(*) $y'' - y' + 2y = \sin x = \operatorname{Im} e^{ix}$

(**) $u'' - u' + 2u = e^{ix}$

Om u är en lösning till (**)
då är $\operatorname{Im} u$ en lösning till (*)

Skriv u som $U + iV$ där U och
V är reella funktioner

$$U'' + iV'' - U' - iV' + 2U + 2iV = \cos x + i \sin x$$

$$(U'' - U' + 2U) + i(V'' - V' + 2V) = \cos x + i \sin x$$

$\Rightarrow$ $U'' - U' + 2U = \cos x$ och $V'' - V' + 2V = \sin x$

Trigonometriska integraler

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$D \ln|\cos x| = \frac{1}{\cos x} D \cos x = -\frac{\sin x}{\cos x}$$

$$\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} \, dx = -\ln|\cos x| + C$$

$$\int \sin x \cos x \, dx = \begin{bmatrix} u = \sin x \\ du = \cos x \, dx \end{bmatrix} = \int u \, du = \frac{u^2}{2} + C$$

$$= \frac{\sin^2 x}{2} + C \qquad D \frac{\sin^2 x}{2} = \frac{\cancel{2} \sin x \cos x}{\cancel{2}}$$

Rationella uttryck i sin och cos kan integreras genom att använda substitutionen $t = \tan\frac{x}{2}$

Ex. $\displaystyle\int \frac{dx}{\cos x} = \left[\begin{array}{l} t = \tan\frac{x}{2} \\ dt = \left(1+\tan^2\frac{x}{2}\right)\frac{1}{2}\,dx = (1+t^2)\frac{1}{2}\,dx \\ dx = \frac{2\,dt}{1+t^2} \end{array}\right] =$

$$\left(\cos x = \cos 2\frac{x}{2} = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} = \right.$$

$$\left. = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{\cos^2\frac{x}{2}\left(1-\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right)}{\cos^2\frac{x}{2}\left(1+\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right)} \right)$$

$$\int \frac{dx}{\cos x} = \int \frac{1}{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}\,dx = \int \frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}\,dx =$$

$$= \left[\begin{array}{l} t = \tan\frac{x}{2} \\ \frac{2dt}{1+t^2} = dx \end{array}\right] = \int \frac{\cancel{1+t^2}}{1-t^2}\,\frac{2\,dt}{\cancel{1+t^2}} = \int \frac{2\,dt}{1-t^2} =$$

$$\left(\frac{2}{1-t^2} = \frac{2}{(1-t)(1+t)} = \frac{1}{(1-t)} + \frac{1}{1+t} = \frac{1+t+1-t}{(1-t)(1+t)} \right)$$

$$= \int \frac{1}{1-t}\,dt + \int \frac{1}{1+t}\,dt = -\ln|1-t| + \ln|1+t| + C$$

$$= \ln\left|\frac{1+t}{1-t}\right| + C = \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C$$

Alt

$$\int \frac{dx}{\cos x} = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx =$$

$$= \left[\begin{array}{l} t = \sin x \\ dt = \cos x\,dx \end{array}\right] = \int \frac{dt}{1-t^2} = \frac{1}{2}\int \frac{1}{1-t} + \frac{1}{1+t}\,dt =$$

$$\left(\frac{1}{1-t^2} = \frac{1}{(1-t)(1+t)} = \frac{\frac{1}{2}}{1-t} + \frac{\frac{1}{2}}{1+t} \right)$$

$$= \frac{1}{2}\left(-\ln|1-t| + \ln|1+t|\right) + C = \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C = \frac{1}{2}\ln\left|\frac{1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}-2\tan\frac{x}{2}}\right| + C =$$

$$\left(\sin x = \sin 2\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \right)$$

$$= \frac{1}{2} \ln \left| \frac{\left(1 + \tan \frac{x}{2}\right)^2}{\left(1 - \tan \frac{x}{2}\right)^2} \right| + C = \ln \left| \left(\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} \right)^2 \right|^{\frac{1}{2}} + C$$

$$= \ln \left| \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} \right| + C$$

Ex: $\displaystyle \int \frac{\cos x}{\cos x + \sin x} dx = \int \frac{\cos x}{\cos x (1 + \tan x)} dx =$

$$= \int \frac{dx}{1 + \tan x} = \left[\begin{array}{l} t = \tan x \\ dt = (1 + \tan^2 x) dx = (1 + t^2) dx \\ dx = \frac{dt}{1 + t^2} \end{array} \right] =$$

$$= \int \frac{1}{1 + t} \frac{dt}{1 + t^2} =$$

$$\left(\frac{1}{(1+t)(1+t^2)} = \frac{A}{(1+t)} + \frac{Bt+C}{1+t^2} = \frac{A(1+t^2)+(Bt+C)(1+t)}{(1+t)(1+t^2)}\right.$$

$$= \frac{(A+B)t^2+(B+C)t+(A+C)}{(1+t)(1+t^2)}$$

$$\Rightarrow \begin{cases} A+B=0 \\ B+C=0 \\ A+C=1 \end{cases} \Longrightarrow \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \\ C=\frac{1}{2} \end{cases}$$

$$\frac{1}{(1+t)(1+t^2)} = \frac{\frac{1}{2}}{1+t} + \frac{-\frac{1}{2}t+\frac{1}{2}}{1+t^2}$$

$$\int \frac{dt}{(1+t)(1+t^2)} = \int \frac{\frac{1}{2}}{1+t}\,dt + \int \frac{-\frac{1}{2}t+\frac{1}{2}}{1+t^2}\,dt =$$

$$= \frac{1}{2}\ln|1+t| + \int \frac{-\frac{1}{2}t}{1+t^2}\,dt + \int \frac{\frac{1}{2}}{1+t^2}\,dt =$$

$$= \frac{1}{2}\ln|1+t| + \left(-\frac{1}{4}\right)\int \frac{2t}{1+t^2}\,dt + \frac{1}{2}\arctan t =$$

$$= \frac{1}{2}\ln|1+t| - \frac{1}{4}\ln|1+t^2| + \frac{1}{2}\arctan t + C =$$

$$= \frac{1}{2}\ln|1+\tan x| - \frac{1}{4}\ln|1+\tan^2 x| + \frac{1}{2}\arctan(\tan x) + C =$$

$$= \frac{1}{2}\ln|1+\tan x| - \frac{1}{4}\ln|1+\tan^2 x| + \frac{1}{2}x + C$$

Om integranden är en rationell funktion i $\tan x$ ger substitutionen $t = \tan x$ ett rationellt uttryck i t.

Ex: $I_n = \int \cos^n x \, dx = \int \cos x \cos^{n-1} x \, dx =$

$= \sin x \cos^{n-1} x - \int \sin x \, (n-1) \cos^{(n-2)} x (-\sin x) \, dx =$

$= \sin x \cos^{n-1} x + (n-1) \int \sin^2 x \cos^{(n-2)} x \, dx =$

$= \sin x \cos^{n-1} x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx =$

$= \sin x \cos^{n-1} x + (n-1) I_{n-2} - (n-1) I_n \Rightarrow$

$\Rightarrow I_n = \frac{1}{n} \left(\sin x \cos^{n-1} x + (n-1) I_{n-2} \right)$

$n-1+1$

Två algebraiska integraler

i) $\displaystyle\int \frac{dx}{\sqrt{a-x^2}} = \int \frac{dx}{\sqrt{a}\sqrt{1-\left(\frac{x}{\sqrt{a}}\right)^2}} = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{1-\left(\frac{x}{\sqrt{a}}\right)^2}} = \left[\begin{array}{l} u = \frac{x}{\sqrt{a}} \\ du = \frac{dx}{\sqrt{a}} \end{array}\right] =$

$\displaystyle = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C = \arcsin \frac{x}{\sqrt{a}} + C$

ii) $\displaystyle\int \frac{dx}{\sqrt{x^2+a}} = \int \frac{D\left(x+\sqrt{x^2+a}\right)}{x+\sqrt{x^2+a}}\,dx = \ln\left|x+\sqrt{x^2+a}\right| + C$

$\displaystyle\int \frac{dx}{\sqrt{x^2+a}} = \left[\begin{array}{l} t = x+\sqrt{x^2+a} \\ dt = \left(1+\frac{1}{2}\frac{1}{\sqrt{x^2+a}}\cdot 2x\right)dx = \left(1+\frac{x}{\sqrt{x^2+a}}\right)dx = \frac{x+\sqrt{x^2+a}}{\sqrt{x^2+a}}\,dx = \frac{t}{\sqrt{x^2+a}}\,dx \end{array}\right.$

$\displaystyle \Rightarrow \frac{dt}{t} = \frac{dx}{\sqrt{x^2+a}}$

$$\int \frac{dx}{\sqrt{x^2+a}} = \left[\begin{array}{l} t = x+\sqrt{x^2+a} \\ \dfrac{dt}{t} = \dfrac{dx}{\sqrt{x^2+a}} \end{array}\right] = \int \frac{dt}{t} = \ln|t| + C$$

$$= \ln\left|x+\sqrt{x^2+a}\right| + C$$

Ex: $$\int \frac{dx}{\sqrt{x^2-2x+2}} = \int \frac{dx}{\sqrt{(x-1)^2+1}} = \left[\begin{array}{l} t = x-1 \\ dt = dx \end{array}\right] =$$

$$= \int \frac{dt}{\sqrt{t^2+1}} = \ln\left|t+\sqrt{t^2+1}\right| + C =$$

$$= \ln\left|x-1+\sqrt{(x-1)^2+1}\right| + C$$

Ex: $\displaystyle\int \frac{dx}{\sqrt{8x-2x^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{4x-x^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{4-(x-2)^2}} =$

$$= \begin{bmatrix} t = x-2 \\ dt = dx \end{bmatrix} = \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{4-t^2}} = \frac{1}{\sqrt{2}}\arcsin\frac{t}{2} + C =$$

$=\sqrt{4}$

$$= \frac{1}{\sqrt{2}}\arcsin\frac{x-2}{2} + C.$$

Minns att $\displaystyle\int \frac{dx}{\sqrt{a-x^2}} = \arcsin\frac{x}{\sqrt{a}} + C$