Visa att $\cos x \geq 1 - \frac{x^2}{2}$ för alla x

$f(x) = \cos x - 1 + \frac{x^2}{2}$

$f(0) = 1 - 1 + \frac{0^2}{2} = 0$

Vill visa $f(x) \geq 0$

$f'(x) = -\sin x + x$

$f'(x) = 0 \Rightarrow x = 0$

Vi använde $|x| \geq |\sin x|$

$f''(x) = -\cos x + 1 \geq 0$

$\Rightarrow f'(x)$ växande

$\Rightarrow$ $f'(x) \leq 0$ $x < 0$

$f'(x) \geq 0$ $x > 0$

$\Rightarrow x = 0$ minpunkt

$\Rightarrow f(x) \geq 0$.

Fråga: Kan man inte använda att $\cos x = 1 - \frac{x^2}{2} + O(x^4)$?

x^4 är ju ≥ 0. $\underbrace{D^4 \cos \}}_{\text{Fungkar}\ \leq 0} \frac{x^4}{4!}$

Problem

$O(x^4)$ betyder att resten går som x^4, dvs att resten är en funktion $\rho(x)$ sådan att

$$\left|\frac{\rho(x)}{x^4}\right| \leq C \qquad \rho(x) \sim Cx^4$$

$$-x^4$$

Rationella integranden

Ex: $\int \frac{x^2-2x+3}{x-1}\,dx = \int (x-1) + \frac{2}{x-1}\,dx$

$$\begin{array}{r l}
 & x - 1 \\
x-1 \,\big) & x^2 - 2x + 3 \\
- & x^2 - x \\
 & -x + 3 \\
- & -x + 1 \\
 & 2
\end{array}$$

$= \int x\,dx - \int 1\,dx + \int \frac{2}{x-1}\,dx$

$= \frac{x^2}{2} - x + 2\ln|x-1| + C$

$D\ln|x| = \frac{1}{x}$ $\quad x>0$ $\; D\ln x = \frac{1}{x}$

$x<0$ $\; D\ln|x| = D\ln(-x) = \frac{-1}{-x} = \frac{1}{x}$

Ex: $\int \frac{x+5}{x^2+x-2}\,dx = \int \frac{x+5}{(x-1)(x+2)}\,dx$

partialbråksuppdelning

$$\frac{x+5}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$= \frac{(A+B)x + (2A-B)}{(x-1)(x+2)}$$

$$\Rightarrow \begin{cases} A+B=1 \\ 2A-B=5 \end{cases} \Rightarrow B = 1-A$$

$$2A-(1-A)=5 \Rightarrow 3A=6 \Rightarrow A=2$$

$$\Rightarrow B=-1$$

$$= \int \frac{2}{x-1} + \frac{-1}{x+2}\,dx = 2\ln|x-1| - \ln|x+2| + C$$

Ex: $\int \frac{x^2}{x^2+x-2}\,dx = \int 1 + \frac{-x+2}{x^2+x-2}\,dx$

$$\int \frac{x^2+x-2-x+2}{x^2+x-2}\,dx =$$

$$= x + \int \frac{-x+2}{x^2+x-2}\,dx = x + \int \frac{\frac{1}{3}}{x-1} - \frac{\frac{4}{3}}{x+2}\,dx$$

$$= x + \frac{1}{3}\ln|x-1| - \frac{4}{3}\ln|x+2| + C$$

$$\frac{-x+2}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} = \frac{(A+B)x + (2A-B)}{(x-1)(x+2)}$$

$$\Rightarrow \begin{cases} A+B = -1 \\ 2A-B = 2 \end{cases} \Rightarrow \begin{array}{l} B = -1-A \\ 2A+1+A = 2 \end{array} \Rightarrow A = \frac{1}{3} \qquad B = -1-\frac{1}{3} = -\frac{4}{3}$$

$c \neq d \qquad \dfrac{ax+b}{(c+x)(x+d)} = \dfrac{A}{x+c} + \dfrac{B}{x+d}$

$$= \frac{A(x+d)+B(x+c)}{(x+c)(x+d)} = \frac{(A+B)x+(Ad+Bc)}{(x+c)(x+d)}$$

$$\Rightarrow \begin{cases} A+B=a \\ Ad+Bc=b \end{cases} \Rightarrow \quad \begin{aligned} &B=a-A \\ &Ad+ac-Ac=b \Rightarrow A=\frac{b-ac}{d-c} \end{aligned}$$

$$\Rightarrow B = a-A = a-\frac{(b-ac)}{(d-c)} = \frac{a(d-c)-b+ac}{d-c} = \frac{ad-b}{d-c} \qquad \text{OBS! } d-c\neq 0$$

$$A=\frac{b-ac}{d-c} = \frac{ax+b}{\cancel{(x+c)}(x+d)}\Bigg|_{x=-c}$$

$$B=\frac{ad-b}{d-c} = \frac{ax+b}{(x+c)\cancel{(x+d)}}\Bigg|_{y=-d}$$

$$A=\frac{1+5}{1+2}=2,\quad B=\frac{-2+5}{-2-1}=-1 \qquad \frac{x+5}{(x-1)(x+2)} = \frac{2}{x-1}+\frac{-1}{x+2}$$

Ex: $\int \frac{x}{(x-1)^2}\,dx = \int \frac{x-1+1}{(x-1)^2}\,dx =$

$= \int \frac{dx}{x-1} + \int \frac{1}{(x-1)^2}\,dx = \ln|x-1| - \frac{1}{x-1} + C$

$\left(\int \frac{1}{(x-1)^2}\,dx = \begin{bmatrix} t = x-1 \\ dt = dx \end{bmatrix} = \int \frac{1}{t^2}\,dt = -\frac{1}{t} = -\frac{1}{x-1} \right)$

$\frac{x}{(x-1)^2} = \frac{1}{x-1} + \frac{1}{(x-1)^2}$

$\frac{ax+b}{(x+c)^2} = \frac{A}{x+c} + \frac{B}{(x+c)^2} = \frac{A(x+c)+B}{(x+c)^2} = \frac{Ax+(Ac+B)}{(x+c)^2}$

$\Rightarrow A = a\,,\ B = b - ac$

Ex: $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + C$

Ex: $\displaystyle\int \frac{x-3}{1+x^2}\,dx = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx - 3\int \frac{1}{1+x^2}\,dx$

$\displaystyle = \frac{1}{2}\ln\left(1+x^2\right) - 3\arctan x + C$

Ex: $\displaystyle\int \frac{1+x}{x^2-2x+5}\,dx = \int \frac{x+1}{(x-1)^2+4}\,dx = \frac{1}{4}\int \frac{x+1}{\left(\frac{x-1}{2}\right)^2+1}\,dx$

(Vi börjar med att söka rötterna till ekvationen $x^2-2x+5=0$. $x = 1 \pm \sqrt{1-5} = 1 \pm 2i$)

$$\frac{1}{4}\int \frac{x+1}{\left(\frac{x-1}{2}\right)^2+1}\,dx = \left[\begin{array}{l} t = \frac{x-1}{2} \\ x = 2t+1 \\ dt = \frac{1}{2}\,dx \end{array}\right] =$$

$$= \frac{1}{4}\int \frac{2t+2}{1+t^2}\,2\,dt = \int \frac{t+1}{1+t^2}\,dt =$$

$$= \frac{1}{2}\int \frac{2t}{1+t^2}\,dt + \int \frac{1}{1+t^2}\,dt =$$

$$= \frac{1}{2}\ln\left(1+t^2\right) + \arctan t + C =$$

$$= \frac{1}{2}\ln\left(1+\left(\frac{x-1}{2}\right)^2\right) + \arctan\left(\frac{x-1}{2}\right) + C$$

Ex: $\displaystyle\int \frac{x}{(x-1)^2(1+x^2)}\,dx = \int \frac{\frac{1}{2}}{(x-1)^2} + \frac{-\frac{1}{2}}{1+x^2}\,dx =$

partialbråksuppdelning

$$\frac{x}{(x-1)^2(1+x^2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{1+x^2}$$

$= x^2-2x+1$

$$= \frac{A(x-1)(1+x^2) + B(1+x^2) + (Cx+D)(x-1)^2}{(x-1)^2(1+x^2)}$$

$$= \frac{(A+C)x^3 + (-A+B-2C+D)x^2 + (A+C-2D)x + (-A+B+D)}{(x-1)^2(1+x^2)}$$

$$\Rightarrow \begin{cases} A+C=0 \\ -A+B-2C+D=0 \\ A+C-2D=1 \\ -A+B+D=0 \end{cases} \Rightarrow \begin{cases} A=0 \\ B=\frac{1}{2} \\ C=0 \\ D=-\frac{1}{2} \end{cases}$$

$$= \frac{1}{2}\int \frac{1}{(x-1)^2}\,dx - \frac{1}{2}\int \frac{1}{1+x^2}\,dx =$$

$$= -\frac{1}{2}\,\frac{1}{x-1} - \frac{1}{2}\arctan x + C$$

Ex: $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1+x^2-x^2}{(1+x^2)^2}\,dx =$

$$= \int \frac{1}{1+x^2}\,dx - \int \frac{x^2}{(1+x^2)^2}\,dx = \arctan x - \int \frac{x}{2}\,\frac{2x}{(1+x^2)^2}\,dx$$

$$= \arctan x + \frac{x}{2}\,\frac{1}{1+x^2} - \int \frac{1}{1+x^2}\,\frac{1}{2}\,dx =$$

$$= \arctan x + \frac{x}{2}\,\frac{1}{1+x^2} - \frac{1}{2}\arctan x + C =$$

$$= \frac{1}{2}\arctan x + \frac{x}{2}\,\frac{1}{1+x^2} + C$$

425 b) Bestäm värdemängden till $f(x) = x^x (1-x)^{(1-x)}$ då $0 < x < 1$.

Lösn: $f(x) = e^{x\ln x} e^{(1-x)\ln(1-x)}$

$$f'(x) = \left(D e^{x\ln x}\right) e^{(1-x)\ln(1-x)} + e^{x\ln x}\left(D e^{(1-x)\ln(1-x)}\right) =$$

$$= e^{x\ln x} D(x\ln x)\, e^{(1-x)\ln(1-x)} + e^{x\ln x} e^{(1-x)\ln(1-x)} D\big((1-x)\ln(1-x)\big)$$

$$= e^{x\ln x}(\ln x + 1) e^{(1-x)\ln(1-x)} + e^{x\ln x} e^{(1-x)\ln(1-x)}\left(-\ln(1-x) + \frac{1-x}{1-x}(-1)\right)$$

$$= x^x (1-x)^{(1-x)}\left(\ln x + 1 - \ln(1-x) - 1\right) = x^x (1-x)^{(1-x)} \ln\left(\frac{x}{1-x}\right)$$

$$f'(x) = 0 \Leftrightarrow \frac{x}{1-x} = 1 \Rightarrow x = 1 - x \Rightarrow x = \frac{1}{2}$$

$$f\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^{\frac{1}{2}}\left(1 - \tfrac{1}{2}\right)^{1-\frac{1}{2}} = \left(\tfrac{1}{2}\right)^{\frac{1}{2}}\left(\tfrac{1}{2}\right)^{\frac{1}{2}} = \tfrac{1}{2}$$

$$\lim_{x \to 0^+} x^x (1-x)^{(1-x)} = \lim_{x \to 0^+} e^{x \ln x} e^{(1-x)\ln(1-x)}$$

$$= e^{\lim_{x \to 0^+} x \ln x} = e^0 = 1$$

$$\lim_{x \to 1^-} e^{x \ln x} e^{(1-x)\ln(1-x)} = e^0 = 1$$

$\Rightarrow$ värdemängden blir antingen

$[\frac{1}{2}, 1]$ eller $[\frac{1}{2}, 1)$

$\frac{1}{2} \leq y \leq 1$ $\qquad$ $\frac{1}{2} \leq y < 1$

Då $1 > x > 0$ blir $x \ln x < 0$ och

$(1-x)\ln(1-x) < 0$ så $f(x)$ blir aldrig

1. SVAR: Värdemängden blir $[\frac{1}{2}, 1)$.