn: $f(x) = \cos x - 1 + \frac{x^2}{2}$

$f'(x) = -\sin x + x \qquad f'(x) = 0$

$\Rightarrow x = 0$

$f'(x) \quad - - - - \overset{0}{|} + + +$

(lokalt) min

$f(0) = 1 - 1 = 0$

$\Rightarrow f(x) \geq 0 \Rightarrow \cos x \geq 1 - \frac{x^2}{2}$.